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Numerical Approximation of Information-Based Model Equation for Bermudan Option with Variable Transaction Costs

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Abstract

Non-linear partial differential equations have been increasingly used to model the price of options in the realistic market setting when transaction costs arising in the hedging of portfolios are taken into account. This paper focuses on finding the numerical solution of the non-linear partial differential equation corresponding to a Bermudan call option price with variable transaction costs for an asset under the information-based framework. The finite difference method is implemented to approximate the option price and its Greeks. Numerical examples are presented and the option prices compared to the closed-form solution of the information-based model and the Black Scholes model with zero transaction costs. The results show that the approximated option prices correspond to the analytical solution of the information-based model but are slightly higher than the prices under Black-Scholes model. These findings validate the finite difference method as an efficient way of approximating the information-based non-linear partial differential equation.

Keywords

Variable Transaction Costs, Information-Based Model, Finite Difference Method, Bermudan Call Option, Greeks

1. Introduction

Option pricing theory has made significant advancements since the development of the Black Scholes and Merton (BSM) pricing framework in 1973 [1]. One such development is the Information-based model (IBM) introduced by [2] which models market filtration explicitly by specifying an information process that provides information about the value of a cash flow process. The standard information processes used include the Brownian bridge, Gamma bridge, Variance Gamma bridge and Lévy Random Bridge (LRB) processes. Most standard option pricing models begin by specifying the law of the price process which is implicitly chosen as the market filtration. For instance, under BSM model, the price process is assumed to be adapted to the Brownian filtration. On the contrary, information-based asset pricing constructs the market filtration explicitly allowing price movements to show more structure, thereby, eliminating the restriction on the distribution of the underlying price process. The IBM has gained traction over the past decade since it allows for more realistic assumptions such as stochastic volatility, stochastic interest rates and other market imperfections such as transaction costs [3]. In the literature, the IBM has been used to price credit risky bonds under stochastic interest rates [4], to model insider trading in an incomplete market [5], as well as price European options and binary bonds [6] [7] for assets with cash flow structures different from the normal distribution.

The information-based model also provides a closed-form pricing formula for European-style contracts under market completeness similar to BSM. However, [7] suggested that the IBM could be used to price early exercise options with American or Bermudan-type exercise rights. In [8], we extended the IBM to include variable transaction costs and derived a partial differential equation (PDE) for valuing a Bermudan call option on an asset driven by the Lévy Random Bridge information process. This is the first attempt to incorporate transaction costs under the IBM and allow for the possibility of pricing an early exercise option. The transaction costs are assumed to be a non-increasing function of the change in the number of shares traded and the LRB-market information process. Specifically, transaction costs are taken to be either exponential or linear non-increasing functions. In the presence of transaction costs, the option cannot be completely hedged thus a closed-form solution is not feasible. Instead, the Bermudan option price is formulated as the value function of an optimal stopping problem which is solved by application of stochastic optimal control.

The Lévy Random Bridge-Information-Based-Model partial differential equation (LRB-IBM-PDE) derived is non-linear due to modified volatility which is a function of time, the underlying stock price, the option price, transaction cost rates and the information process. The non-linear volatility makes it difficult to solve the equation analytically, hence numerical approaches can be explored. Many articles have been published on the numerical approximation of non-linear partial differential equations arising in pricing options. The numerical approaches include tree methods [9] [10] [11], Monte-Carlo methods [12] [13] [14], spectral methods [15] [16] [17] [18], finite element discretization [19] [20] and finite difference schemes [21] [22] [23] [24]. In particular, numerical approximations of the Black-Scholes type equations with non-linear volatility has been performed using finite difference method (FDM) because of its simplicity [25] [26] [27] [28]. For this work, we apply the finite difference method for ease of implementation and faster convergence, since it is the first attempt to represent the option price under the information-based model in the form of a non-linear PDE.

The rest of the paper is organized as follows. In Section 2, the Bermudan call option price equation with variable transaction costs under the information-based model is introduced in detail. In Section 3, the finite difference scheme for the LRB-IBM-PDE is presented. In Section 4, a simulation study is performed to demonstrate the practicability and validity of the FDM in approximating the price of the Bermudan call option. The numerical approximation and results of the option Greeks for the Bermudan call option are then presented in Section 5. Finally, some concluding remarks and direction for future work are given in Section 6.

2. Option Pricing under the LRB-Information Based Model

Consider a financial market consisting of three assets: a risk free asset B which acts as a bank account and grows according to the risk free rate of interest r, a risky asset S which is a stock and an option with the value V. Consider the time interval $[0,T]$, $T\in\mathbb{N}$ such that $0\le t\le T$. The risky asset S is assumed to generate cash flows X_t such that the sequence $S_t=\{X_1,X_2,\cdots,X_T\}$ of random variables can be modeled as measurable mappings $S_t:\Omega\to\mathbb{R}$. In addition, X_t is assumed to be integrable and has *a priori* continuous distribution ν. The uncertainty is modeled by the risk-neutral probability space $(\Omega,\mathcal{F}_t,\mathbb{Q})$. The value of X over the time interval $[0,T]$ is completely determined by the information available in the market. $\mathcal{F}_t^X$ is termed as the market filtration, which is the information generated by observing X over the time interval $[0,t]$ and is cadlag. Therefore, the process X_t is adapted to the filtration $\mathcal{F}_t^X$. Under the information-based framework, the filtration $\{\mathcal{F}_t\}$ is constructed explicitly where the market information process that provides information about X_T is assumed to be a Lévy Random Bridge process defined by

$$\xi_t=\lambda tX_T+\beta_t \text{ for } 0\le t\le T \quad (1)$$

where $\lambda=\frac{t}{T}$ is the rate of information flow to market participants, $\{\beta_t\}$ denotes a Brownian bridge process with mean zero and variance $\frac{t(T-t)}{T}$ and X_T is the terminal cash flow. It follows that, $\{\mathcal{F}_t\}=\sigma\left(\{\xi_s\}\right)_{0\le s\le t}$ *i.e.* $\{\mathcal{F}_t\}$ is a σ-algebra generated by $\{\xi_s\}_{0\le s\le t}$. Given the new market filtration ξ_t, the stock price process S is shown in [7] to evolve according to the following stochastic differential equation (SDE)

$$\mathrm{d}S=\mu S\mathrm{d}t+\sigma_t\mathrm{d}W \quad (2)$$

where $\mu > 0$ is the mean return of the stock, W is a weiner process defined as

$$W_t = \xi_t - \int_0^t \frac{1}{T-s}\left(\lambda t X_T - \xi_s\right)\mathrm{d}s \tag{3}$$

and σ_t is the diffusion parameter defined as

$$\sigma_t = \exp\left(-r\left(T-t\right)\right)\frac{\lambda T}{T-t}V_t \tag{4}$$

where V_t is the conditional variance of X_t with the following dynamics

$$\mathrm{d}V_t = -\lambda^2\left(\frac{T}{T-t}\right)^2 V_t^2\mathrm{d}t + \frac{\lambda T}{T-t}K_t\mathrm{d}W_t \tag{5}$$

with K_t denoting the conditional skewness of X_t

Suppose the *a priori* distribution of X_t is normal, then the solution to the SDE in Equation (2) is derived in [6] as

$$S_t = S_0\exp\left(rt - \frac{1}{2}\sigma_t^2 T + \frac{1}{2}\frac{\sigma_t}{\lambda^2\kappa+1} + \frac{\lambda\kappa\sigma_t\sqrt{T}}{t\left(\lambda^2\kappa+1\right)}\xi_t\right) \tag{6}$$

where $\kappa = \frac{tT}{T-t}$.

The no-arbitrage call option price at time t is then given by

$$V\left(S,t\right) = \mathbb{E}^{\mathbb{Q}}\left(\left(S_t - K\right)^+ \mid \xi_t\right) \tag{7}$$

for a strike price K. The exact solution of Equation (7) for a European option is derived in [29] and given by the following closed-form formula

$$V\left(S,t\right) = S_0\mathrm{e}^{\psi+\frac{1}{2}\delta^2}\Phi\left(d_1\right) - K\Phi\left(-d_2\right) \tag{8}$$

where $\Phi(.)$ is the cumulative distribution function of the standard normal distribution and

$$d_1 = \frac{\log\left(\frac{S_0}{K}\right)+\psi}{\delta} + \delta \tag{9}$$

$$d_2 = d_1 - \delta \tag{10}$$

$$\psi = rt - \frac{1}{2}\sigma_t^2 T + \frac{1}{2}\frac{\sigma_t\sqrt{T}}{\lambda^2\kappa+1} \tag{11}$$

$$\delta^2 = \left(\frac{\lambda\kappa\sigma_t\sqrt{T}}{t\left(\lambda^2\kappa+1\right)}\right)^2\left(\lambda^2 t^2 + \frac{t\left(T-t\right)}{T}\right) \tag{12}$$

Equation (8) can only be used to price a European call option under the assumptions of no-arbitrage and in a frictionless market. To include transaction costs into the IBM and allow for the possibility of pricing early exercise options, the delta hedging strategy is applied so that we have the LRB-IBM-PDE for pricing an option similar to the BSM type equations.

Consider a Bermudan call option with a maturity at time T. The option can be exercised at $P \geq 1$ discrete times or stages that are equally spaced. This means that $t_1 = \Delta t, t_2 = 2\Delta t, \cdots, t_P = P\Delta t = T$, with $\Delta t = \frac{T}{P}$ and $t_1 < t_2 < \cdots < t_P$ for $t_p \in [0,T]$; $p = 1,2,\cdots,P \in \mathbb{N}$. When $P = 1$, the Bermudan call corresponds to a European call option with only one exercise date at maturity. For $P > 1$, the valuation of the Bermudan option transforms to an optimal stopping problem which involves finding the optimal stopping time for exercising the option.

Let the stopping time τ be defined as $\{\tau = t_p\} \in \xi_{t_p}$. The optimal stopping problem is formulated as

$$V(S,t_p) = \max_{t_p \le \tau \le T} \mathrm{e}^{-r(\tau - t_p)} \mathbb{E}\left[(S_\tau - K)^+ \mid \xi_{t_p}\right] \tag{13}$$

and the optimal stopping time by Snell Envelope theorem (see in [30]) for discrete time is defined as

$$\hat{\tau} = \min\left\{t_p \ge 0; V(S,t_p) \le (S_{t_p} - K)^+\right\} \tag{14}$$

The optimal stopping problem in Equation (13) is transformed to a stochastic control problem such that the option price $V(S,t_p)$ satisfies the discrete Bellman equation:

$$\begin{cases} \frac{\partial V(S,t_p)}{\partial t} + \mathcal{L}V(S,t_p) = 0 & \forall\, t_p \in [0,T] \\ V(S,t_p) = (S_{t_p} - K)^+, & \forall\, S \in \mathbb{R} \end{cases} \tag{15}$$

where $V(S,t_p) = (S_{t_p} - K)^+$ is called the exercise or the stopping region, $-\frac{\partial V}{\partial t} = \mathcal{L}^a V(S,t_p)$ is the continuation region and $\mathcal{L}$ is the partial differential operator of the LRB-Information based pricing model. The subscript p in t_p is dropped henceforth, for ease of notation. The partial differential operator is derived in [8] for a call option with variable transaction costs and given as

$$\mathcal{L} = \frac{\partial}{\partial S} rS + \frac{1}{2}\frac{\partial^2}{\partial S^2}\hat{\sigma}_t^2 - rV \tag{16}$$

with the LRB-IBM-PDE defined by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\hat{\sigma}_t^2 + rS\frac{\partial V}{\partial S} - rV = 0 \tag{17}$$

where S evolves according to the dynamics in Equation (2) and the non-linear volatility $\hat{\sigma}_t$ is defined as

$$\begin{aligned} \hat{\sigma}_t^2 = \sigma_t^2 \Bigg(1 - \frac{SC_0}{\sigma_t\sqrt{dt}} \exp\Bigg\{ -C_1\left(\lambda t X_T + \frac{t(T-t)}{2T}\right) \\ - C_2 \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\frac{\sigma_t}{\sqrt{dt}}\Bigg\} \sqrt{\frac{2}{\pi}} sign\left(\frac{\partial^2 V}{\partial S^2}\right)\Bigg) \end{aligned} \tag{18}$$

for non-increasing exponential transaction costs of the form

$$C(|\Delta\Pi_S|, \xi_t) = C_0 \mathrm{e}^{-C_1\xi_t - C_2|\Delta\Pi_S|} \tag{19}$$

or

$$\hat{\sigma}_t^2 = \sigma_t^2 \left(1 - \frac{S}{\sigma_t} \sqrt{\frac{2}{\pi dt}} sign\left(\frac{\partial^2 V}{\partial S^2} \right) \left(C_0 - C_1 \lambda t X_T \right) + SC_2 \frac{\partial^2 V}{\partial S^2} \right) \tag{20}$$

for non-increasing linear transaction costs of the form

$$C\left(\left| \Delta \Pi_S \right|, \xi_t \right) = C_0 - C_1 \xi_t - C_2 \left| \Delta \Pi_S \right| \tag{21}$$

where Π_S is the number of shares of the underlying stock, $C_0 > 0$ denotes the constant cost of trading, $C_1 \geq 0$ denotes the reduced cost per unit time as $t \to T$ due to increased information about the value of the terminal cash flow and $C_2 \geq 0$ denotes the reduced cost per amount of share traded.

3. Numerical Approximation of the LRB-Information-Based Model

Finding a numerical approximation to the solution of the non-linear LRB-IBM-PDE can be challenging. The numerical scheme must converge and be stable to ensure the numerical solution is close to the exact solution. The finite difference method is used to approximate the solution of the LRB-IBM-PDE because it is able to achieve high precision with less computational power as long as some stability and convergence conditions are met. The convex nature of the value function associated with the LRB-IBM-PDE ensures convergence of the scheme, which in turn guarantees stability. Thus, a price-time mesh (uniform grid) as displayed in Figure 1 is created to implement the finite difference method on the LRB-IBM-PDE defined in Equation (17). The horizontal axis represents the time while the vertical axis represents the stock price. Each node on the grid has a horizontal index *i* and vertical index *j* such that a node represents the option price for a given stock price and time.

The uniform grid consists of *M* and *N* equally spaced points satisfying $0 < S_0 < S_1 < \cdots < S_M = S_{\max}$ and $0 < t^1 < t^2 < \cdots < t^N = T$ respectively, where $S_{\max}$ represents the underlying price unlikely to be reached. The step size is defined as $\delta S = S_{j+1} - S_j$ for the spatial domain and $\delta t = t^{i+1} - t^i$ for the time domain. For each node, $j\delta S$ is equal to the stock price, $i\delta t$ is equal to the time and the grid notation for the option price is defined as $V_i^j = V\left(j\delta S, i\delta t \right)$.

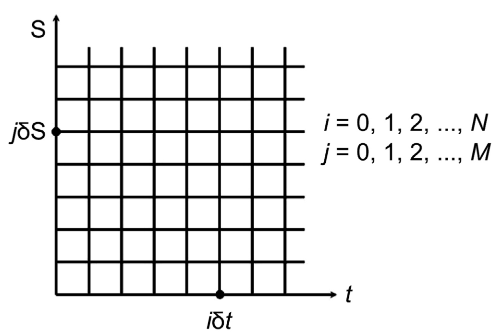

Figure 1. Illustration of price-time mesh used for implementing finite difference method.

Applying the FDM on the derivatives of the LRB-IBM-PDE on the uniform grid created gives the following approximations of the derivatives:

Forward approximation

$$\frac{\partial V}{\partial t} = \frac{V_j^{i+1} - V_j^i}{\delta t} \tag{22}$$

Backward approximation

$$\frac{\partial V}{\partial t} = \frac{V_j^{i} - V_j^{i-1}}{\delta t} \tag{23}$$

Central approximation

$$\frac{\partial V}{\partial t} = \frac{V_j^{i+1} - V_j^{i-1}}{2\delta t} \tag{24}$$

$$\frac{\partial V}{\partial S} = \frac{V_{j+1}^{i} - V_{j-1}^{i}}{2\delta S} \tag{25}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{V_{j+1}^{i} - 2V_j^i + V_{j-1}^{i}}{\left(\delta S\right)^2} \tag{26}$$

These approximations of derivatives can be used to rewrite the LRB-IBM-PDE using the explicit, implicit or Crank-Nicholson (CN) methods. The explicit method uses a forward approximation for time and central approximation for spacial derivative (FTCS). It is least computational intensive but conditionally stable as long as small time steps are used [31]. The implicit method uses Backward Time Centered Space (BTCS) scheme and is more stable than the explicit method but computationally intensive. Lastly, the Crank-Nicholson method combines both the implicit and the explicit methods making it the most stable with the fastest convergence rate especially for larger time steps. The FTCS is used for solving the LRB-IBM-PDE because of its high accuracy with less computational time but with careful selection of the time-step size to ensure stability of the scheme.

The following three boundary conditions are imposed on the grid so that the call option price can be approximated.

$$\begin{gathered} V\left(S,T\right) = \max\left(S-K,0\right) \\ V\left(0,t\right) = 0 \\ V\left(S_{\max},t\right) = \left(S_{\max}-K\right)\exp\left(-rt\right) \end{gathered} \tag{27}$$

First, there exists a terminal boundary condition such that the option price must be equal to the payoff function. Second, for the lower boundary condition, the stock price is set to be equal to zero such that the option value is also zero. Third, the upper boundary where the option value for the maximum stock value, $S_{\max}$, in the mesh is approximated. $S_{\max}$ is chosen large enough so that the option value is equivalent to the discounted value of the payoff function for all times in the mesh. The option price is then calculated for each time step backwards from

$t = T$ until the value at time $t = 0$ is approximated. The three boundary conditions are applicable for the case of the Bermudan call option with $P = 1$ corresponding to a European call option.

For $P \in (1, \mathbb{N}]$, the exercise boundary conditions as specified in the Bellman Equation (15) for each exercise point p is introduced together with optimal stopping rule such that

$$V(S, t_p) \leq \max(S_{t_p} - K)^+ \tag{28}$$

For a Bermudan call option with finitely many exercise points equivalent to an American option, the following free boundary condition at all time points is introduced

$$\max\left(V(S,t), (S_t - K)^+\right) \tag{29}$$

The two conditions are necessary because the Bermudan or American call option can be greater than its European counterpart because of the possibility of early exercise.

Applying the FTCS scheme on the derivatives of the LRB-IBM-PDE in the uniform grid gives the following discretization of the modified volatility

$$\begin{aligned}\hat{\sigma}_t^2 = \sigma_t^2 \Bigg(1 - \frac{SC_0}{\sigma_t \sqrt{dt}} \exp\Bigg\{-C_1\left(\lambda t X_T + \frac{t(T-t)}{2T}\right) \\ - C_2 \frac{1}{2} \frac{\sigma_t}{\sqrt{dt}} \frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{(\delta S)^2}\Bigg\} \sqrt{\frac{2}{\pi}} sign\left(\frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{(\delta S)^2}\right)\Bigg)\end{aligned} \tag{30}$$

for the case of non-increasing exponential transaction costs and

$$\begin{aligned}\hat{\sigma}_t^2 = \sigma_t^2 \Bigg(1 - \frac{S}{\sigma_t} \sqrt{\frac{2}{\pi dt}} sign\left(\frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{(\delta S)^2}\right)(C_0 - C_1 \lambda t X_T) \\ + SC_2 \frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{(\delta S)^2}\Bigg)\end{aligned} \tag{31}$$

for non-increasing linear transaction cost function. Consequently, the approximation of the option price with variable transaction costs is given as

$$\frac{V_j^{i+1} - V_j^i}{\delta t} + rj\delta S \frac{V_{j+1}^i - V_{j-1}^i}{2\delta S} + \frac{1}{2} \hat{\sigma}_t^2 \frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{(\delta S)^2} - rV_j^i = 0 \tag{32}$$

This is simplified to

$$\begin{aligned}V_j^{i+1} = \left(1 + \delta t\left(r + \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right)\right)V_j^i + \frac{1}{2}\delta t\left(rj - \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right)V_{j-1}^i \\ + \left(-\frac{1}{2}\delta t\left(rj + \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right)\right)V_{j+1}^i\end{aligned} \tag{33}$$

changing from V_j^{i+1} to V_j^i yields

$$V_j^i = \frac{1}{2}\delta t\left(rj - \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right)V_{j-1}^{i-1} + \left(1 + \delta t\left(r + \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right)\right)V_j^{i-1} + \left(-\frac{1}{2}\delta t\left(rj + \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right)\right)V_{j+1}^{i-1} \tag{34}$$

Equation (34) implies that approximation of the option price at $T - \delta t$ is explicitly dependent on known information. Let the coefficients of V_{j-1}^{i-1}, V_j^{i-1} and V_{j+1}^{i-1} be defined as a_j, b_j and c_j respectively such that

$$\begin{aligned} a_j &= \frac{1}{2}\delta t\left(rj - \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right) \\ b_j &= 1 + \delta t\left(r + \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right) \\ c_j &= -\frac{1}{2}\delta t\left(rj + \frac{\hat{\sigma}_t^2}{(\delta S)^2}\right) \end{aligned} \tag{35}$$

The option price at each point in the grid is given by the linear equation

$$V_j^i = a_j V_{j-1}^{i-1} + b_j V_j^{i-1} + c_j V_{j+1}^{i-1} \tag{36}$$

Equation (36) can be formulated in matrix form and solved numerically. The matrix notation is given by

$$V^i = BV^{i-1} \tag{37}$$

where

$$V^i = \begin{bmatrix} V_1^i \\ V_2^i \\ \vdots \\ V_{M-1}^i \end{bmatrix} \tag{38}$$

$$B = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & \cdots & 0 \\ 0 & a_3 & b_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{M-2} \\ 0 & 0 & \cdots & a_{M-1} & b_{M-1} \end{bmatrix} \tag{39}$$

The matrix B is tridiagonal and can be solved efficiently using the Thomas algorithm [32]. The option value at time 0 is interpolated for a given underlying share price.

4. Simulation Results

In this section, we present the simulation results to demonstrate the numerical solution of the LRB-IBM-PDE. The initial values are arbitrarily chosen and set as follows: $S_0 = \$10$, $r = 4.5\%$, $T = 1$, $X_T = 14$, trading days = 252, $dt = 1/252$ and assume that the *apriori* distribution of the process X_t is a normal distribution, say $X_t \sim N(10, 0.5)$. The interval $[0,1]$ is divided into $N = 252$ equally spaced sub-intervals of length δt. The trading days is chosen as the time step such that $\delta t = \frac{1}{252}$. With a current share price of 10, the strike prices can be constructed as 8, 9, 10, 11, and 12. The price of the underlying asset takes values from the unbounded interval $[0, \infty)$. We use an artificial limit of $S_{\max} = 25$ which is chosen to be around two to three times the exercise price. The interval $[0, S_{\max}]$ is also divided into $M = 252$ equal sub-intervals of length δS such that $\delta S = 0.1$. The choice of $N = M$ is for simplicity, however, higher values may be used. From the literature, values of $N, M \geq 30$ lead to convergence of the finite difference scheme. For comparison, the call prices under the exact solution of the IBM are also computed based on the same initial values to validate the numerical approximation. The comparison is also made with the commonly used Longstaff-Schwartz Least-Squares Monte Carlo Method for pricing Bermudan options for assets evolving according to the Black-Scholes Model. All results are calculated using Matlab version R2019a.

4.1. Simulation of Stock Price Process under LRB-IBM

Given the starting values previously stated, we simulate the sample path for the information process, volatility without transaction costs, and asset price process under the IBM. All simulations are based on Euler discretization. Figure 2 displays the sample path of the LRB-information process, ξ_t. It can be observed that the information process increases with time as a result of the information flow rate parameter λ which strictly increases with time. At time T, all market participants have full information about the value of the cash flow such that $\xi_T = X_T$. The minimal upward and downward fluctuations observed represent market rumors brought about by the Brownian Bridge process.

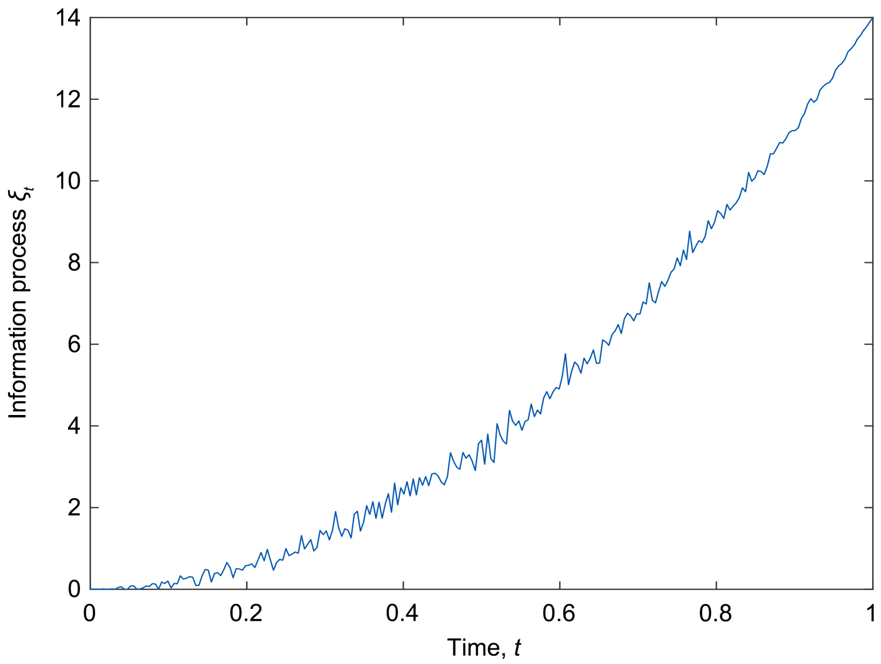

Figure 2. Sample path for the LRB information process ξ_t with $X_T = 14$.

Next, the simulation of stochastic volatility requires that the initial conditional variance V_0 in Equation (5) is carefully chosen. Simulation tests suggest that values between 0 and 0.1 lead to stable values of the stochastic volatility. The optimal initial variance chosen was 0.03. **Figure 3** shows the sample path of the volatility of the stock price.

The results demonstrate that stochastic volatility increases with time as a consequence of the information process which is an increasing diffusion process as given in Equation (1). The huge spike towards the end can be linked to the increased amount of information related to the value of the cash flow at time T. From a buyer's perspective, more information about the price of the stock would lead to increased purchases which would in turn cause a sharp increase in the stock price. From a seller's perspective, more information about the price of the stock would lead to more sales which in turn result in a sharp decrease in the stock price. Using the simulated values of stochastic volatility, the stock price process is simulated and presented in **Figure 4**. The price path looks like a reasonable presentation of a random price process showing trend and fluctuations over time.

4.2. Option Price without Transaction Costs

The price of a Bermudan call option without transaction costs for an asset under the LRB-IBM-PDE as simulated in Section 4.1 is computed. In this case, the transaction cost rates are all zero with $C_0 = C_1 = C_2 = 0$ for the pricing Equation (17) such that the market is complete with the stochastic volatility equivalent to Equation (4).

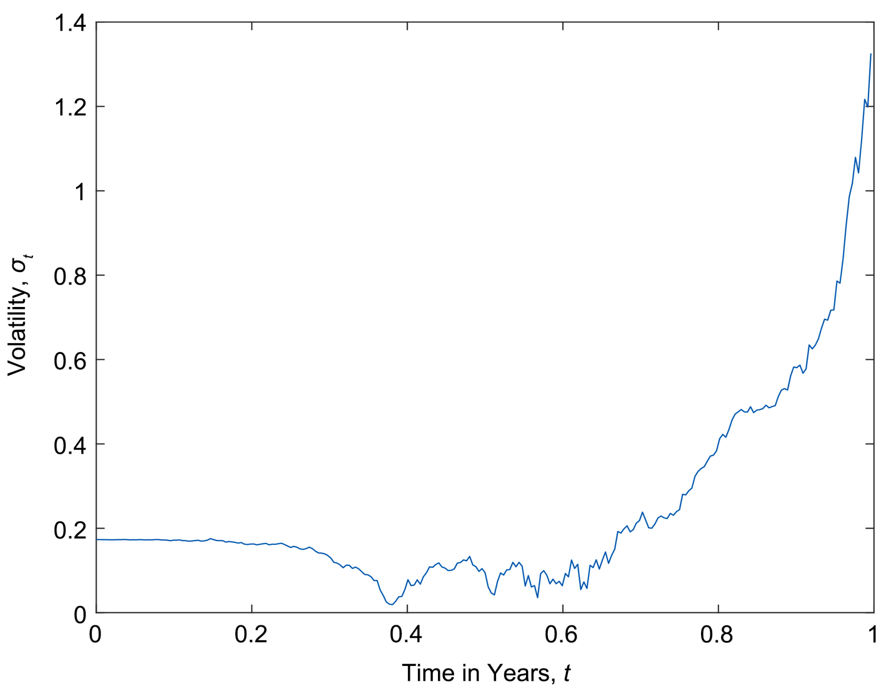

Figure 3. Sample path for the volatility process σ_t.

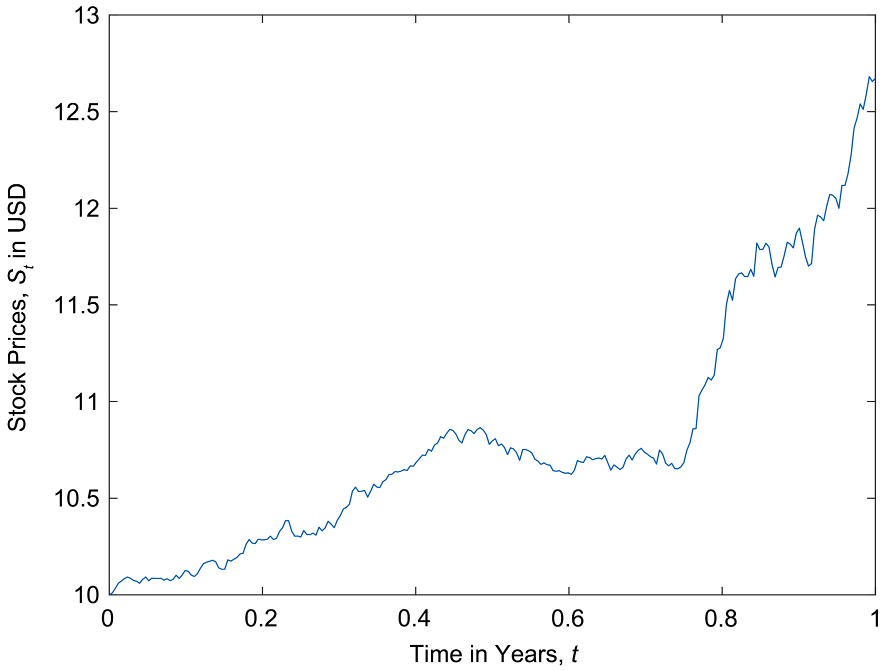

Figure 4. Sample path for the stock price S_t under LRB-IBM.

4.2.1. European-Style Exercise Rights

For $P = 1$, the Bermudan call option has a European exercise style. Using the uniform grid, the price of the option is calculated through backward iteration in time with the boundary conditions stated in Equation (27) imposed on the grid. The option price is 1.1928 when it is at the money (ATM), that is when $K = 10$. Figure 5 illustrates the European call option price surface as a function of the initial asset price and the time to maturity with zero transaction costs under the LRB-IBM-PDE. The option surface indicates that the option price increases as the stock price increases. Similarly, the call value increases as $t \to T$, because as the maturity time approaches, there is more information about the price of the asset and the call price must go up to reflect the possibility of profit.

The accuracy of the FDM for solving the LRB-IBM-PDE is checked by comparing the results with the exact solution of the IBM and the BSM under zero transaction costs and for the same initial values. Since BSM assumes constant volatility, the option prices are simulated for different values of volatility. For illustration, we use volatility values of 20%, 30%, 40% and 50%. Table 1 summarizes the call prices under the LRB-IBM-PDE, closed-form IBM and BSM for different strike prices.

The call prices are observed to reduce as the strike prices increases. The simulated option prices under LRB-IBM-PDE are very close to the exact solution of the IBM with a percentage error within 1%, hence, validating the finite difference approximation. However, the BSM prices vary for different strike prices but are generally slightly lower than the approximated IBM prices. This is expected because of the assumption of constant volatility which leads to some discrepancies in the theoretical price of the BSM such as overpricing ATM options and

Table 1. Prices for a Bermudan call option without transaction costs, with one exercise date, under the LRB-IBM-PDE, closed-form IBM and Black Scholes models. The parameters used are $T = 1$, $S_0 = 10$, $X_T = 14$, $r = 4.5\%$.

Strike	IBM-PDE	IBM-CLOSED	BSM (20%)	BSM (30%)	BSM (40%)	BSM (50%)
8	3.0003	3.0002	2.4589	2.6462	2.8976	3.1793
9	2.0462	2.0454	1.6699	1.9697	2.2985	2.6373
10	1.1928	1.1919	1.0451	1.4231	1.8023	2.1793
11	0.7264	0.7270	0.6040	1.0020	1.4004	1.7962
12	0.4042	0.4039	0.3247	0.6904	1.0806	1.4783

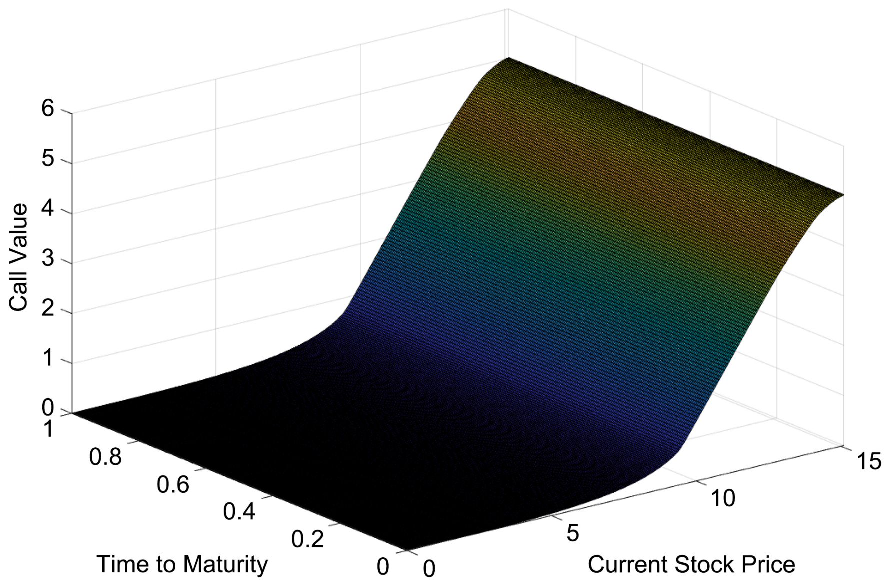

Figure 5. European call option price under LRB-IBM.

under pricing ITM and OTM options. It is also observed that the option is most expensive when it is in the money (ITM), followed by ATM and least expensive when it is out of the money (OTM).

4.2.2. Early-Exercise Premium

The finite difference scheme is also applied in solving the price of an early exercise option for the case when $P > 1$. The option prices are computed for different values of P. For illustration, we randomly select four values of P to represent a Bermudan call option that can be exercised semi-annually, quarterly, monthly, and daily such that $P = (2, 4, 12, 252)$, respectively. When $P = 252$, the option can be considered to be an approximation of an American style option that can be exercised on any day within the life of the option. The simulated call prices under the LRB-IBM-PDE are then compared to simulated values obtained using the Longstaff-Schwartz least squares method for an underlying evolving according to the BSM. The BSM prices are calculated based on a volatility of 20%, which is closest to the IBM prices obtained when $P = 1$. This early exercise premium is calculated using the function optstockbyls() in Matlab.

Table 2 displays the call premiums under LRB-IBM-PDE and BSM for Bermudan option with different exercise times. The call premiums increase with an increase in the number of possible exercise dates under both models. The early exercise priviledge for the Bermudan option with $P > 1$ makes it more expensive as compared to the European style option. These results are consistent with the literature such that the price of the Bermudan option is higher than the European option but lower than the American option.

To illustrate the early exercise value (EV) and continuation values (CV) of the option, we consider $P = 4$ where that option can be exercised at four equal times apart. The continuation values are calculated directly from the PDE at the exercise points on the grid. At each exercise date, one needs to compare the immediate exercise value with the continuation value and decide to exercise as soon as the exercise value is greater than or equal to the continuation value as given in Equation (14). Table 3 displays the continuation values and exercise values for the Bermudan call option at each of the four possible exercise dates.

Table 2. Prices for a Bermudan call option without transaction costs, expiry $T = 1$ with 2, 4, 12 and 252 exercise times, under the LRB-IBM-PDE and Black Scholes models.

Strike		$P = 2$	$P = 4$	$P = 12$	$P = 252$
8	IBM	3.0005	3.0011	3.0015	3.0017
	BSM	2.5180	2.5356	2.5516	2.5725
9	IBM	2.0463	2.0469	2.0473	2.0475
	BSM	1.6904	1.7541	1.7828	1.7962
10	IBM	1.1929	1.1934	1.1938	1.1940
	BSM	1.0868	1.1127	1.1757	1.1805
11	IBM	0.7265	0.7269	0.7273	0.7275
	BSM	0.6379	0.6636	0.6708	0.6740
12	IBM	0.4043	0.4047	0.4051	0.4053
	BSM	0.3583	0.3738	0.3845	0.3897

Table 3. Continuation values and exercise values for Bermudan call option without transaction costs and with four possible exercise dates for an asset under IBM.

Moneyness		t_1	t_2	t_3	t_4
ITM	CV	2.7637	2.5162	2.2631	2
$K = 8$	EV	2.3043	2.7788	2.6616	4.6717
ATM	CV	0.9452	0.6861	0.4141	0
$K = 10$	EV	0.3043	0.7788	0.6616	2.6717
OTM	CV	0.2610	0.1310	0.0303	0
$K = 12$	EV	0	0	0	0.6717

When the option is ITM or ATM, the results indicate that the option holder may exercise the option early at t_2 which is the smallest time when the exercise value is greater than the continuation value. While this is the smallest optimal time to exercise the Bermudan option, it may not be the best since the option is worth more as it approaches maturity. However, when the option is OTM, it is only optimal to exercise it at maturity.

4.3. Option Price with Variable Transaction Costs

From Section 4.2, it is observed that option prices approximated using the finite difference scheme under zero transaction costs compare with the closed-form IBM prices, demonstrating accuracy of the numerical approximation. Therefore, the option prices are also approximated for the pricing equation in the presence of variable transaction costs (TC). The transaction costs rates are such that $\{C_0, C_1, C_2\} \geq 0$ for the pricing Equation (17) and the modified stochastic volatilities are defined in Equation (18) and Equation (20) for non-increasing exponential transaction costs and non-increasing linear transaction costs respectively. The numerical approximation of the modified volatilities is respectively given in Equation (30) and Equation (31).

The simulation experiment is done for a Bermudan option with 4 exercise dates and with initial values as previously stated: $T = 1$, $r = 4.5\%$, $S_0 = 10$. The transaction cost rates, C_0, C_1 and C_2 are chosen arbitrarily while keeping the cumulative transaction costs at 10% to account for taxes, brokerage fees, bid-ask spreads, stamp duties, and exchange fees [33]. The call values are obtained for different transaction costs rates and strike prices. The constant cost of trading, C_0 is used as the basis and is examined at 1%, 5% and 10% in order to determine the effect of changing transaction costs rates on call premium.

Under the non-increasing exponential transaction cost function, different values of C_1 and C_2 are arbitrarily chosen to examine the behavior of the option prices. As long as $C_0 \geq 0$, any value of $C_1, C_2 \in [0,1]$ can be chosen to yield reasonable values of transactions costs. On the other hand, under the linear transaction cost function, the following values of C_1 and C_2 result in reasonable values for transaction costs given the constant cost of trading C_0. Any values above the limits lead to negative transaction costs.

$$\text{For } C_0 = 1\% \quad C_1, C_2 \leq 0.06\%$$

$$\text{For } C_0 = 5\% \quad C_1, C_2 \leq 0.3\%$$

$$\text{For } C_0 = 10\% \quad C_1, C_2 \leq 0.6\%$$

Table 4 and **Table 5** display the Bermudan call prices with 4 exercise dates, under the non-increasing exponential transaction cost function and linear transaction cost function respectively, for different transaction costs rates and exercise prices.

The results show that transaction costs have a negative relationship with the price of the Bermudan call option. The call prices reduce when the option is ATM and OTM as one increases the constant cost of trading from 1% to 10%.

Table 4. Bermudan call prices for different transactions costs rates under the non-increasing exponential transaction cost function for $P = 4$, $T = 1$, $r = 4.5\%$ and $S_0 = 10$.

Transaction Costs Rates	Moneyness of the Option		
	ITM ($K = 8$)	ATM ($K = 10$)	OTM ($K = 12$)
$C_0 = C_1 = C_2 = 0$	3.0011	1.1934	0.4047
$C_0 = 1\%$, $C_1 = 5\%$, $C_2 = 0\%$	3.0003	1.1796	0.3887
$C_0 = 1\%$, $C_1 = 0\%$, $C_2 = 5\%$	3.0003	1.1787	0.3869
$C_0 = 1\%$, $C_1 = C_2 = 0\%$	3.0003	1.1780	0.3863
$C_0 = 1\%$, $C_1 = 5\%$, $C_2 = 3\%$	3.0003	1.1796	0.3887
$C_0 = 1\%$, $C_1 = C_2 = 5\%$	3.0003	1.1794	0.3889
$C_0 = 1\%$, $C_1 = 3\%$, $C_2 = 5\%$	3.0003	1.1790	0.3878
$C_0 = 5\%$, $C_1 = 5\%$, $C_2 = 3\%$	3.0003	1.1319	0.3205
$C_0 = 5\%$, $C_1 = 5\%$, $C_2 = 5\%$	3.0003	1.1316	0.3201
$C_0 = 5\%$, $C_1 = 3\%$, $C_2 = 5\%$	3.0003	1.1284	0.3145
$C_0 = 10\%$, $C_1 = 5\%$, $C_2 = 3\%$	3.0002	1.0996	0.2288
$C_0 = 10\%$, $C_1 = 5\%$, $C_2 = 5\%$	3.0002	1.0993	0.2286
$C_0 = 10\%$, $C_1 = 3\%$, $C_2 = 5\%$	3.0002	1.0943	0.2148

Table 5. Bermudan call prices for different transactions costs rates under the non-increasing linear transaction cost function for $P = 4$, $T = 1$, $r = 4.5\%$ and $S_0 = 10$.

Transaction Costs	Moneyness of the Option		
	ITM ($K = 8$)	ATM ($K = 10$)	OTM ($K = 12$)
$C_0 = C_1 = C_2 = 0$	3.0011	1.1934	0.4046
$C_0 = 1\%$, $C_1 = C_2 = 0.06\%$	3.0011	1.1934	0.4046
$C_0 = 1\%$, $C_1 = 0.06\%$, $C_2 = 0\%$	3.0011	1.1934	0.4046
$C_0 = 1\%$, $C_1 = 0\%$, $C_2 = 0.06\%$	3.0011	1.1934	0.4045
$C_0 = 1\%$, $C_1 = C_2 = 0\%$	3.0011	1.1934	0.4045
$C_0 = 5\%$, $C_1 = C_2 = 0.3\%$	3.0011	1.1932	0.4040
$C_0 = 10\%$, $C_1 = C_2 = 0.6\%$	3.0011	1.1930	0.4034

From a holder's perspective, the portfolio wealth of an investor will be eroded the more the cost of hedging increases. These costs are compensated through a decrease in the price of the call option. Thus, it is observed that as the rate of transaction costs increases the call option value will decrease. On the other hand, the value when the option is ITM is not very sensitive to changes in the transaction costs rates since the price difference is negligible. The results also show that the option price changes differently for different values of C_1 and C_2 for the exponential case. The call prices are higher for $C_1 > C_2$ as compared to when $C_2 > C_1$ or $C_1 = C_2$. This means that information flow to market traders has a significant role in lowering transaction costs of trading which in turn, increase the call prices. However, the call prices under the linear transaction cost function are not very sensitive to changes in transaction cost rates as they are much closer to the prices under zero transaction costs. For instance, the prices when the option is ITM under the linear transaction cost function are indifferent for all values of C_0.

For illustration, consider the ATM option (K = 10) with $C_0 = C_1 = C_2 = 5\%$ for the exponential transaction function and $C_0 = 10\%$, $C_1 = C_2 = 0.6\%$ for the linear transaction function. For the same initial values, $T = 1$, $S_0 = 10$, $X_T = 14$, $r = 4.5\%$, and $P = 4$, **Figure 6** displays the price of the Bermudan call option with variable transaction costs under the Information-based model. It considers the effect of the two transaction cost functions on the price of a Bermudan call option from the holder's perspective. Generally, the option price increases as the stock price increases which is consistent with the literature. The option prices are lower under the exponential transaction cost function as compared to the linear case but converge as the share price increase.

The continuation values (CV) in the presence of transaction costs also decrease at all exercise points as illustrated in **Table 6**. The smallest optimal time to exercise the option is still at the second exercise point where the exercise value (EV) is greater than the continuation value.

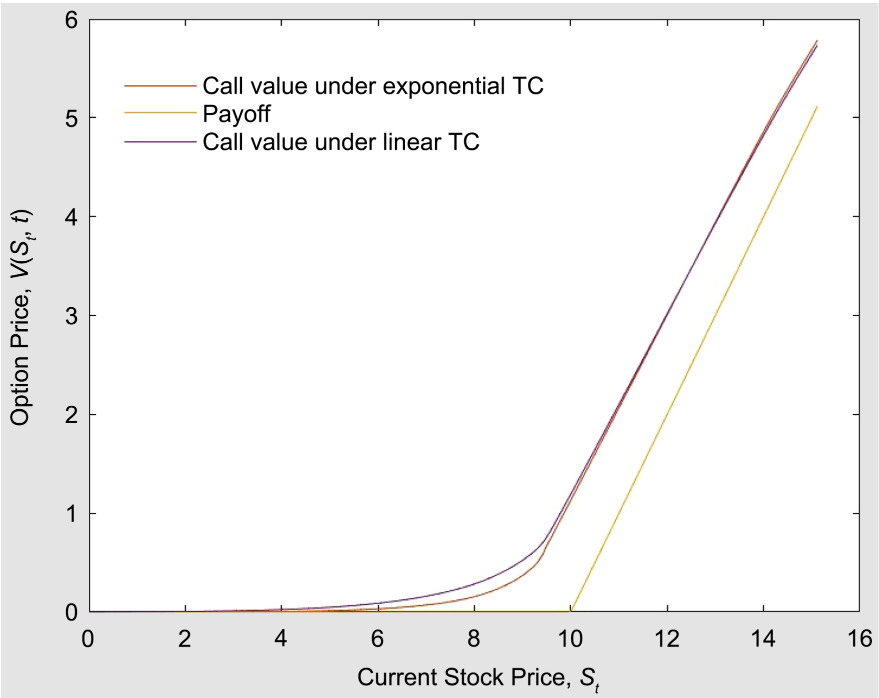

Figure 6. ATM Bermudan option price with variable transaction costs under the LRB-IBM.

Table 6. Continuation values and exercise values for an ATM Bermudan call option with variable transaction costs and with four possible exercise dates for an asset under IBM.

	t_1	t_2	t_3	t_4
CV-Without TC	0.9452	0.6861	0.4141	0
CV-Exponential TC	0.8798	0.6197	0.3556	0
CV-Linear TC	0.9450	0.6859	0.4139	0
EV	0.3043	0.7788	0.6616	2.6717

5. Numerical Approximation of Bermudan Option Greeks under the LRB-Information Based Model

The Greeks provide a means of measuring the sensitivity of the price of an option to quantifiable factors. The finite difference scheme is also applied in estimating the Greeks associated with the Bermudan call price under the IBM. The most common Greeks used include delta, vega, gamma, theta, and rho. Delta measures the sensitivity of the value of the option with respect to changes in price of the asset *S*. The notation Δ^s is used to differentiate the options delta from change in a variable. Delta for the Bermudan call option is numerically approximated as

$$\Delta^s = \frac{\partial V}{\partial S} = \frac{V(S+\Delta S,t) - V(S,t)}{\Delta S} \tag{40}$$

where ΔS represents a unit change in the underlying stock price and $V(.)$ is the option price approximated using the FTCS for a given stock price at time zero. Gamma measures the rate of change of delta with respect to a unit change in the price of the underlying asset. It is obtained directly from delta as the stock price moves either up or down. The numerical gamma is approximated as follows:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\Delta^s_{up} - \Delta^s_{down}}{0.5 * (S_{up} - S_{down})} \tag{41}$$

where S_{up} and S_{down} denote the unit upward and downward movement in the stock price respectively, and Δ^s_{up} and Δ^s_{down} are estimated using the finite difference method in Equation (40). Vega measures the sensitivity of the option price relative to the expected volatility of the underlying asset. Volatility under the IBM is stochastic hence the initial variance is bumped by 1% to determine the change in the option price. The numerical approximation for vega is given by

$$\nu = \frac{\partial V}{\partial \hat{\sigma}_t} = \frac{V(S,t)^{\hat{\sigma}_t}_{new} - V(S,t)^{\hat{\sigma}_t}_{old}}{\Delta \hat{\sigma}_t} \tag{42}$$

where $V(S,t)^{\hat{\sigma}_t}_{new}$ is the option price estimated under the bumped volatility and $V(S,t)^{\hat{\sigma}_t}_{old}$ is the option price under the initial volatility. Theta measures the sensitivity of the option value relative to the option's time to maturity. It is negative for long positions and positive for short positions. For the Bermudan option, theta quantifies the risk that time poses to an option holder when the exercise dates are increased or reduced. It is defined as

$$\theta = \frac{\partial V}{\partial t_p} = \frac{V\left(S, t_{p+1}\right) - V\left(S, t_p\right)}{\Delta t_p} \tag{43}$$

Rho measures the sensitivity of option value relative to changes in interest rate *r*. Its numerical approximation is obtained as

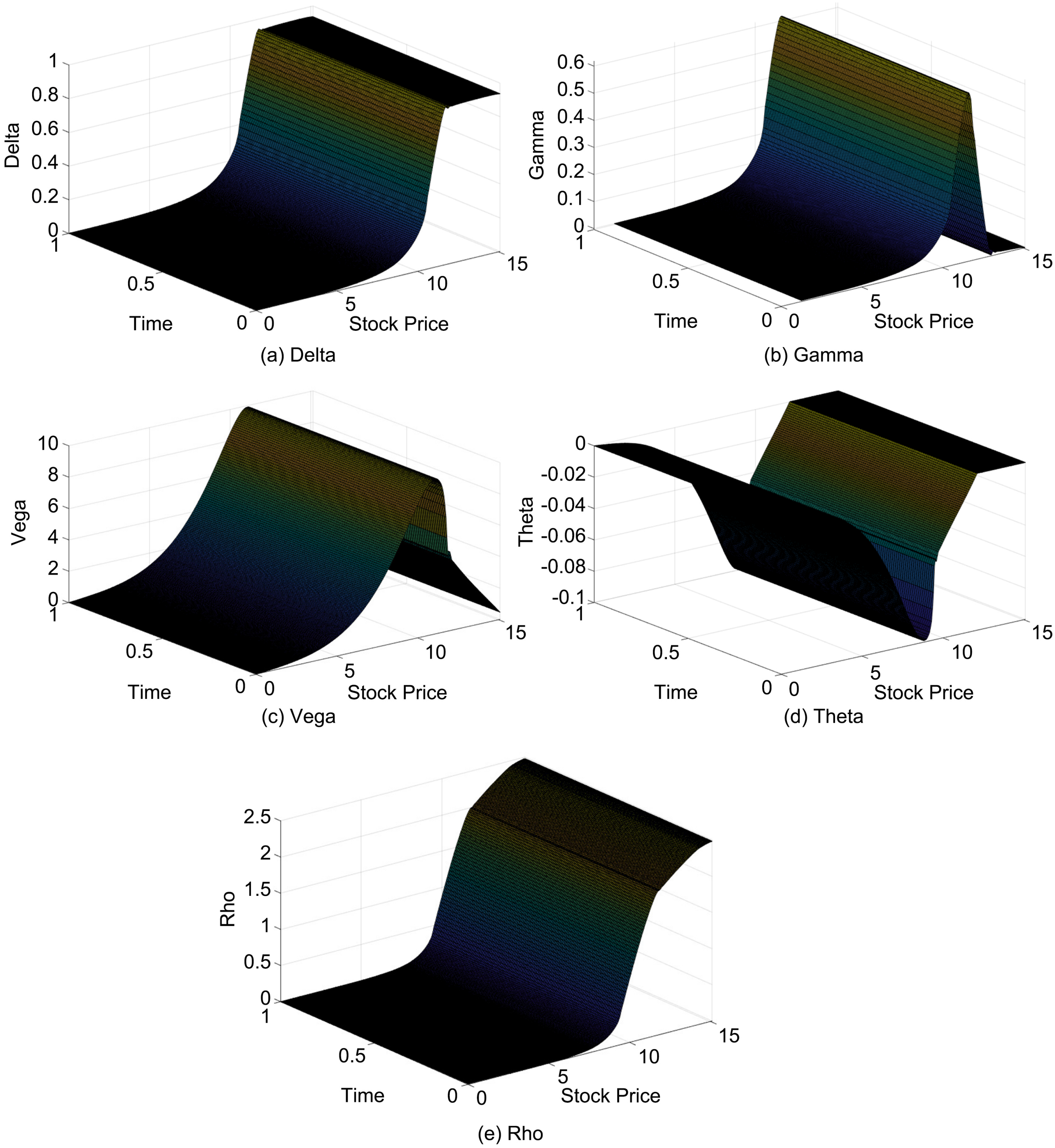

Figure 7. Option Greeks for Bermudan call with variable transaction costs under the LRB-IBM.

$$\rho = \frac{\partial V}{\partial r} = \frac{V(S,t)^{r_{new}} - V(S,t)^{r_{old}}}{\Delta r} \tag{44}$$

Simulation tests shows that the numerical approximations give reasonable results for the option sensitivities under the LRB-IBM. For illustration, the option Greeks are simulated for the Bermudan call option with four exercise dates under non-increasing exponential transaction costs where $C_1 = C_2 = C_3 = 5\%$. Figure 7 displays the Greeks as a function of asset price and time for the Bermudan call option with 4 exercise dates, strike at \$10, $r = 4.5\%$ and $T = 1$. Delta is positive as expected and increases as the stock price gets closer to the strike price. The values of delta oscillate around 0 if option is OTM, 0.5 for ATM and 1 if it is ITM. This implies that the option premium is expected to rise if the stock price increases over the contract period causing delta to be higher for ITM option.

Consequently, the rate of change of delta measured by gamma, is also positive. The results show that gamma is highest when the option is ATM and as it approaches maturity. This implies greatest volatility because every single move in the underlying asset will change the value of delta. The value of vega is also highest if the option is ATM as compared to ITM and OTM because of the high volatility expected. In addition, the shape for theta implies that the effect of time decay is higher when the option is ATM because of the highest volatility potential. Lastly, rho is positive for the Bermudan call option and increases with time and price of the underlying. Thus, the call option is more favorable in a high interest environment. Overall, the patterns displayed by the different Greeks are consistent with the literature on option sensitivities under standard models such as Black Scholes and Heston models.

6. Conclusions and Further Research

Finite difference schemes have been shown to be straightforward in obtaining the numerical approximation of partial differential equations used to model the price of options with non-linear volatility. In this paper, we implement the explicit finite difference scheme to obtain the numerical solution of the non-linear information-based model partial differential equation arising in pricing when variable transaction costs are taken into account. Specifically, we consider the numerical approximation of the price of a Bermudan call option for an asset driven by Lévy Random Bridge market information process within the information-based framework where transaction costs are either exponentially or linearly non-increasing. A uniform grid with small time and space steps is chosen for ease of implementation and to ensure the convergence and stability of the explicit finite difference scheme. Simulation tests are performed to show the practicability of the finite difference scheme on the LRB-IBM-PDE and the results compared to the exact solution of the IBM and the BSM prices. The numerical approximation of the Bermudan option Greeks derived from the estimated option prices is also presented.

The simulation results of the study can be summarized as follows. Firstly, the computed option prices approximated by the finite difference scheme are close to the closed-form solution of the information-based model with zero transaction costs, thus verifying the accuracy of the numerical approximation. Secondly, the IBM prices are slightly higher than the Black Scholes prices. This can be attributed to the assumption of constant volatility under BSM which leads to under pricing of ITM and OTM options. Thirdly, the results indicate that transaction costs decrease the call value and the change in transaction costs rates is sensitive to the call value when the option is ATM or OTM. However, the option is insensitive to changes in transaction costs rates when it is ITM. In general, the non-increasing exponential transaction costs give lower values of the call as compared to the linear case. The continuation values of the Bermudan option are also reduced in the presence of transaction costs. Furthermore, the findings show that information flow to market participants has a significant role in decreasing transaction costs of trading. Finally, the simulated option Greeks under the IBM with variable transaction costs are consistent with call option sensitivity movements in the literature.

For future work, other numerical methods such as spectral methods and Monte-Carlo methods can be explored in obtaining the numerical solution of the LRB-information based model equation with variable transaction costs in order to compare their convergence rate and accuracy power. Moreover, further research can be focused on applying the finite different scheme or other numerical approximation methods on the LRB-IBM-PDE to real options data, specifically in pricing exchange-traded American or European options which are special forms of the Bermudan options.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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