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Unraveling Market Inefficiencies: Weak Arbitrage and the Information-Based Model for Option Pricing

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Abstract

Discrepancies between theoretical option pricing models and actual market prices create arbitrage opportunities in financial markets. Despite being widely used in option pricing, the famous Black-Scholes model estimates option values based on the strict assumption of no arbitrage. In addition, its assumptions of constant volatility and log-normal asset price distribution may not fully capture real-world market dynamics, resulting in mispricing and potential arbitrage opportunities. The Information-based model is adopted as an alternative to address this, allowing for stochastic volatility, non-specific asset price distributions, and variable transaction costs. This study extends the IBM by developing a pricing equation incorporating weak arbitrage possibilities using the weaker form of no-arbitrage termed as the Zero Curvature condition. The equation incorporates an adjusted risk-free rate, influenced by an arbitrage measure and option derivatives. Empirical findings based on the iShares S&P 100 ETF American call options dataset demonstrate that capturing weak arbitrage improves theoretical option price estimates, reducing discrepancies and potential arbitrage opportunities. Further research can focus on validating and enhancing the Information-based model using alternative financial assets data.

Keywords

Weak Arbitrage, Variable Transaction Costs, Information-Based Model, Zero Curvature, American Call Options

1. Introduction

In financial markets, market inefficiencies emerge from discrepancies between theoretical option pricing models and actual market prices. Theoretical option pricing models, like the Black-Scholes-Merton (BSM) model, employ certain assumptions and parameters to calculate the fair value of options based on underlying asset prices, option strike prices, time to expiration, interest rates, and constant volatility. However, real-world markets are influenced by multiple factors that diverge from the assumptions of these models, giving rise to differences between theoretical prices and observed market prices. These disparities create opportunities for traders and investors to exploit potential mispricings.

Various reasons contribute to the divergence between theoretical option pricing models and market prices. Market frictions, such as transaction costs, bid-ask spreads, and liquidity constraints, can impact option prices and introduce inefficiencies. Moreover, the volatility parameter used in option pricing models may differ from the actual market volatility, leading to pricing errors. Changes in interest rates may also not align with the model assumptions, influencing the pricing of options. In addition, investor sentiment and behavioural biases, such as fear or optimism, can drive demand for options, affecting their prices. Traders with superior information can also influence market prices, leading to deviations from theoretical values. These inefficiencies create arbitrage opportunities for market participants, albeit often short-lived, as trading activity corrects the discrepancies over time.

The no-arbitrage assumption is a fundamental concept in modern financial econometrics which specifies that markets are efficient and there are no opportunities to make risk-free profits. According to [1], the assumption of *No-arbitrage* (NA) summarized under the First Fundamental Theorem of Asset Pricing (FFTAP) for discrete-time finite-state stochastic models, assumes that the absence of arbitrage opportunities is equivalent to the existence of an Equivalent Martingale Measure (EMM) for the risky asset. The extension to the finite-horizon infinite state and infinite-horizon was proved by [2] and [3] respectively. More versions of the theorem were proven by [4] [5] [6] and [7], take into account continuous-time models. For these extensions, the condition of NA turns out to be weak and has to be replaced by a stronger assumption. The studies show that the existence of an EMM is equivalent to the stronger version called the *No Freel Lunch with Vanishing Risk* (NFLVR) for asset price processes that follow a general semi-martingale.

The NFLVR condition has become popular and most option pricing models are based on its assumptions despite the options market experience being inconsistent with this concept since arbitrage possibilities occur naturally for short periods. In addition, the NFLVR represents a very strong assumption about market dynamics as it does not take into account market frictions, such as transaction costs and liquidity constraints. In practice, these frictions can create arbitrage opportunities when an option's market price deviates from its theoretical price, allowing traders to buy and sell options to make a risk-free profit. The BSM model revolutionized option pricing under the NFLVR condition. The model assumes that there are no arbitrage opportunities in the market, meaning that the price of the option should be equal to the expected value of its future payoff. Consequently, a derivative can be replicated through the creation of a risk-neutral portfolio consisting of the underlying asset and the risk-free asset.

Several studies have investigated the empirical validity of the NFLVR condition, in particular the BSM model assumptions. In a study by [8], a sample of traded options was used to examine whether the BSM model's NFLVR assumption holds in practice. The results of the study indicated significant deviations in the model's predictions from the market prices, suggesting the presence of arbitrage opportunities. Another study by [9] investigated the use of the BSM model in the presence of trading frictions. The transaction costs had a significant impact on option pricing, leading to mispricing by the BSM, and hence arbitrage opportunities. [10] examined the American options market for the S&P 100 Exchange-Traded Fund (ETF) and found that the prices of call options were often higher than the theoretical values calculated using the BSM option pricing model. The results implied that there were arbitrage opportunities in the market, and these opportunities could be exploited using a riskless trading strategy involving buying a call option and short-selling a combination of the underlying ETF and a risk-free bond. [11] examined the pricing of European call options on the French stock market. The study found evidence of price discrepancies which were positively correlated with the option's time to expiration. These results suggest that longer-dated options are more susceptible to market frictions.

[12] also identified price deviations from the BSM predictions on the S&P 500 index American call option market. The study found that the BSM model predictions were reasonably accurate for short-dated options, but significant deviations occurred for longer-dated options. These deviations resulted from the presence of transaction costs and the difficulty in replicating the option's payoff using the underlying asset. [13] analyzed the S&P 500 options market from 1987 to 2009 and found that the BSM model significantly overpriced deep out-of-the-money options, leading to potential arbitrage opportunities for traders. They also found evidence of a volatility smile pattern, where the implied volatility of options varies with the strike price, contradicting the constant volatility assumption of the BSM model. [14] and [15] examined the relationship between option prices and implied volatility on index options based on Heston's model. The results indicated that Heston's model is more likely to overprice out-of-the-money options and underprice in-the-money options albeit it gives more accurate estimates than the BSM model. However, BSM outperformed the Heston model for short-term in-the-money options where the latter was unable to capture the high implied volatility. More recent studies that provide evidence of model mispricing include the works of [16] [17] [18] [19] and [20]. The findings suggested that pricing inefficiency was more pronounced during periods of high market volatility.

The preceding empirical evidence that invalidates the assumption of the FFTAP prompted further research on the development of option pricing theory that does not depend on the existence of an equivalent martingale measure. Some studies focus on developing models which take into account short-lived arbitrage possibilities. One of the earliest attempts to include arbitrage possibilities in derivative pricing was by physicists [21] [22] and [23]. Arbitrage is characterised as a random or virtual arbitrage return which follows the mean-reverting Ornstein-Uhlenbeck (OU) process. The approach was later adopted by mathematicians in the field of finance. [24] and [25] considered the arbitrage return to be a component of short-term stochastic interest rates and applied it to the BSM model. The main limitation of this approach was that classical hedging was impossible because the random interest rate was not tradable. [26] modified the BSM model to include endogenous arbitrage opportunities modelled using the OU process. [27] later developed an asymptotic pricing theory by assuming that the option price and random arbitrage opportunities change on different time scales. The approach resulted in pricing bands that were independent of the detailed statistical characteristics of the random arbitrage return.

Other studies have provided a weaker characterization of the strong no-arbitrage NFLVR assumption that admits the possibility of arbitrage opportunities. The works of [28] [29] [30] and [31] demonstrated that the full strength of NFLVR is not needed in order to solve option valuation problems for models that incorporate market frictions. Over the past three decades, other alternative forms of no-arbitrage conditions have been proposed in the literature, which are weaker than the classical and strong NFLVR and NA conditions. Some of the weaker forms of no-arbitrage conditions include the *no unbounded profit with bounded risk* (NUPBR), *No Increasing Profit* (NIP), *No Strong Arbitrage* (NSA), and *No Arbitrage of First Kind* (NA1) [32] and [33] show that the NUPBR condition is equivalent to the existence of a strict martingale deflator for the price process, which is a concept weaker than the EMM. Similarly, [34] shows that the NIP, NSA and NA1 can be fully characterized in terms of the semi-martingale characteristics of the discounted underlying asset process, which is not possible for the strong NA and NFLVR conditions because the latter depends on the structure of the filtration. The findings from these studies suggest that the strong no-arbitrage condition (NFLVR) implies the weaker no-arbitrage conditions.

Another representation of the weaker form of no-arbitrage is based on Geometric Arbitrage Theory (GAT) which gives a geometric interpretation of arbitrage. GAT allows the modelling of the financial market as a stochastic principle fibre bundle and arbitrage corresponds to its curvature. [35] provide a general measure of arbitrage curvature for any financial market model governed by an Itô process. The arbitrage measure is shown to be invariant under the change of numéraire and equivalent probability. [36] shows that the NFLVR implies the vanishing of the curvature and refers to this phenomenon as the Zero Curvature (ZC) condition. Later, [37] prove the equivalence between the ZC and the

NUPBR condition. Mathematical results showed that the ZC condition is equivalent to a weaker form of economic equilibrium, and can therefore be considered a form of market efficiency. An extension of the BSM model is also given that accounts for arbitrage under the ZC condition.

The existing literature on option pricing with arbitrage primarily revolves around the well-known Black-Scholes model. However, the model still has significant limitations due to its assumptions of constant volatility and the log-normal distribution for asset prices. As a result, it has faced criticism for its accuracy in pricing options. To address these limitations, [38] introduced the Information-Based Model (IBM), which allows for stochastic volatility and any distribution for asset price movements. The Information-based model proposes that the asset price process is 'information-driven' rather than following a specific law. Since its inception, IBM has undergone extensions and modifications to accommodate various market dynamics in option pricing and to price different types of options. In [39], the first attempt was made to price early exercise Bermudan-style options under the information-based framework while considering variable transaction costs of trading.

In this study, the main objective is to expand on the IBM approach, enabling the pricing of plain-vanilla options considering both variable transaction costs and weak arbitrage possibilities. To achieve this, an arbitrage measure is introduced, that helps quantify small arbitrage opportunities present in the market. By doing so, a more accurate representation of the real market setting can be achieved, leading to improved price estimates and a reduction in the price discrepancies between theoretical and market prices. Consequently, addressing and managing the existing inefficiencies. The strong no-arbitrage assumption of NFLVR commonly used in option pricing models is relaxed to allow the pricing of options under a weaker notion of no-arbitrage specified by the ZC condition. Subsequently, empirical evidence is presented to showcase that financial markets indeed encounter arbitrage opportunities, which can be successfully incorporated into theoretical option pricing models by integrating an arbitrage measure to account for these weaker arbitrage possibilities.

The motivation behind this work stems from several aspects. Firstly, there is a lack of existing studies that consider arbitrage possibilities in option pricing within the information-based modelling framework. Additionally, the current literature on option pricing with arbitrage under the Black-Scholes model primarily provides abstract descriptions of weaker forms of no-arbitrage, overlooking the crucial aspect of quantifying arbitrage to develop pricing models that account for weak arbitrage. Furthermore, there are few empirical studies demonstrating option pricing while considering an arbitrage measure.

The rest of the paper is structured as follows: Section 2 provides an explanation of the primitives of the classical market model in stochastic finance. It also covers various concepts related to arbitrage in option pricing and the foundation of pricing under the information-based model. Section 3 presents the derivation of the information-based model equation, considering the presence of weak arbitrage. The derivation is based on the ZC condition under GAT and extends the assumption to include variable transaction costs of trading characterized by the non-increasing exponential transaction cost function. In Section 4, an empirical study is conducted to demonstrate the practical implementation of the model and present empirical evidence regarding the implications of estimating theoretical option prices while accounting for weak arbitrage possibilities. Finally, Section 5 gives concluding remarks and suggestions for future research.

2. Arbitrage and the Information-Based Model in Option Pricing

2.1. The Classical Market Model

In this paper, continuous time trading is assumed, where the set of trading dates is considered to be the interval $\left[0, T \in +\infty\right]$. This assumption encompasses cases of both finite and infinite discrete times, as well as a finite horizon represented in continuous time. Real-world trading occurs at specific moments in time, which are not known a *priori*. Therefore, adopting the continuous time assumption proves to be a suitable choice as it simplifies mathematical analysis and supports the development of robust financial models.

Let the uncertainty be modelled by the filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{t>0}\right)$, where Ω is the non-empty set representing the sample space of all possible future states of the financial market, $\mathcal{F}$ is the σ-algebra representing subsets of Ω, and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. The sequence $\{\mathcal{F}_t : t \in T\}$ of σ-algebras such that $\mathcal{F}_t \subset \mathcal{F}_{t+1} \subset \mathcal{F}$ for every $t \in T$ is the market filtration which satisfies the condition of right continuity. Suppose the market consists of a bank account denoted by *B*, which acts as the numéraire and grows according to a deterministic risk-free rate of interest *r*, and finitely many risky assets denoted by S^i. The risky assets are indexed by $i = 1, 2, 3, \cdots, n$ and are assumed to be semi-martingales. The discounted prices of all risky assets are represented in terms of the numéraire and defined as the normalized asset processes $\bar{S}_t^i = \frac{S_t^i}{B_t}$.

To build upon the concept of arbitrage in option pricing, several definitions and notations extracted from [7] are introduced.

A portfolio or a strategy is an $\mathcal{F}_t$-predictable stochastic processes $\hat{\Pi}_S(t)$ defined by

$$\hat{\Pi}_S(t) = \Pi_S^1(t), \Pi_S^2(t), \cdots, \Pi_S^n(t) \tag{1}$$

where each $\Pi_S^i(t)$ represents number of shares of the risky asset *i* held at time *t* and $i = 1, 2, \cdots, n$. The time *t*-value of the portfolio denoted by $\Pi(t)$ is given by

$$\Pi(t) = \sum_{i=1}^{n} \Pi_S^i(t) S_t^i \tag{2}$$

A portfolio $\hat{\Pi}_S(t)$ is called admissible if its value is a.s. non-negative. This means

$$\Pi(t) \geq 0, \mathbb{P} - a.s. \tag{3}$$

The portfolio $\hat{\Pi}_S(t)$ is said to be self-financing if

$$\mathrm{d}\Pi(t) = \sum_{i=1}^{n} \Pi_S^i(t)\mathrm{d}S_t^i \tag{4}$$

A self-financing portfolio process is called admissible if it is *S*-integrable such that if there exists a constant $a > 0$, the stochastic integral satisfies

$\Pi(t) \cdot S_t \geq -a$ if $\Pi(0) = 0$ and

$(\Pi(t) \cdot S_t)_\infty = \lim_{t \to \infty}(\Pi(t) \cdot S_t)$ exists a.s

A portfolio $\hat{\Pi}_S(t)$ is said to be an arbitrage opportunity if $\hat{\Pi}_S(0) = 0$ and there exists a time $t \in [0, T]$ such that $P(\Pi(t) \geq 0) > 0$.

Definition 2.1 *Let the stochastic process* $\{S_t\}$ *be a semi-martingale,* $\Pi(t)$ *be an admissible self-financing portfolio for the time horizon* $[0, T]$*. Suppose* L^0 *denotes the vector space of real valued measurable functions defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, the space* L^∞ *is the subspace of* L^0 *of all bounded functions. Define the sets:*

- $\mathcal{K} = \{(\Pi \cdot S)_\infty \mid \Pi \text{ is admissible}\}$;
- $\mathcal{K}_a = \{(\Pi \cdot S)_\infty \mid \Pi \text{ is } a\text{-admissible}\}$;
- $\mathcal{A}_0 = \mathcal{K} - L_+^0$;
- $\mathcal{A} = \mathcal{K}_0 \cap L^\infty$.

The semi-martingale *S* is said to satisfy the

1) No-arbitrage (NA) condition if $\mathcal{K} \cap L_+^\infty = \{0\}$ *i.e.* there is no *a*-admissible strategy.

2) No Free Lunch with Vanishing Risk (NFLVR) condition if $\bar{\mathcal{K}} \cap L_+^\infty = \{0\}$. where the bar represents the closure of $\mathcal{A}$ in L^∞.

3) No unbounded profit with bounded risk (NUPBR) condition if $\mathcal{K}_a$ is bounded in L^0.

The relationship between these three types of arbitrages is derived in [32] giving a proof of the result in Equation (5).

$$\text{NFLVR} \Leftrightarrow \text{NA} + \text{NUPBR} \tag{5}$$

Equation (5) indicates that the NFLVR is a stronger no-arbitrage condition as compared to the NA and the NUBPR conditions. The assumption of NFLVR is widely used in most option pricing models which assume the absence of arbitrage in the financial market.

Definition 2.2. *A probability measure on* $\mathcal{F}_t$ *is called an equivalent martingale measure if it has the properties:*

- $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_t$.
- The price process *S* is a martingale under $\mathbb{Q}$ on the time interval $[0, T]$.

Theorem 2.1. ***The First Fundamental Theorem of Asset Pricing:*** *Given a fixed numéraire, the market satisfies the NFLVR property if and only if there exists an equivalent martingale measure* $\mathbb{Q}$.

The First Fundamental Theorem of Asset Pricing (FFTAP) provides the necessary and sufficient conditions for a financial market to be arbitrage-free. The Theorem assumes that a market does not admit arbitrage based on the existence of an EMM which is defined in Definition 2.2. Theorem 2.1 has two implications: Firstly, the existence of an EMM implies absence of arbitrage. Secondly, the absence of arbitrage implies existence of an EMM. The proofs of both implications can be found in [40].

2.2. LRB-Information-Based Modelling Framework

The Information-based model of Brody, Hugston and Macrina (BHM), also known as X-factor theory examines the role of information flow as the main driver of price dynamics. The main result of information-based modelling is that the risky assets S_t^i follow an Itô process which is derived from individual market information processes ξ_t^i. Suppose we consider a single asset S_t which generates cash flows X_t such that the sequence $S_t = \{X_1, X_2, \cdots, X_T\}$ of random variables can be modeled as measurable mappings $S_t : \Omega \to \mathbb{R}$. In addition, the terminal cash flow X_T is assumed to be integrable and has *a priori* continuous distribution ν and is adapted to the filtration $\mathcal{F}_t$. This implies that the asset price process S_t is also adapted to the filtration.

In information-based modelling, the filtration $\mathcal{F}_t$ is assumed to be equivalent to, and generated by a market information process. Suppose $\xi_t \in \mathbb{R}$ represents the market information process, then the σ-algebra generated by ξ_t over the time interval $[0,t]$ is given as

$$\mathcal{F}_t^{\xi_t} = \sigma\{\xi_s; s \le t\} \tag{6}$$

By the assumption of equivalance, $\mathcal{F}_t \equiv \xi_t$.

Let the market information process driving the asset price dynamics be a Lévy Random Bridge (LRB) defined by

$$\xi_t = \lambda t X_T + \beta_t \quad \text{for } 0 \le t \le T \tag{7}$$

where $\lambda \in [0,1]$ denotes the rate of information flow to market participants

$$\lambda = \begin{cases} 0 & \text{if } t = 0 \\ \dfrac{t}{T} & \text{if } 0 < t < T \\ 1 & \text{if } t = T \end{cases} \tag{8}$$

β_t denotes a Brownian bridge process with mean zero and variance $\dfrac{t(T-t)}{T}$ such that $\beta_0 = \beta_T = 0$, and X_T is the terminal cash flow assumed to be known from the onset.

The LRB market information process consists of two parts. The term, $\lambda t X_T$ represents the true information about the value of the cash flow with λ denoting the speed at which the true information about X_T is revealed to market participants and grows in magnitude as t increases. The second part, β_t, represents rumours or partial information about the value of X_T which is not directly accessible to all market participants. Since $\beta_T = 0$ and $\lambda = 1$ at time T, the value of X_T is known at time T and there is no remaining noise such that $\xi_T = X_T$.

The terminal cash flow X_T and the Brownian bridge process β_t are assumed to be independent.

Given a deterministic rate of interest *r*, the martingale price of the risky asset S_t is defined by

$$S_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[X_T \mid \xi_t\right], \quad 0 \le t \le T \tag{9}$$

One can solve Equation (9) by employing first principles, which involves expanding the expectation using the concept of Bayes' rule while considering the *a priori* marginal density of the cash flows X_t denoted by ν. The resulting asset price process is given in Equation (10)

$$S_t = P_t \frac{\int_0^\infty x \exp\left(\frac{T}{T-t}\left[\lambda x \xi_t - \frac{1}{2}\lambda^2 x^2 t\right]\right)\nu \mathrm{d}x}{\int_0^\infty \exp\left(\frac{T}{T-t}\left[\lambda x \xi_t - \frac{1}{2}\lambda^2 x^2 t\right]\right)\nu \mathrm{d}x} \tag{10}$$

where $P_t = e^{-r(T-t)}$ and $\xi_t = \lambda t X_T + \beta_t$ is the LRB market information process. By application of Itô formula and Fujisaki-Kallianpur-Kunit (FKK) theory, the stochastic differential equation for the risky asset S_t given the LRB market information process is derived in [38] and [41] as follows:

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t \mathrm{d}W_t; \quad S_0 = s \tag{11}$$

where $\mu = r$ is the mean return on the stock, W_t is an $\mathcal{F}_t$-Brownian motion defined by

$$W_t = \xi_t - \int_0^t \frac{1}{T-s}\left(\lambda t X_T - \xi_s\right)\mathrm{d}s \tag{12}$$

and σ_t is the absolute volatility process denoted by

$$\sigma_t = \exp\left(-r\left(T-t\right)\right)\frac{\lambda T}{T-t}V_t \tag{13}$$

V_t is the conditional variance of X_t with dynamics defined by

$$\mathrm{d}V_t = -\lambda^2\left(\frac{T}{T-t}\right)^2 V_t^2 \mathrm{d}t + \frac{\lambda T}{T-t}K_t \mathrm{d}W_t \tag{14}$$

where K_t is the conditional skewness of X_t. The conditional variance defined in Equation (14) is a function of W, t and λ showing that the volatility process is stochastic and depends on both time and information flow rate.

The no-arbitrage price of a call option V on an underlying asset that satisfies the SDE in Equation (11) can be obtained through the application of dynamic programming to yield the second-order linear partial differential equation (PDE):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2 + rS\frac{\partial V}{\partial S} - rV = 0 \tag{15}$$

The representation of the option price as a PDE allows one to incorporate transaction costs within the pricing equation through the utilization of Itô formula and a hedging argument. Suppose market frictions characterized by a realistic variable transaction costs are considered. The cost per transaction is assumed to be a non-increasing exponential function of the change in the number of shares denoted by $|\Delta\Pi_S|$ and the LRB market information process, ξ_t per unit of time. This cost is defined by

$$C\left(|\Delta\Pi_S|,\xi_t\right)=C_0\mathrm{e}^{-C_1\xi_t-C_2|\Delta\Pi_S|} \tag{16}$$

where $C_0>0$ is the constant cost of trading, $C_1\geq 0$ is the reduced cost per unit time as $t\to T$ (information cost) and $C_2\geq 0$ is the reduced cost per amount of share traded. $C_1,C_2\in[0,1]$ for all positive values of ξ_t and $|\Delta\Pi_S|$ such that $C\left(|\Delta\Pi_S|,\xi_t\right)\geq 0$. The decision to use non-increasing exponential transaction costs is based on their reflection of realistic transaction costs commonly encountered in financial markets. Nonetheless, other forms of transaction costs can also be employed for analysis.

To derive the PDE with variable transaction costs, a self-financing portfolio Π is structured comprising of Π_S shares of the underlying asset S and one call option V. When accounting for transaction costs, the change in the value of the portfolio is expressed as

$$\mathrm{d}\Pi=\mathrm{d}V+\Pi_S\mathrm{d}S-\Delta C \tag{17}$$

where C denotes the variable transaction costs of trading given by the non-increasing exponential function defined in Equation (16). The PDE can then be obtained through the application of Itô formula together with the assumption that the portfolio is risk-less and must earn the risk-free rate of interest. For a comprehensive derivation of this PDE along with the existence and uniqueness proofs, please refer to [39]. The resultant no-arbitrage pricing equation, accounting for variable transaction costs, is presented as follows:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\hat{\sigma}_t^2+rS\frac{\partial V}{\partial S}-rV=0 \tag{18}$$

where

$$\hat{\sigma}_t^2=\sigma_t^2\left(1-\frac{SC_0}{\sigma_t\sqrt{dt}}\exp\left\{-C_1\left(\lambda tX_T+\frac{t(T-t)}{2T}\right)-C_2\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\frac{\sigma_t}{\sqrt{dt}}\right\}\sqrt{\frac{2}{\pi}}sign\left(\frac{\partial^2 V}{\partial S^2}\right)\right) \tag{19}$$

The utilization of non-increasing exponential costs in the model modifies the PDE into a second-order non-linear equation. This equation incorporates adjusted volatility, which depends on various factors, such as transaction cost rates representing trade size and information cost, time, asset price, first and second-order derivatives of the option price, information flow rate parameter, and terminal cash flow.

3. The Information-Based Model PDE in the Presence of Arbitrage

Pricing with market frictions may introduce short-lived arbitrage opportunities in the market. The inclusion of variable transaction costs in option pricing may cause temporary discrepancies in option prices, which astute investors can exploit to generate risk-free profits through arbitrage. In addition, stochastic volatility in IBM can potentially lead to volatility arbitrage. This may result in situations where the option is either overpriced or underpriced relative to its market value. In order to tackle the potential arbitrage opportunities in information-based modelling, the option price is calculated while incorporating an arbitrage measure that quantifies these possibilities. Nevertheless, it is crucial to recognize that the existence of arbitrage opportunities contradicts the presence of equilibrium in the market. Consequently, the pricing model developed must adhere to an appropriate weak no-arbitrage condition to ensure market efficiency is maintained.

The conceptual structure called Geometric Arbitrage Theory (GAT) is applied to model arbitrage in financial markets in the presence of variable transaction costs and stochastic volatility. GAT involves modelling the market consisting of financial instruments and their term structures as principal fibre bundles. It requires a geometrical reformulation of the market model in order to measure arbitrage (See [35] for more details). For instance, the risky asset acts as a deflator and the risk-free asset as the term structure. This leads to the relaxation of the FFTAP since GAT assumes the existence of a generalization of the EMM given by a martingale deflator. Arbitrage in the market is characterized as a curvature such that the no-arbitrage condition holds when the curvature vanishes. This is termed the Zero Curvature (ZC) condition and provides a weaker assumption of no-arbitrage compared to the NFLVR condition. Unlike other weaker notions of no-arbitrage such as the NUBPR, NA1, NIP and NSA which only give an abstract description of weak no-arbitrage conditions, the ZC assumption quantifies an arbitrage measure that can be tested empirically. As proved by [37], the ZC and NUBPR are equivalent

$$\text{ZC} \Leftrightarrow \text{NUBPR} \tag{20}$$

Thus, using the well-known results in [32] given in Equation (5) leads to following

$$\text{NFLVR} \Leftrightarrow \text{NA} + \text{ZC} \tag{21}$$

The definition of the ZC condition under the Lévy Random Bridge-information-based model (LRB-IBM) is restated as follows:

Definition 3.1. *Suppose the dynamics of the market model follow an Ito process as in Equation* (11) *where the drift parameter* μ *and diffusion process* σ_t *are predictable, and the term structure, B grows according to the rate* $r > 0$. *Let* $e = [1,1,\cdots,1]$, *then the market model satisfies the ZC condition if and only if*

$$Span(\mu + r) = Range(\sigma_t) = Span(e) \tag{22}$$

Equation (22) implies that the market model satisfies the Zero Curvature condition if and only if the curvature vanishes a.s. Based on Definition 3.1 an arbitrage measure on the market model is defined as

$$\alpha = \sum_{n=1}^{N} G_t^n (\mu + r) \equiv 0 \in \mathbb{R} \tag{23}$$

where N is the number of assets in the market model and $G_t = \{G_t^1, G_t^2, \cdots, G_t^N\}$ is the orthonormal basis of $ker(\sigma_t)$. Equation (23) means that arbitrage is measured as a linear combination of orthonormal basis vectors. For a general market model, the method of finding the orthonormal basis vectors as outlined in [35] is given in Definition 3.2

Definition 3.2. *Let* Σ *be a real and symmetric* $N \times N$ *covariance matrix with components* $\Sigma_{ij} = \sum_{n=1}^{N} \sigma_{t_i} \sigma_{t_j}$, *where N is the number of assets in the market model. In addition, define U as an* $N \times N$ *matrix of ones. Then the matrix M is defined by*

$$M = \Sigma - \frac{1}{dim(\Sigma)}(U\Sigma + \Sigma U) + \frac{1}{(dim(\Sigma))^2} Tr(U\Sigma)U \tag{24}$$

Let $\mathcal{N}$ be the null space of matrix M such that the orthonormal basis vectors $\sum_i G_{t_i} = 0$ are orthogonal to the vector of ones, say $e = (1,1,1,\cdots)$.

The determination of the call option price in the presence of arbitrage is summarised under Proposition 3.1.

Proposition 3.1. *Let the market model consist of three assets: a bank account B which acts as the numeráire, a stock S which is driven by the LRB-market information process* ξ_t *and evolves according to the SDE in Equation* (11), *and a call option V written on the underlying asset S. Suppose that there are variable transaction costs of trading represented by the cost function,* $C(|\Delta\Pi|, \xi_t): \mathbb{R}^+ \to \mathbb{R}$ *which is a non-increasing exponential function of the form in Equation* (16), *for all* $\xi_t > 0$, $|\Delta\Pi_S| > 0$. *Then the risk-neutral price of a call option under the weak no-arbitrage Zero Curvature condition solves the non-linear second-order partial differential equation*:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\hat{\sigma}_t^2 + \frac{\partial V}{\partial S} S \left(r + \alpha \left(\frac{1 - \frac{\partial V}{\partial S}}{\sqrt{2}\sqrt{1 + \left(\frac{\partial V}{\partial S}\right)^2 - \frac{\partial V}{\partial S}}} \right) \right) - rV \approx 0 \tag{25}$$

where α measures the arbitrage allowed in the market. The corresponding partial differential operator is defined as

$$\mathcal{L} = \frac{\partial}{\partial S} S(r + \tilde{\alpha}) + \frac{1}{2}\frac{\partial^2}{\partial S^2}\hat{\sigma}_t^2 - rV \tag{26}$$

where $\tilde{\alpha} \approx \alpha \left(\frac{1 - \frac{\partial V}{\partial S}}{\sqrt{2}\sqrt{1 + \left(\frac{\partial V}{\partial S}\right)^2 - \frac{\partial V}{\partial S}}} \right)$.

The modified stochastic volatility $\hat{\sigma}_t$ is defined by Equation (19).

Proof of Proposition 3.1. The market model comprises of three assets namely the risk-free asset (B), risky asset (S), and a call option (V) with dynamics respectively given by

$$\mathrm{d}B = rB\mathrm{d}t \tag{27}$$

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \tag{28}$$

$$\mathrm{d}V=\underbrace{\left(\frac{\partial V}{\partial t}+\frac{\partial V}{\partial S}\mu S+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2\right)\mathrm{d}t+\frac{\partial V}{\partial S}\sigma_t\mathrm{d}W}_{aV\mathrm{d}t+b_t\mathrm{d}W} \tag{29}$$

where a is the drift parameter and b_t is the diffusion process of the option. Following Definition 3.1, we need to obtain the null space of the market $\mathcal{N}$ and obtain its corresponding orthonormal basis. First, the covariance matrix Σ relating to the market model is determined.

$$\Sigma=\begin{pmatrix}0 & 0 & 0\\ 0 & \sigma_t^2 & \sigma_t b_t\\ 0 & \sigma_t b_t & b_t^2\end{pmatrix} \tag{30}$$

Using Σ, the M matrix is obtained using Equation (24) and is given by

$$M=\frac{1}{9}\begin{pmatrix}\sigma_t^2+b_t^2+\sigma_t b_t & -2\sigma_t^2+b_t^2-\sigma_t b_t & \sigma_t^2-2b_t^2-\sigma_t b_t\\ -2\sigma_t^2+b_t^2-\sigma_t b_t & 4\sigma_t^2+b_t^2-4\sigma_t b_t & -2\sigma_t^2-2b_t^2+5\sigma_t b_t\\ \sigma_t^2-2b_t^2-\sigma_t b_t & -2\sigma_t^2-2b_t^2+5\sigma_t b_t & \sigma_t^2+4b_t^2-4\sigma_t b_t\end{pmatrix} \tag{31}$$

The resulting matrix M has a complex form as given in Equation (31) making it difficult to obtain the null space. Instead, the concept of eigen values and eigen vectors are used to indirectly obtain the null space of the matrix.

The eigen values for M are $\lambda_1=\frac{2\left(\sigma_t^2-\sigma_t b_t+b_t^2\right)}{3}, \lambda_2=0$ and $\lambda_3=0$. The corresponding eigen vectors are

$$V_1=\begin{bmatrix}\frac{\sigma_t+b_t}{\sigma_t-2b_t}\\ \frac{-2\sigma_t+b_t}{\sigma_t-2b_t}\\ 1\end{bmatrix}, V_2=\begin{bmatrix}\frac{2\sigma_t-b_t}{\sigma_t+b_t}\\ 1\\ 0\end{bmatrix}, V_3=\begin{bmatrix}\frac{-\sigma_t+2b_t}{\sigma_t+b_t}\\ 0\\ 1\end{bmatrix} \tag{32}$$

where $MV_2=MV_3=0$ and $MV_1\neq 0$. The basis of the null space is then determined by projecting V_2 or V_3 onto the space orthogonal to $e=(1,1,1)$.

Definition 3.3 *For any given matrix G of order* $n\times p$, $n\leq p$, *where* $G'G$ *is non-singular, the projection matrix P is defined by*

$$P=G\left(G'G\right)^{-1}G' \tag{33}$$

It follows that $P^2=P$ and $P(1-P)=O$.

Based on Definition 3.3, the projection matrix P onto the range space of the 1 × 3 matrix $G\equiv G_t$ is given by

$$P=\frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\Rightarrow 1-P=\frac{1}{3}\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix} \tag{34}$$

The normalized null vector corresponding to V_3 is thus given by

$$G_t=\frac{(1-P)V_3}{\left\|(1-P)V_3\right\|}=\frac{(1-P)V_3}{\sqrt{\left((1-P)V_3\right)^2}}=\frac{1}{\sqrt{2}\sqrt{\sigma_t^2+b_t^2-\sigma_t b_t}}\begin{pmatrix}-\sigma_t+b_t\\ -b_t\\ \sigma_t\end{pmatrix} \tag{35}$$

where the condition $\sum G_t=0$ is satisfied implying that the curvature vanishes.

However, to account for arbitrage which corresponds to the realistic case where $\alpha \neq 0$, we introduce some randomness in the drift terms given in Equations (28)-(29). The deterministic drift terms are decomposed as opposed to the diffusion terms because the tangent space $\mathrm{d}S_t$ and $\mathrm{d}V$ has a natural decomposition into the direction which contains the randomness $\sigma_t \mathrm{d}W_t$ and the ones orthogonal to it. The decomposition of the drift terms allowing for arbitrage under the weak no-arbitrage ZC condition is given by

$$\bar{\mu} = \zeta \bar{\sigma}_t + \alpha G_t' \tag{36}$$

where

$$\bar{\mu} = \begin{bmatrix} 0 \\ a_t \\ \mu \end{bmatrix}, \bar{\sigma}_t = \begin{bmatrix} 0 \\ \sigma_t \\ b_t \end{bmatrix} \tag{37}$$

ζ is the market price of risk and α is the arbitrage measure defined by Equation (23). The dynamics of the market model can be rewritten as

$$\mathrm{d}B = rB\mathrm{d}t \tag{38}$$

$$\mathrm{d}S_t = \left(\zeta \sigma_t + \alpha_t G_t^2\right) S_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \tag{39}$$

$$\mathrm{d}V = \left(\zeta b_t + \alpha_t G_t^3\right) V \mathrm{d}t + b_t \mathrm{d}W \tag{40}$$

$$= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\left(\zeta \sigma_t + \alpha_t G_t^3\right) S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2\right)\mathrm{d}t + \frac{\partial V}{\partial S}\sigma_t \mathrm{d}W \tag{41}$$

Recall that the change in the value of the portfolio considering transaction costs is given by

$$\mathrm{d}\Pi = \mathrm{d}V + \Pi_S \mathrm{d}S - \Delta C \tag{42}$$

where ΔC represents the change in transaction costs and is defined in [39] as follows:

$$\Delta C = \frac{S}{2} C\left(\left|\Delta \Pi_S\right|, \xi_t\right) \cdot \left|\Delta \Pi_S\right| \tag{43}$$

$$= \frac{S}{2}\sigma_t \left(C_0 \left|\frac{\partial^2 V}{\partial S^2}\right| \sqrt{\frac{2}{\pi \mathrm{d}t}} - C_1 \lambda t X_T \left|\frac{\partial^2 V}{\partial S^2}\right| \sqrt{\frac{2}{\pi \mathrm{d}t}} - C_2 \sigma_t \left(\frac{\partial^2 V}{\partial S^2}\right)^2\right)\mathrm{d}t \tag{44}$$

Substituting the new dynamics of $\mathrm{d}S$ and $\mathrm{d}V$ in Equation (42) representing change in the value of the portfolio yields

$$\begin{aligned} \mathrm{d}\Pi = &\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\left(\zeta \sigma_t + \alpha G_t^3\right) S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2\right)\mathrm{d}t + \frac{\partial V}{\partial S}\sigma_t \mathrm{d}W \\ &+ \Pi_S\left(\left(\zeta \sigma_t + \alpha G_t^2\right) S_t \mathrm{d}t + \sigma_t \mathrm{d}W_t\right) - \frac{S}{2} C\left(\left|\Delta \Pi_S\right|, \xi_t\right) \cdot \left|\Delta \Pi_S\right| \end{aligned} \tag{45}$$

The last part of Equation (45) given by $\frac{S}{2} C\left(\left|\Delta \Pi_S\right|, \xi_t\right) \cdot \left|\Delta \Pi_S\right|$ remains unchanged since

$$\Delta \Pi_S \approx -\frac{\partial^2 V}{\partial S^2}\mathrm{d}S \qquad \Rightarrow \Delta \Pi_S = \left(-\zeta \sigma_t \frac{\partial^2 V}{\partial S^2} - \alpha G_t^2 \frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t - \frac{\partial^2 V}{\partial S^2}\sigma_t \mathrm{d}W \qquad \text{but}$$

$\mathrm{d}t = 0$ hence the decomposed drift term is zero. This means that arbitrage does not have an effect on the transaction costs of trading.

Rearranging Equation (45) and using $\Pi_S = -\frac{\partial V}{\partial S}$ gives

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2 + \frac{\partial V}{\partial S}\alpha G_t^3 S - \frac{\partial V}{\partial S}\alpha G_t^2 S\right)\mathrm{d}t - \frac{S}{2}C\left(|\Delta\Pi_S|, \xi_t\right)\cdot|\Delta\Pi_S| \quad (46)$$

Inserting the values of G_t^2, G_t^3 defined in Equation (35) into Equation (46) yields

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2 + \frac{\partial V}{\partial S}\alpha S\left(\frac{\sigma_t - \frac{\partial V}{\partial S}\sigma_t}{\sqrt{2}\sqrt{\sigma_t^2 + \left(\frac{\partial V}{\partial S}\right)^2\sigma_t^2 - \sigma_t^2\frac{\partial V}{\partial S}}}\right)\right)\mathrm{d}t - \frac{S}{2}C\left(|\Delta\Pi_S|, \xi_t\right)\cdot|\Delta\Pi_S| \quad (47)$$

Substituting the dynamics of $\frac{S}{2}C\left(|\Delta\Pi_S|, \xi_t\right)\cdot|\Delta\Pi_S|$ previously derived for the variable transaction costs yields

$$\mathrm{d}\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\hat{\sigma}_t^2 + \frac{\partial V}{\partial S}\alpha S\left(\frac{1 - \frac{\partial V}{\partial S}}{\sqrt{2}\sqrt{1 + \left(\frac{\partial V}{\partial S}\right)^2 - \frac{\partial V}{\partial S}}}\right)\right)\mathrm{d}t \quad (48)$$

where $\hat{\sigma}_t^2$ is defined in Equation (19).

Assuming the portfolio grows approximately according to the risk-free rate of interest *r*, the Equation (48) is equated to $r\Pi\mathrm{d}t$ to obtain

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\hat{\sigma}_t^2 + \frac{\partial V}{\partial S}\alpha S\left(\frac{1 - \frac{\partial V}{\partial S}}{\sqrt{2}\sqrt{1 + \left(\frac{\partial V}{\partial S}\right)^2 - \frac{\partial V}{\partial S}}}\right) - rV \approx 0 \quad (49)$$

Equation (49) can be rewritten as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\hat{\sigma}_t^2 + \frac{\partial V}{\partial S}rS - rV \approx -\frac{\partial V}{\partial S}S\tilde{\alpha} \quad (50)$$

where

$$\tilde{\alpha} = \alpha\left(\frac{1 - \frac{\partial V}{\partial S}}{\sqrt{2}\sqrt{1 + \left(\frac{\partial V}{\partial S}\right)^2 - \frac{\partial V}{\partial S}}}\right) \quad (51)$$

This completes the proof.

Remark 1. *In practice, the arbitrage measure* α *is not directly observable in the market making it difficult to solve Equation* (50). [35] *suggests an estimate of* α *to be the largest eigenvalue of the matrix M. When* $\alpha = 0$, *then Equation*

(50) *reduces to the standard no-arbitrage price with variable costs defined in Equation* (18).

The resulting Equation (50) is referred to as the weak no-arbitrage Lévy Random Bridge Information-based Model Partial Differential Equation (LRB-IBM-PDE) for pricing call options on an underlying driven by the LRB market information process. Rearranging this equation yields

$$\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\hat{\sigma}_t^2+\frac{\partial V}{\partial S}r^*S-rV\approx 0 \quad (52)$$

where $r^*=r+\tilde{\alpha}$ is the adjusted risk-free rate of interest and $\tilde{\alpha}$ is a function of the arbitrage measure α and the first derivatives of the option as defined in Equation (51).

Any numerical approach, like Finite Difference Schemes, Finite Element Methods, and Spectral Methods, can be utilized to obtain the numerical solution for the pricing equation. For the specific case of solving Equation (18) using Finite Difference Methods, refer to [42]. This approach can be extended to incorporate the arbitrage measure into the equation.

4. Empirical Results

4.1. Data Description

The data set utilized consists of the daily closing prices of iShares S&P 100 ETF (OEF) American options. The data spans an 8-month period from January 21, 2020, to September 18, 2020, and was retrieved from https://www.ivolatility.com/landing/index.html. The dataset comprises 4901 data points, representing daily observations for 29 distinct call options on OEF shares. These call options have strike prices ranging from \$134 to \$162 and share a common expiration period of 8 months. iShares S&P 100 ETF Options are among the most traded American-style options in the NYSE market. The fund tracks the investment results of the S&P 100 index composed of 100-large capitalization U.S equities. ETF options are known for their high liquidity, which facilitates efficient trading and reliability of the empirical analysis. Despite high liquidity, ETF markets can suffer from temporary inefficiencies, especially during volatile periods or when new information affects market sentiment. These inefficiencies can create opportunities for arbitrageurs to capitalize on the price discrepancies.

The summary statistics for both the OEF share prices and option prices over the 8-month period are presented in **Table 1**. During this period, the share price of OEF exhibited a range of values, a minimum recorded at \$104.41 and a maximum at \$167.97. The distribution of these share prices displayed a negative skewness of −0.56, indicating that the majority of prices were higher and making it reasonable to consider pricing call options rather than put options. This observation is also evident from the histogram depicted in **Figure 1**. Moreover, the negative skewness suggests that OEF share price deviates from the normal distribution, with most prices exceeding the mean value. Furthermore, the kurtosis of the share prices was 2.81, which is less than 3. This also implies that the distribution of share prices had a lower peak compared to the normal distribution.

Table 1. Summary statistics of the stock price and empirical option price for iShares S&P 100 ETF.

	Share Price	Option Price
Min	104.41	2.50
Max	167.97	17.60
Mean	141.31	8.12
Median	143.50	7.05
Range	63.56	15.10
Std. Deviation	13.41	5.01
Kurtosis	2.81	1.85
Skewness	-0.56	0.44

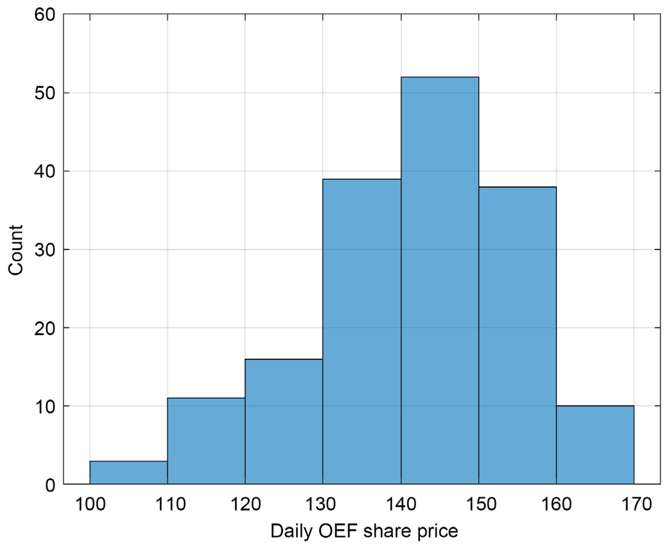

Figure 1. Histogram of daily closing share prices of iShares S&P 100 ETF (OEF) from Jan-Sept 2020.

The summary in **Table 1** also shows that the empirical option prices for the different strikes ranged between $2.5 and $17.6 for the same period. A positive skewness for the option price implies that most of the option prices were lower than the mean value while a positive kurtosis of less than 3 indicates that most prices though lower, were closer to the mean. Overall, these observations highlight the non-normal distribution of both the OEF share prices and the empirical option prices

Figure 2 presents the historical share prices of iShares S&P 100 ETF from January 21, 2020, to September 18, 2020. During this period, there was a notable decrease in the share price around March 2020, likely attributable to the impact of the Covid-19 pandemic. It can be deduced that the share price plunged at this time due to a combination of factual information and market rumours circulating regarding the potential ripple effects of Covid-19. Subsequently, the share price embarked on an upward trajectory with intermittent fluctuations from April to September.

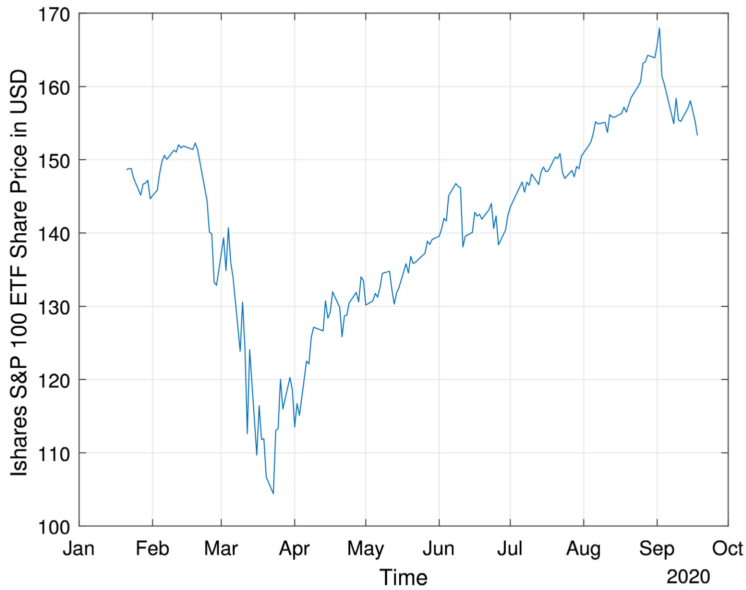

Figure 2. iShares S&P 100 ETF historical daily closing share prices from Jan-Sept 2020.

In **Figure 3**, the plot illustrates the time series of daily log returns of iShares S&P 100 ETF. It is evident that these returns exhibit mean reversion and volatility clustering, which are key features, commonly observed in time series analysis. There are significant spikes observed around March suggesting notable fluctuations in volatility during this time.

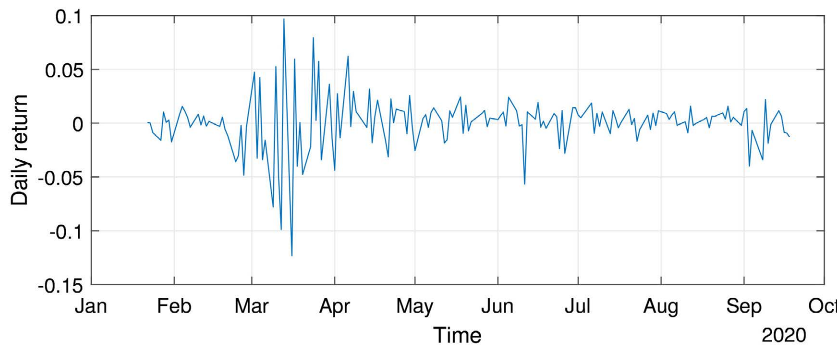

Figure 3. iShares S&P 100 ETF daily returns from Jan-Sept 2020.

4.2. Market Parameters for the Information-Based Model

In the context of information-based modelling, the primary focus is on the terminal cash flow denoted by X_T. This cash flow is generated by the asset price process S_t within the time interval $[0,T]$, where $T = 8$ months. For this analysis, the initial asset price is represented by OEF's share price at the beginning of the contract, denoted as S_0, which is equal to \$148.62. Similarly, the share price at the end of the contract denoted as S_T, is equivalent to the terminal cash flow X_T, with a value of \$153.31. In order to determine the risk-free rate of interest, we use the average one-year U.S. Treasury Bill rate in 2020, which is recorded as

0.37% (source: Macrotrends).

Variable transaction costs included in the Lévy Random Bridge Information-based model pricing equations are estimated based on various factors. These factors include the bid-ask spreads obtained from the dataset, the Options fee schedule provided by the NYSE market, and the information available in the iShares S&P 100 ETF prospectus. Based on the NYSE American Options Fee Schedule for the year 2023, electronic options transaction fees range from 0.25% to 1.50% of the total industry customer equity and ETF option average daily volume. These fees cover broker-dealer fees and other facilitation fees. Most brokerage firms operating under the NYSE charge a commission based on the trade value, which typically falls within the range of 0.1% and 0.5% (source: U.S. News). Additionally, the iShares S&P 100 ETF charges an expense ratio of 0.20%, which represents a management fee automatically deducted from the fund's value.

To account for the constant cost of trading denoted by C_0, the average NYSE electronic options transaction fee of 0.875% is used, along with an additional 0.20% for the expense ratio, resulting in a total of $C_0 = 1.075\%$. The information cost denoted as C_1, is considered to be reflected by the bid-ask spreads, which indicate the presence of information asymmetry in the market. For the considered 8-month period and all the 29 call options related to the iShares S&P 100 ETF, the average bid-ask spread is approximately 0.3799%, which is rounded to 0.38%. The transaction cost rate applied to the trade volume denoted as C_2, is assumed to be 0.3%, which represents the average broker fees charged.

4.3. iShares S&P 100 ETF Volatility and Volatility Surface

The estimated stochastic volatility for OEF during the 8-month period is based on the Euler discretization of Equation (13). The results displayed in **Figure 4** show volatility exhibits stochastic behaviour, displaying variation and fluctuations over time. The volatility ranges between approximately 8% and 18% across the observed period.

The volatility surface of options on iShares S&P 100 ETF is also displayed to depict market expectations regarding future volatility and to evaluate any significant deviations from the estimated volatility. Implied volatility is estimated using the Newton-Raphson method at different strike prices and time to maturity. The outcomes are showcased in **Figure 5**.

The volatility surface of iShares S&P 100 ETF exhibits a downward slope, indicating a negative or reverse skew. This volatility skew suggests that market participants anticipate higher future volatility for options with lower strike prices compared to options with higher strike prices. This can be attributed to a number of factors. Firstly, the distribution of OEF's price movements is left-skewed, meaning that large downward price movements are more probable than large upward movements. This may indicate a greater concern for downside risks rather than upside risks, potentially due to the belief that downside risks are more severe or an expectation of a higher likelihood of negative market

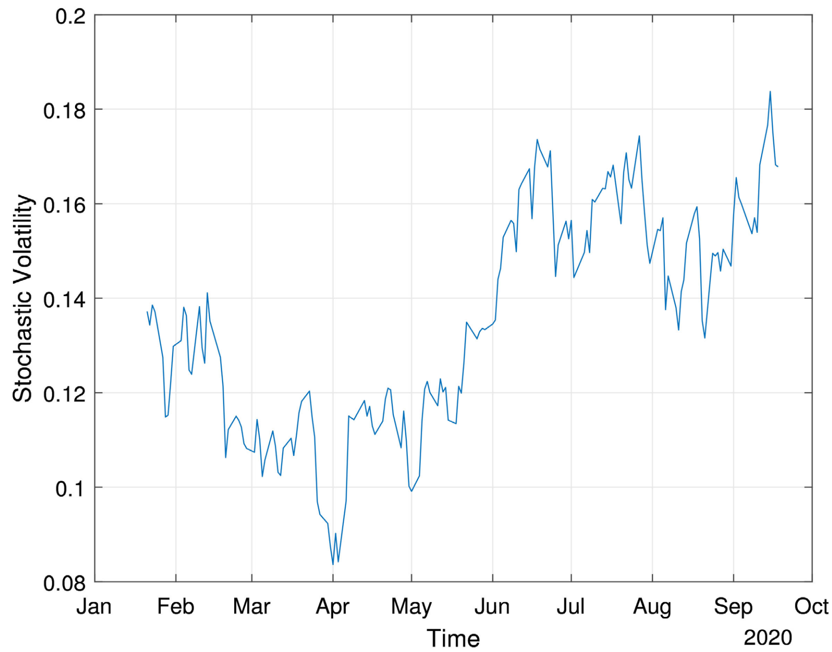

Figure 4. Estimated stochastic volatility for iShares S&P 100 ETF for the period Jan-Sep 2020.

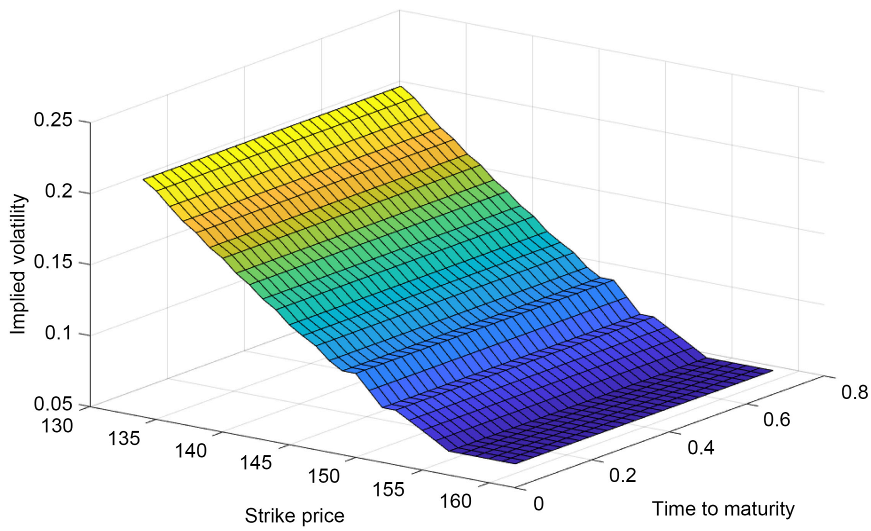

Figure 5. Implied volatility surface for iShares S&P 100 ETF.

shocks. In such a case, in-the-money (ITM) options, are more exposed to potential downward price movements than out-of-the-money (OTM) options. This increased downside risk increases the demand for protection, leading to higher implied volatility. Consequently, market participants are willing to pay a higher premium for options with lower strike prices, anticipating significant price movements in the underlying asset. Conversely, options with higher strike prices have lower implied volatility as market participants anticipate smaller price fluctuations.

Secondly, ITM options have greater intrinsic value than OTM options. This intrinsic value contributes to the total option value and increases the sensitivity of the option's price to changes in the underlying asset's price. As a result, ITM options are more influenced by fluctuations in the underlying asset's price, leading to higher implied volatility. Thirdly, investors and market participants generally exhibit risk aversion, being more concerned about potential losses than gains. This risk aversion is reflected in the higher implied volatility of ITM options as they provide more downside protection. Market participants are willing to pay a higher premium for ITM options due to their higher perceived risk associated with potential adverse price movements. Fourthly, market demand and supply dynamics can also contribute to higher implied volatility for ITM options. The demand for ITM options may be driven by investors seeking to hedge existing positions or speculators anticipating further price movements. The increased demand for ITM options can drive up their prices and, consequently, their implied volatility.

Overall, the implied volatility deviates from the estimated volatility at various points in time. At the inception of the options contract, the implied volatility tends to be slightly higher, while towards maturity, it decreases. This behaviour is expected, as IBM may not fully account for factors such as investor sentiments, market inefficiencies, or unforeseen events that can significantly impact volatility. Moreover, it is noteworthy that the implied volatility generally reaches its lowest level as the option approaches maturity. The reason behind this pattern is that as an option nears its expiration date, there is less time for the underlying asset to undergo substantial price movements. Consequently, the level of uncertainty and the potential for large price swings diminish, leading to a decrease in implied volatility. In information-based modelling, the decline in implied volatility towards maturity can be attributed to the increased availability of information regarding the asset's price, which reduces uncertainty for market participants.

4.4. Pricing with Arbitrage

The Lévy Random Bridge Information-based model is used to price American call options on iShares S&P 100 ETF. It incorporates stochastic volatility, which is adjusted to account for variable transaction costs. Equation (16) defines the modified volatility, considering non-increasing exponential costs. [42] provides additional details on how the numerical approximation of the modified volatility is conducted using the explicit finite difference scheme. By considering transaction costs, the model effectively reduces stochastic volatility.

For the empirical illustration, a near-the-money option is chosen with a strike price of \$149 because the initial underlying asset price is close to \$149 (\$148.65). Additionally, one option each is selected to represent in-the-money (ITM), deep ITM, out-of-the-money (OTM), and deep OTM options, with strike prices differing by intervals of \$5. This selection results in 5 out of the 29 iShares S&P 100 ETF call options with the following strikes: $K = \{\$139, \$144, \$149, \$154, \$159\}$.

The selected market data is utilized to evaluate whether the LRB-IBM-PDE satisfies the weak no-arbitrage Zero Curvature condition. The measure of arbitrage curvature, denoted by α, is determined based on the orthonormal basis of the null space of the market G_t, as defined in Equation (35). At each time step, the value of G_t is calculated, and the resulting values of G_t fulfill the condition $\sum_{n=1}^{3} G_t^n = 0$ for every time step, implying that α is also zero. This observation indicates that the LRB-IBM-PDE satisfies the ZC condition and that market efficiency holds, taking into account the iShares S&P 100 ETF market data.

However, the volatility surface of the iShares S&P 100 ETF, as illustrated in **Figure 5** differs from the estimated stochastic volatility under LRB-IBM-PDE. This discrepancy suggests the existence of potential weak arbitrage opportunities in the market, indicating that α is not equal to zero in reality. Consequently, the market estimate for α is taken as the largest eigenvalue for the matrix M in Equation (24) which is important in determining the orthonormal basis of the null space of the market. The eigenvalues of matrix M are calculated as follows: $\lambda_1 \approx 0.002$, $\lambda_2 = \lambda_3 \approx 0$. Hence, the market estimate for the arbitrage curvature is approximately $\hat{\alpha} \approx 0.002$ across all strike prices. This suggests that potential weak arbitrage opportunities are quantified using a positive arbitrage curvature of 0.002. Subsequently, call prices are calculated by considering variable transaction costs and accounting for these weak arbitrage opportunities. The resulting prices are presented in **Table 2**, where the values in parentheses represent the absolute percentage errors between the estimated theoretical prices and the empirical prices.

Table 2. Comparison of iShares S&P 100 ETF call prices: LRB-IBM with transaction costs and weak arbitrage vs. No Arbitrage.

Strike	Empirical Prices	Non-increasing exponential costs	
		Without arbitrage	With arbitrage
139	13.55	12.57	12.98
		(0.0723)	(0.042)
144	9.95	9.14	9.46
		(0.0814)	(0.0492)
149	6.50	6.28	6.52
		(0.0339)	(0.0031)
154	4.00	4.07	4.23
		(0.0175)	(0.0575)
159	2.50	2.50	2.60
		(0.0000)	(0.0400)
	MAPE	4.10%	3.84%
	MAE	0.4160	0.2820

Based on the results, incorporating positive arbitrage curvature leads to an increase in call prices across all strikes. This adjustment brings the LRB-IBM prices with non-increasing exponential transaction costs closer to the empirical prices, particularly for in-the-money, deep in-the-money, and at-the-money calls, as evidenced by the reduced absolute percentage errors. However, the inclusion of positive curvature results in overpricing of out-of-the-money and deep out-of-the-money calls, which typically have a lower likelihood of accommodating weak arbitrage opportunities due to their low intrinsic value and implied volatility. These results suggest that including arbitrage curvature is not necessary when estimating OTM or deep OTM calls, as it reduces the accuracy of the theoretical estimates. However, including arbitrage curvature becomes crucial when estimating ATM, ITM and deep ITM calls, as it significantly improves the accuracy of the theoretical estimates. Consequently, this helps to reduce the price discrepancies within information-based modelling, thereby removing potential arbitrage opportunities that arise froms such discrepancies.

Additionally, the sMAPE and MAE values for LRB-IBM continuation values were computed to analyze the impact of incorporating arbitrage on these values. The results, presented in **Table 3**, provide insights into the differences between scenarios with and without arbitrage curvature. The results suggest that the estimates of continuation values are improved, especially for ITM, deep ITM, and ATM calls with strikes of $139, $144, and $149, respectively. This improvement is indicated by the reduced sMAPE and MAE values for these options. However, the effect of the positive arbitrage curvature is insignificant for OTM and deep OTM calls with strikes of $154 and $159, respectively. In these cases, the change in sMAPE and MAE is negligible. Thus, by accounting for arbitrage in the computation of continuation values, the theoretical estimates are refined. These improved estimates can then be used to determine the optimal exercise times for the American options.

Table 3. Comparison of sMAPE and MAE for continuation values: LRB-IBM with transaction costs and weak arbitrage vs. No Arbitrage.

	Option Strike	Price without arbitrage	Price with arbitrage
sMAPE	139	25.50%	23.40%
	144	29.77%	28.42%
	149	36.39%	36.14%
	154	56.10%	56.12%
	159	71.02%	71.04%
MAE	139	1.2848	1.2415
	144	0.9903	0.9367
	149	0.9100	0.9105
	154	1.0545	1.0555
	159	1.1517	1.1524

5. Conclusions and Further Research

This study extends the Information-based model with variable transaction costs by developing a pricing equation incorporating weak arbitrage possibilities. The successful attainment of this objective involved a departure from the classical no-arbitrage condition to assuming the Zero Curvature condition for weak arbitrage. Pricing under the ZC condition allowed for the quantification of potential weak arbitrage opportunities using an arbitrage measure. Under the ZC condition, the dynamics of the option and asset price were adjusted to incorporate the arbitrage measure through the decomposition of the respective drift terms. Consequently, the second-order non-linear partial differential equation that represents the fair value of a call option under the information-based model was modified. This modification entailed the inclusion of an adjusted rate of interest earned on the underlying asset, which was expressed as a function of the arbitrage measure, as well as the first and second derivatives of the option. Remarkably, a positive arbitrage measure resulted in an adjusted interest rate exceeding the assumed risk-free rate.

The pricing equation, incorporating variable transaction costs and weak arbitrage possibilities, was validated to satisfy the weak no-arbitrage Zero Curvature condition. This validation was based on the market data from the OEF dataset, where the estimated curvature was determined to be zero. However, significant disparities were observed between the model's estimated stochastic volatility and the OEF's volatility surface, suggesting the presence of potential weak arbitrage opportunities. A positive curvature estimate of 0.002 was used to quantify these opportunities, leading to an improved theoretical call estimate for ATM, ITM, and deep ITM options, but overpricing for OTM and deep OTM options. The findings imply that a positive arbitrage measure is suitable for ATM, ITM, and deep ITM options, as they are more likely to be mispriced due to demand. However, for OTM and deep OTM options, a zero or lower arbitrage measure is recommended. The information-based model, accounting for variable transaction costs and weak arbitrage, offered better accurate estimations of call values and continuation values compared to the no-arbitrage scenario. Therefore, it can be concluded that capturing weak arbitrage in theoretical option pricing can improve price estimates, reducing price discrepancies, and limiting opportunities for arbitrageurs. This holds particularly true when analyzing iShares S&P 100 ETF market data.

Future research can explore alternative approaches for estimating the arbitrage measure since the current method proves unsuitable for all types of options based on moneyness. Such investigations could enhance the model's usability and adoption within the financial industry. Alternatively, the research could focus on identifying other methods to account for weak arbitrage possibilities in the market while preserving market efficiency within the information-based or other option pricing frameworks. For robustness and consistency, it is recommended to conduct further investigations using alternative historical options data. This encompasses a range of datasets, such as those spanning various time periods, diverse market segments, different geographical markets, varying market regimes, and distinct market volatility states. This extended analysis will yield valuable insights into the model's performance across different datasets, reinforcing its credibility and reliability.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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