Optimal Rate Delay Tradeoffs for Multipath Routed and Network Coded Networks

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***Abstract*— Via multiterminal information theory, we present fundamental rate delay tradeoffs that delay mitigating codes must have when utilized over multipath routed and network coded networks. We formulate plotting the rate delay tradeoff as a calculus problem on a capacity region of a related abstracted broadcast channel. This calculus problem simplifies to an integer programming problem, which for small numbers of packets may be solved explicitly, or for larger values of packets, may be accurately approximated through the calculus of variations by relaxing the integer constraint. We prove the utility of our techniques by plotting the rate delay tradeoff for networks in which the packets experience independent exponentially distributed propagation and queuing delays while traversing the network.**

I. Introduction

Imagine a typical multipath routed network in which a particular source hands M packets $\mathbf{x}_1, \ldots, \mathbf{x}_M$ of equal size and all headed to the same destination in order to the lower network layers for transmission at time 0. Because of queuing and propagation delays along the different routes, the packets $(\mathbf{x}_1, \ldots, \mathbf{x}_M)$ arrive at the sink at different time instants $(t_1, \ldots, t_M)$, possibly giving a different order. We will model the source to destination effects of the multi-path routed network transmission, then, as selecting these arrival times $t_1, \ldots, t_M$ randomly according to some joint distribution $\mathsf{p}_{t_1,\ldots,t_M}$.

Denote by $\pi(i)$ the index of the transmitted packet that is the i received packet, so that the packets arriving at the receiver in order are $\mathbf{x}_{\pi(1)}, \mathbf{x}_{\pi(2)}, \ldots, \mathbf{x}_{\pi(M)}$, with[1] $t_{\pi(1)} < t_{\pi(2)} < \ldots < t_{\pi(M)}$, which we denote by $\tau_i = t_{\pi(i)}$. We can then think of the probability distribution $\mathsf{p}(t_1, \ldots, t_M)$ on the arrival times as being composed of a probability distribution $\mathsf{p}(\pi)$ on the arrival order and a probability distribution on the ordered arrival instants $\mathsf{p}(\tau_1, \ldots, \tau_M)$.

Suppose further that the network source node under consideration wishes to use the packets $\mathbf{x}_1, \ldots, \mathbf{x}_M$ to convey some temporally ordered data to the receiver. For instance, the source node may wish to convey a digital multimedia file that is the output of a multimedia source encoder which after compressing the source returns data organized into temporally ordered frames $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N$ corresponding to successive chunks of time from the multimedia signal. Assuming that the source encoder has done a good job of compressing, the data in $\mathbf{s}_1, \ldots, \mathbf{s}_N$ will be independent. Most practical multimedia source encoders such as MPEG movies order data in this way. Alternatively, the source node may be controlling the destination network node remotely through the network, with a sequence of temporally ordered instruction frames $\mathbf{s}_1, \ldots, \mathbf{s}_N$ which must be executed in order. As another possibility, the source node may be engaged in a voice over IP (VoIP) conversation with the receiver. Indeed, almost any connection oriented network source application could be considered as relevant for what we are about to discuss.

All of these temporally ordered source signals share the common characteristic that later data (i.e. with higher frame indices) is not useful until earlier data (i.e. with lower frame indices) has been received. However, since the effective point to point channel that the multi-path routed network creates reorders packets, were we simply to transmit the source data directly as is over the network channel, we would have to wait at the receiver, storing packets in a buffer, and reordering them as we played them out. Alternatively, one could consider a single source multicast network coded network employing random linear network coding [1], [2]. In this case $(\mathbf{x}_1, \ldots, \mathbf{x}_M)$ arriving at the sink will be random linear combinations of original transmitted packets (letting M be the dimension of the global encoding vector), and we will have to wait, buffering the received packets, until all M innovative (i.e. with linearly independent encoding vectors) packets have been received before we can start decoding even the earliest source frame $\mathbf{s}_1$.

While buffering has been extensively studied as a technique for connection oriented transmission over networks, we wish to consider here an alternative approach based on (forward erasure correction) coding. In particular, rather than simply copying (and possibly repeating) the source data $\mathbf{s}_1, \ldots, \mathbf{s}_N$ into the transmitted packets, we propose to use a linear code, which we call a *delay mitigating code* to create the data in the transmitted packets from the source packets in such a way as to minimize (an appropriately defined notion of) the delay incurred while playing the source out at the receiver.

To achieve the goal of exhaustively characterizing and designing such delay mitigating codes, we wish to identify in Section II the fundamental performance limits via multiterminal information theory by performing rate delay calculus on the capacity region of an associated abstracted degraded broadcast channel. Because the uniform permutation channel [3] and uniform network coded channel [4] share the same capacity region for this degraded broadcast channel, they share the same fundamental rate delay tradeoffs. After identifying these fundamental rate delay tradeoffs, one may focus on rate

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[1]Since we are using continuous time, and thus continuously distributed arrival instants, the probability that any of two or more of the arrival instants must be zero, so we ignore that case in our discussion.

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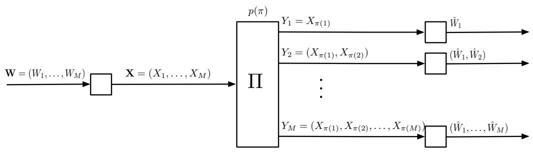

Fig. 1. The permutation channel reinterpreted as a abstracted degraded broadcast channel for the purpose of analyzing rate delay tradeoffs.

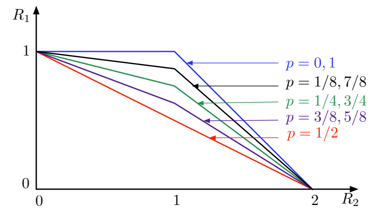

Fig. 2. The permutation channel's capacity region for $M = 2$, with p the probability that the packet numbered 1 arrives first at the receiver.

optimal code design, showing that, despite the asymptotic (and thus delay notion defeating) block lengths employed in proving the degraded broadcast channel capacity region, surprisingly there exist codes with very short block lengths exhibiting the desired rate delay tradeoffs. While we do not focus on the types of codes to use here beyond the fact that they are time-shared maximum distance separable (MDS) erasure codes of different rates which we determine, [3] and [4] discuss how priority encoded transmission (PET) [5], [6] and a slight variant that uses rank-metric MDS component codes [7] for the network coded case provide practical codes achieving the rate delay tradeoffs we provide. Thus, the point of this article is to provide the necessary analysis for an end user who needs a particular bounded delay to determine the component code rates of a time shared erasure (PET) code in order to satisfy this delay. These rates can then be used in any of the previous related references to obtain the practical capacity achieving codes.

II. Fundamental Rate-Delay Tradeoffs

To tackle the tradeoffs between rate and delay in such a channel, let us first focus on the distribution $\mathsf{p}(\pi)$ on the permutation determining the arrival order π by determining the region $\mathcal{R}$ of possible amounts of information that may be decoded upon each successive packet arrival at the sink. In particular, denote by R_i the amount of new information that can be decoded after i packets have arrived at the sink per block of packets transmitted at the source. Collect these rates R_i into a vector $\mathbf{r}$. We can determine the region $\mathcal{R}$ of rate vectors $\mathbf{r}$, and thus mutually satisfiable rates as the capacity region of an abstracted degraded broadcast channel shown in Figure 1. Different receivers in the abstracted broadcast channel correspond to the cumulative packets received upon each successive packet reception at the sink: for instance the observed values at the ith receiver are $\mathbf{y}_i := (\pi(1), \mathbf{x}_{\pi(1)}, \pi(2), \mathbf{x}_{\pi(2)}, \ldots, \pi(i), \mathbf{x}_{\pi(i)})$. Note that the network, like many practical networks, through the header indicates the identity of the transmitted packet. That this broadcast channel is degraded may be seen from the fact that the time reversed observations form a Markov chain through erasure operations $\mathbf{y}_M \to \mathbf{y}_{M-1} \to \cdots \to \mathbf{y}_1$. The capacity region of the generic degraded broadcast channel is known [8] to be the the closure of the convex hull of the region $\mathcal{R}$ of rates satisfying $R_1 \leq \mathcal{I}(\mathbf{y}_1; \mathbf{U}_1), R_2 \leq \mathcal{I}(\mathbf{y}_2; \mathbf{U}_2 | \mathbf{U}_1), \ldots, R_M \leq \mathcal{I}(\mathbf{y}_M; \mathbf{U}_M | \mathbf{U}_{M-1})$ for a sequence of dummy discrete random variables $\mathbf{U}_1 \to \cdots \to \mathbf{U}_M$ with bounded range set cardinality. Here $\mathbf{U}_M$ is a vector composed of the inputs to the permutation channel, so that $\mathbf{U}_M^1 := \mathbf{x}_1, \mathbf{U}_M^2 = \mathbf{x}_2, \ldots, \mathbf{U}_M^M = \mathbf{x}_M$. In the particular case of the permutation channel, the generic degraded broadcast channel capacity region expression can be simplified to

$$\begin{aligned} R_1 &\leq \sum_{\pi} \mathsf{p}(\pi) \left(H(\mathbf{U}_M^{\pi(1)}) - H(\mathbf{U}_M^{\pi(1)} | \mathbf{U}_1) \right), \ldots \\ R_k &\leq \sum_{\pi} \mathsf{p}(\pi) \Big(H(\mathbf{U}_M^{\pi(1)}, \ldots, \mathbf{U}_M^{\pi(k)} | \mathbf{U}_{k-1}) \\ &\qquad - H(\mathbf{U}_M^{\pi(1)}, \ldots, \mathbf{U}_M^{\pi(k)} | \mathbf{U}_k) \Big), \ldots \\ R_M &\leq H(\mathbf{U}_M | \mathbf{U}_{M-1}) \end{aligned} \tag{1}$$

which can be obtained by substituting into the definition of mutual information the definitions of the received signals in the abstracted degraded broadcast channel. From this expression we can view the capacity region as the image under a linear transformation whose coefficients are determined by the permutation distribution of the set of all possible entropy vectors created from joint entropies of subsets of the dummy random variables $\mathbf{U}_1, \ldots, \mathbf{U}_M^1, \ldots, \mathbf{U}_M^M$. In the two special cases of interest we presently discuss in sections II-.1 and II-.2, the expression for the rate region can be simplified even further.

1) Special Case: Single Permutation: In the first special case, only one permutation has any probability. It is easy to see in this case that the amount of information we receive upon the ith packet arrival is only limited by the cardinality of the set that the packets lie in. In particular, if the packets $\mathbf{x}_1, \ldots, \mathbf{x}_M$ are each K bits long, then the rate region is specified by the inequalities

$$\sum_{i=1}^{p} R_i \leq pK, \quad \forall p \in \{1, \ldots, M\}$$

This rate region is plotted for the case $M = 2$ in Figure 2.

2) Special Case: Uniform Permutation Channel: In the second special case, all of the permutations are equally likely, and we call the associated permutation channel the uniform permutation channel. Assuming again that all of the packets $\mathbf{x}_1, \ldots, \mathbf{x}_M$ are each K bits long, the rate region for this case simplifies to

$$\sum_{i=1}^{M} \frac{R_i}{i} \leq K \tag{2}$$

This rate region is plotted for the case $M = 2$ in Figure 2.

A. Network Coded Case

In the random linear network coded case where the packets that arrive at time instants $\{\tau_i\}$ are linear combinations of the transmitted packets, we are again interested in a similar

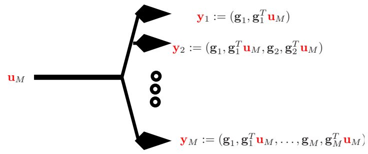

Fig. 3. Point to point network coded channel as a degraded broadcast channel.

abstracted degraded broadcast channel (DBC), depicted in Figure 3 where the different receivers correspond to successive packet arrivals and the vectors $\mathbf{g}_k$ correspond to the global encoding vector of the kth received packet. The capacity region of this DBC was solved in [4]. There it was shown that when the global encoding vectors are chosen uniformly from the set of innovative encoding vectors, this channel shares the same capacity region as the uniform permutation channel discussed in the previous section.

B. Rate Delay Calculus Problem Formulation

Now that we have determined the region $\mathcal{R}$ describing the amounts of information available after each successive packet arrival at the sink, we can incorporate the inter-arrival statistics. In particular, in keeping with the connection oriented context, let us suppose that the source decoder at the sink node needs to decode a new frame $\mathbf{s}_i$ every T_s seconds. Then, the overall rate at which data is decoded at the source decoder at the sink is

$$\rho(\mathbf{r}) := \frac{1}{NT_s} \sum_{i=1}^{M} R_i$$

Continuing the assumption that we are working with a constant bit rate source encoder, then this means we are trying to decode $\frac{1}{N} \sum_{i=1}^{M} R_i$ bits every T_s seconds.

A delay metric may then be associated with a rate vector $\mathbf{r}$. Here we choose a delay metric that penalizes the delay incurred by every frame according to the expression

$$D(\mathbf{r}) = \sum_{i=1}^{N} \left(\tau_{g(i,\mathbf{r})} - iT_s\right)^+, \ (x)^+ := \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where $g(i, \mathbf{r}) := \inf \left\{ k \left| \sum_{n=1}^{k} R_n \geq iT_s \rho(\mathbf{r}) \right. \right\}$ so that $g(i)$ is the number of packets that must be received at the receiver before we can decode the first i frames of the source.

At the heart of the idea of rate delay calculus is the intent of finding the point $\rho(d)$ in the coding achievable rate region $\mathbf{r} \in \mathcal{R}$ which maximizes the overall rate among all codes with delay less than or equal to some bound d. Because the arrival instants $\{\tau_i\}$ are randomly distributed, so is the delay, so we bound its mean, generating the rate delay tradeoff as

$$\rho^*(d) := \max \left\{ \rho(\mathbf{r}) \left| \mathbf{r} \in \mathcal{R}, \ \mathbb{E}\left[D(\mathbf{r})\right] \leq d \right. \right\}$$

C. Structural Properties of the Rate Delay Tradeoff in Uniform Permutation Case

In many common situations, e.g. when the packet arrival times $\{t_i\}$ are i.i.d., the permutation distribution will be uniform, yielding the corresponding rate region given by (2). Since the delay metrics $D(\mathbf{r})$ depend on $\mathbf{r}$ only through $g(i, \mathbf{r})$, we can break the optimization of $\mathbf{r}$ up into two coupled optimization problems, so that the optimal rate vector $\mathbf{r}^*$ is the solution to

$$\rho^*(\mathbf{r}^*) = \max_{h(i)|\mathbb{E}[D(h(i))] \leq d} \ \max_{\mathbf{r} \in \mathcal{R} | g(i,\mathbf{r}) = h(i)} [\rho(\mathbf{r})] \qquad (3)$$

subject to $R_i \geq 0$ and (2). Next, by rewriting the constraint $h(i) = g(i, \mathbf{y})$ as

$$h(i) = \inf \left\{ k \left| \frac{\sum_{n=1}^{k} R_n}{\sum_{i=1}^{M} R_i} \geq \frac{i}{N} \right. \right\}$$

we can see that $\rho(\mathbf{r})$ depends only on the magnitude ($\|\cdot\|_1$) of $\mathbf{r}$, while $h(i)$ depends only on the direction $\mathbf{r}$ points in. The constraint (2), on the other hand, depends on both its magnitude and direction. This observation motivates a change of variables to $\gamma_i \geq 0$ and ρ such that $\sum_{i=1}^{M} \gamma_i = 1$ and $R_i = NT_s \rho \gamma_i$. The optimization in (3) can then be written as

$$\max \left\{ \rho \left| NT_s \rho \sum_{i=1}^{M} \frac{\gamma_i}{i} \leq K, \ \sum_{i=1}^{M} \gamma_i = 1, \ g(i, \mathbf{r}) = h(i) \ \forall i \right. \right\}$$

From the constraints, we see that the maximum value of ρ is easily determined as

$$\rho(\mathbf{r}) = \frac{K}{NT_s} \left(\sum_{i=1}^{M} \frac{\gamma_i}{i} \right)^{-1} \qquad (4)$$

and the problem is thus transformed into the minimization of $\sum_{i=1}^{M} \frac{\gamma_i}{i}$ over the parameters $\{\gamma_i\}$.

Since several successive values of $h(i)$ may be equal with $h(i) \geq h(i-1)$, starting with $k_1 = \max\{k | h(k) = h(1)\}$ recursively define the sequence $k_j = \max\{k | h(k) = h(k_{j-1} + 1)\}, \quad j = 2, 3, \ldots$. Because the cost $\sum_{i=1}^{M} \frac{\gamma_i}{i}$ penalizes lower index γ_i more than higher index γ_i, and γ_i must be positive and sum to one, to simultaneously satisfy the constraints and minimize the cost we must choose the only non-zero γ_is to be $\gamma_{h(k_1)} = \frac{k_1}{N}$, $\gamma_{h(k_j)} = \frac{k_j - k_{j-1}}{N}$, $j \in \{2, 3, \ldots\}$. Undoing the change of variables, with $\gamma_{h(k_j)}$ so determined, the corresponding values of $R_{h(k_j)}$ are determined using $R_{h(k_j)} = \rho \gamma_{h(k_j)}$ with ρ given by (4).

We have thus shown that the rate delay tradeoff, although correctly posed as a continuous optimization, in the uniform case can be found as the solution to an integer programming problem in the decoding deadlines $\{h(i) \in \{1, \ldots, M\} | i \in \{1, \ldots, N\}, h(i+1) \geq h(i)\}$.

D. Variational Approximation based Rate Delay Tradeoff for Large M, N

For reasonably small values of M, N, this structural transformation in the previous section to an integer programming problem over $h(i)$ allows direct calculation of the rate delay tradeoffs by explicit enumeration over all possible non-decreasing sequences $h : \{1, \ldots, N\} \to \{1, \ldots, M\}$. However, for larger M, N this explicit enumeration becomes computationally unfeasible, requiring an approximate solution. Such an approximate distribution can be obtained by turning the integer programming problem into a variational calculus problem over a family of non-decreasing continuous functions, whose solution, via Lagrangian techniques, can be found as a solution to an algebraic equation. Towards this end, given the selected delay metric $\mathsf{D}(\{h(i)\}) \in [0,\infty)$ of interest, we first define a normalized rate and delay, which for all $N, M, T_s, \{h(i)\}$ lie in between 0 and 1. The normalized rate can be written as

$$\bar{\rho}(\{h(i)\}) = \frac{NT_s}{MK}\rho = \left(M \sum_j \frac{k_j - k_{j-1}}{N} \right)^{-1}$$

The normalized delay can be obtained by noting that all of the delay metrics we are considering must be monotone non-decreasing in $h(i)$, and thus the greatest delay is obtained with $h(i) = M, \quad \forall i \in \{1, \ldots, N\}$. Thus, to obtain a normalized delay lying in between 0 and 1, we need only to divide the non-normalized delay metric of choice by its value at $h(i) = M, \quad \forall i \in \{1, \ldots, N\}$

$$\bar{d}(\{h(i)\}) := \frac{\mathsf{D}(\{h(i)\})}{\mathsf{D}(\{h(i) = M \ \forall i \in \{1, \ldots, N\}\})}$$

In order to approximate the solution to the integer programming problem by the solution to a variational calculus problem, first define a zero order hold piecewise continuous function $\mathsf{z} : [0,1] \to [0,1]$ such that $h(i) = M\mathsf{z}(i/N)$

$$\mathsf{z}(x) = \sum_{k=1}^{N} \frac{h(k)}{M} \mathbf{1}_{\frac{k-1}{N} \le x < \frac{k}{N}}$$

We can then rewrite the normalized rate as the functional

$$\bar{\rho}[\mathsf{z}(\cdot)] = \left(\int_0^1 \frac{1}{\mathsf{z}(x)} dx \right)^{-1}$$

The normalized delay is then (if need be approximated by) a functional $\bar{d}[\mathsf{z}(\cdot)]$ of $\mathsf{z}(\cdot)$ as well. We presently consider two special cases where the normalized delay functional can be further specified with ease.

1) Normalized Delay Functional for Constant Inter-Arrival Times: [3] determines the normalized delay functional in the special case when $N = M$ and the packets arrive at evenly spaced, non-random, time instants, so that $\tau_i = iT_s$. In that special case, the normalized delay functional can be written as

$$\begin{aligned} \bar{d}(\{h(i)\}) &:= \frac{\sum_{i=1}^{M} (h(i) - i)^+}{\sum_{i=1}^{M} (M - i)} = \frac{2M}{M-1} \sum_{i=1}^{M} \frac{(h(i) - i)^+}{M^2} \\ \bar{d}[\mathsf{z}(\cdot)] &= \frac{2M}{M-1} \int_0^1 (\mathsf{z}(t) - t)^+ \mathsf{d}t \end{aligned} \quad (5)$$

2) Normalized Delay Functional for IID Arrival Times: As another special case, the form of the normalized delay functional may be further specified when the uniform permutation selection arises from i.i.d. arrival times t_i distributed according to a distribution with density p_t and cumulative distribution P_t. The asymptotic theory of order statistics ([9] Theorem 10.3 pp. 288) asserts that $M^{-\frac{1}{2}} \left(\tau_{M\mathsf{z}} - \mathsf{P}_t^{-1}(\mathsf{z}) \right)$ has an asymptotic normal distribution with variance $\frac{\mathsf{z}(1-\mathsf{z})}{\left(\mathsf{p}_t(\mathsf{P}_t^{-1}(\mathsf{z}))\right)^2}$ (as $M \to \infty$, with z fixed):

$$M^{-\frac{1}{2}} \left(\tau_{M\mathsf{z}} - \mathsf{P}_t^{-1}(\mathsf{z}) \right) \to \mathcal{N}\left(0, \frac{\mathsf{z}(1-\mathsf{z})}{\left(\mathsf{p}_t(\mathsf{P}_t^{-1}(\mathsf{z}))\right)^2} \right) \quad (6)$$

This holds whenever $0 < \mathsf{p}_t(\mathsf{P}_t^{-1}(\mathsf{z})) < \infty$ (an easy to satisfy condition, indeed). This implies that the limiting form of the delay functional for the mean sum delay metric is

$$\bar{d}[\mathsf{z}] = \frac{\int_0^1 \left(\mathsf{P}_t^{-1}(\mathsf{z}(w)) - wNT_s \right)^+ \mathsf{d}w}{\int_0^1 \mathbb{E}\left[(\tau_M - wNT_s)^+ \right] \mathsf{d}w}. \quad (7)$$

Here, the denominator may also be approximated through the asymptotic distribution of the extreme values ([9], Ch. 11), which, after scaling can take only one of three forms.

3) Find Approximate Rate Delay Tradeoffs via Variational Calculus: Once we have selected the delay metric and obtained the normalized delay functional $\bar{d}[\mathsf{z}]$, the idea is to obtain the approximate rate delay tradeoffs for large M by solving the calculus of variations [10] problem

$$\bar{\rho}^*(\bar{d}^*) := \max_{\mathsf{z}:[0,1]\to[0,1] \ni \bar{d}[\mathsf{z}] < \bar{d}^*, \ \frac{\partial \mathsf{z}(t)}{\partial t} \ge 0} \bar{\rho}[\mathsf{z}] \quad (8)$$

where z is no longer constrained to be a piecewise constant function, and, in fact, is now considered among the class of continuously differentiable non-decreasing functions. For large M, which are the cases which we would like to calculate because the explicit enumeration of $h(i)$ is too difficult, the intervals on which z was to be piecewise constant should be very small, so one would expect to get a good approximation for these cases. Using monotonicity of the function $\frac{1}{x}$, we may observe that the solution to (8) may alternatively be determined as the solution to

$$\bar{\rho}^*(\bar{d}^*) := \left(\min_{\mathsf{z}:[0,1]\to[0,1] \ni \bar{d}[\mathsf{z}] < \bar{d}^*, \ \frac{\partial \mathsf{z}(t)}{\partial t} \ge 0} \int_0^1 \frac{1}{\mathsf{z}(t)} \mathsf{d}t \right)^{-1} \quad (9)$$

For those special cases (e.g., the constant inter-arrival time and mean sum delay metric cases above) when the normalized delay function $\bar{d}$ may be written as

$$\bar{d}[\mathsf{z}(\cdot)] = \int_0^1 \mathsf{Q}(t, \mathsf{z}(t)) \mathsf{d}t \quad (10)$$

then the solution to the variational problem (8) must satisfy the algebraic equation

$$-\frac{1}{(\mathsf{z}^*(t))^2} + \mu \left. \frac{\partial \mathsf{Q}(t,x)}{\partial x} \right|_{x=\mathsf{z}^*(t)} + \alpha(t) \mathbf{1}_{\frac{\partial \mathsf{z}(t)}{\partial t} = 0} = 0 \quad \forall t \in [0,1]$$

with $\mu, \mathsf{z}(0)$ such that $\bar{d}[\mathsf{z}^*] \le \bar{d}^*$ and $\alpha(t)$ obviously selected so as to make z^* constant on intervals where it is constant (and thus we need not actually solve for it). The rate delay tradeoff is then given by $\bar{\rho}^*(\bar{d}^*) = \bar{\rho}[\mathsf{z}^*_{\bar{d}^*}]$.

E. Example Rate Delay Tradeoffs for Common Arrival Time Distributions

In order to get greater practical insight, we now consider the rate delay tradeoff for a two common arrival time distributions, constant periodic packet arrivals, and IID exponentially distributed packet arrival times.

1) Constant Inter-arrival Times: [3] investigates the case $\tau_i = iT_s$ and $N = M$. The normalized delay function and functional has already been given for this case in (5), and the asymptotic rate delay tradeoff is given by

$$\bar{\rho}^*(\bar{d}^*) = \left(1 - \log\left(\sqrt{\frac{M-1}{M}\bar{d}}\right)\right)^{-1}, \quad \bar{d}^* = \frac{M}{M-1}e^{2-\frac{2}{\bar{\rho}^*}}$$

which is proved and plotted together with the exact rate delay tradeoffs in [3].

2) Trading Rate for Average Sum Delay with Exponential Arrival Times: Consider the case where the arrival times t_i are i.i.d. exponential random variables with parameter λ. Symmetry suggests that the permutation distribution $\mathsf{p}(\pi)$ will be uniform, thereby selecting the capacity region solved in II-.2. If we are alternatively considering the network coded context, we assume that the global encoding vectors are chosen such that the global encoding matrix (after M packet arrivals) is sampled uniformly from the set of invertible $M \times M$ matrices over the finite field in use, and the t_i is the arrival of the kth network coded packet.

For small values of M and N numerical calculation of the rate delay tradeoffs through enumeration of $h(i)$ is possible, and some normalized rate and normalized delay tradeoffs are shown for some example cases in Figure 4. For larger values of M and N, an approximation is necessary, and we use the variational technique discussed in Section II-D to get close approximations to the rate delay tradeoff. Specializing (7) for i.i.d. exponential arrival times the normalized delay functional for large M is well approximated by

$$\bar{d}[\mathsf{z}] := \frac{\int_0^1 \left[-\lambda^{-1}\log(1-\mathsf{z}) - wNT_s\right]^+ \mathsf{d}w}{\int_0^1 \mathbb{E}\left[(\tau_M - wNT_s)^+\right] \mathsf{d}w}$$

which may be used to obtain the solution to the calculus of variations problem as M gets large. This gives a rate delay tradeoff (for normalized delay requirement $\bar{d} \in [0,1]$):

$$\bar{\rho}^*(\bar{d}) := \left(\frac{c\sqrt{\bar{d}}}{1-\exp(-\lambda NT_s c\sqrt{\bar{d}})} + \frac{1}{\lambda NT_s}\log\left(\frac{\exp(\lambda NT_s)-1}{\exp(\lambda NT_s c\sqrt{\bar{d}})-1}\right)\right)^{-1} \quad (11)$$

where for large M $c \approx \sqrt{\frac{2\log(M)-\gamma}{3\lambda^2 NT_s} - \frac{1}{3\lambda}}$, with $\gamma \approx 0.5772156649$, which comes from the limiting form of the extremum. This is derived with a few steps of calculus to find the associated optimal decoding deadline function, which takes the form

$$\mathsf{z}^*(w) := \begin{cases} 1-\exp(-\lambda NT_s c\sqrt{\bar{d}}) & w \le c\sqrt{\bar{d}} \\ 1-\exp(-\lambda NT_s w) & w > c\sqrt{\bar{d}} \end{cases}$$

III. Conclusions

We have provided a general technique for obtaining the optimal fundamental rate versus mean sum normalized delay tradeoffs in multipath routed and network coded networks. We did this by introducing a related degraded broadcast channel whose capacity region we obtained through multi-terminal information theory. We demonstrated the utility of our technique by applying it to obtain the rate delay tradeoffs in the case where the packet propagation times were i.i.d. according to an exponential distribution. An end user wishing to obtain a code with maximal rate subject to a delay bound of δ may find the associated point on our rate delay curve, and its associated decoding deadlines z, $\{h(i)\}$, or rates $\mathbf{r}$. These may then be used as parameters of a (time shared erasure) PET code [5], [6] in the permutation channel case as discussed in [3], or a modified (time shared rank metric) PET code using rank metric codes as discussed in [4] and [7] to get small block length codes achieving the specified rate delay tradeoff.

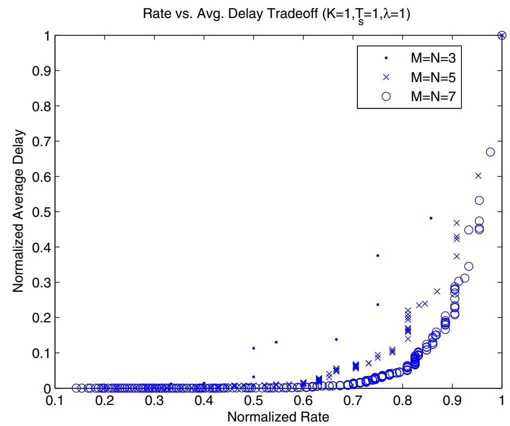

Fig. 4. Rate delay tradeoff for $N = 3, 5, 7$ and i.i.d. exponential arrival times.

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