ASYMPTOTIC MODELLING OF FLUID FLOW PHENOMENA

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Asymptotic Modelling of Fluid Flow Phenomena

by

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*"A tiny bit of reflection on and, from time to time,
revolt against the recognized ideas is a good and necessary
matter in the physical world!"*

To *Natalia* and *Christine*.

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PREFACE AND ACKNOWLEDGMENTS

My purpose in this book concerns the role of that part of formal (applied) mathematics which is called asymptotic techniques in the art of the asymptotic modelling of Newtonian laminar fluid flows. From my point of view, I consider it sufficient, for later use, to define asymptotic modelling of fluid flows as:

The scientific activity which consists in deriving fluid flow model problems in such a rational way that they become amenable to mathematical analysis and to numerical simulation.

More precisely, with this point of view, I may define asymptotic modelling as:

"The art of modelling assisted by the spirit of asymptotic techniques".

But I insist that the main goal is modelling and not finding solutions! Naturally, it may happen that the solution emerges from the very process of modelling but this is quite exceptional. Usually, a process of some mathematical analysis and numerical simulation is expected to *follow after this modelling* whether it is assisted or not by asymptotic techniques. Formerly, asymptotic techniques were used in order to obtain closed-form (approximate) solutions, but this aspect of applied mathematics has been rendered somewhat obsolete by the tremendous increase in capabilities of numerical simulation and also by the increasingly more difficult and complicated problems arising from various technological flows.

A limited number of fluid flow problems may still be solved (approximately) by closed form solutions through highly idealized mathematical models, but most interesting (and, in fact, important) are either unsolvable or only amenable to numerical simulation after some appropriate asymptotic modelling. The role of classical asymptotic techniques will probably be limited in scope to some part of the mathematical work which occurs before the numerical simulation and one may conjecture that:

For some time the growth in capabilities of numerical simulation will be dependent on, or at least related to, the development of asymptotic modelling.

I list, below, a few of the key reasons for which asymptotic analysis and modelling are suspected to hold a very prominent position in the future development of technological research.

(1) Naturally, the derivation of closed form (approximate) solutions will continue to be useful in some special circumstances.

(2) A number of important technological problems will remain, for beyond the capabilities of full-scale numerical simulation for some time to come, and asymptotic modelling is expected to generate simpler, adequate and consistent models amenable to numerical simulation. In most cases an efficient tool for achieving this goal will be to retain more the spirit of asymptotic analysis than its complete formal structure nothing being said of the mathematics.

(3) Very often asymptotic analysis will be quite closely tied to the preparation of the numerical simulation especially when the computations involve a wide range of important scales and simultaneously dominant and negligible effects of stiff local problems.

(4) One should not underestimate the very substantial aid that asymptotic analysis and modelling may provide to the understanding of either the underlying physics or the mathematical structure of the models under investigation.

The main governing idea of the present book is that the various asymptotic models for Newtonian fluid flows must be obtained rationally as significant degeneracies of the so-called: Navier-Stokes-Fourier (NS-F) equations governing a compressible and thermally conducting viscous fluid.

Generally speaking, it just so happens that the various dimensionless parameters (Reynolds, Strouhal, Mach, Prandtl, Froude, Boussinesq, Rossby/Kibel,... numbers and various ratios of lengths and times) which come into the NS-F equations and also in the boundary conditions and the geometry of the problems are, often, small or large singular perturbation parameters.

Indeed, asymptotic analysis and modelling is situated in an intermediate position between: rigorous mathematical, studies (well-posedness, solvability, existence, regularity and uniqueness proofs) - which are generally carried out on “typical simplified and relatively unrealistic problems” - and studies which are “more physical and realistic in nature”, but have not as yet been defined formally in the sense of singular perturbation techniques.

The book is divided in *twelve Chapters.* In the *first* Chapter: “Introductory comments and summary”, the reader can find a short summary of Chapters 2 to 12. The cited references for the Chapters 2 to 12 are (alphabetically) listed at the end of the book.

The choice of these twelve Chapters and their ordering are, at least from our point of view, quite natural. The presentation of the material, the relative weight of the various arguments, and the general style reflects the tastes of the author and his knowledge - it cannot be otherwise.

Naturally, the present book constitutes a collection of “Advanced Topics” rather than a classical Course on Fluid Dynamics and, concerning the various Chapters of this book, I have been highly selective in my choice of topics and in many cases the choice of subjects is based on my own interest and judgment.

As a consequence, obviously, the present text is a personal expression of my view on Asymptotic Modelling in Theoretical Fluid Dynamics.

The above ‘Table of Contents’ is ambitious (!), and I don’t know if the reader will consider that I have (even partially) attained my objective - which was a modern presentation of some key problems in Newtonian fluid flows in the light of asymptotic analysis and modelling - chiefly devoted to emphasizing the considerable support that mastering asymptotic tools can afford to researchers embarking on modelling, very difficult, but fundamental, problems of fluid flows.

Obviously, only the derivation of a consistent, significant, model problem, in place of a physically ‘exact’, stiff, problem, makes it possible to obtain a ‘good’ numerical simulation!

It is hoped that this book will serve the purpose of providing a good rational introduction to the asymptotic analysis and modelling of fluid flows, both for graduate students and young researchers in fluid mechanics, applied mathematics and theoretical/mathematical physics.

Several readers made many useful suggestions and criticisms during the elaboration of this book, for which I am grateful, although I must accept the final responsability for remaining errors and omissions.

I would like to emphasize that, beyond what appears through the list of references, copublished with Jean-Pierre Guiraud, from 1970 onwards a good deal of my views on asymptotics and fluid mechanics rests on endless exchanges with him, both manuscript and oral, over a period of almost twenty years. Our last cosigned paper appeared in 1986, but we continued to exchange ideas and notes at a slackened rate and I benefited from this long-lived scientific and personal friendship. In addition, I would like to point out that I increased my understanding of the asymptotic modelling of thin viscous liquid films subject to the Marangoni effect (considered in Chapter 10), thanks to my visit to the “Unidad de Fluidos” of the Instituto

Pluridisciplinar UCM de Madrid (May-July 2000), thanks to an invitation from Professor Manuel G. Velarde and the grant SAB1999-0109 of the "Direccion General de Enseñanza Superior e Investigación Cientifica" of the Spanish "Ministerio de Educación y Cultura".

I express, first, my gratitude to Professor René Moreau, as the series Editor of "Fluid Mechanics and its Applications", who have prompt me to write the present book. I appreciate also the important and fruitful work carried out by Dr. Neil Edwards (of the University of Southampton) on the original English manuscript of this book in correcting my "French oriented" English. Finally, my thanks to Dr Arno Schouwenburg, Publishing Editor in Engineering Kluwer Academic Publishers (Physical Sciences Unit), for the publishing of this volume.

Paris
February 2001

R. Kh. Zeytounian

CHAPTER 1

INTRODUCTORY COMMENTS AND SUMMARY

1.1. INTRODUCTION

This book for the Kluwer series in "Fluid Mechanics and its Applications", concerns the derivation of asymptotic models for Newtonian fluid flow problems.

In this Volume the various models that have been derived from the early 1970's are described from a consistent asymptotic point of view. In addition, the reader can find also the more classical asymptotic models and also some new approaches for the low Mach number and large Reynolds number asymptotics.

The present book is not an "English version" of my 1994 book, in French, "Modelisation asymptotique en mécanique des fluides newtoniens", edited by Springer-Verlag (In the series: Mathématiques & Applications, Vol. 15).

Indeed, in the present book, to formal asymptotic methods, we devote only a short chapter; Chapter 3, and the main subject of this book is the derivation of various rational and consistent asymptotic models from the NS-F 'exact' Newtonian fluid flow model.

It is important to note that actually some singular problems in fluid dynamics are still unresolved! Among others we can mention, for example:

1) The *initialization* (in time!) of the unsteady, compressible, Prandtl boundary-layer equations, by means of the viscous unsteady Rayleigh-Howarth model (valid in the vicinity of the initial-time and near the wall) coupled with the one-dimensional unsteady inviscid gas-dynamics model (which replaces the Prandtl boundary-layer equations close to initial time), by an *unsteady adjustment* process. This problem is discussed in §7.4 of Chapter 7, but it is far from being resolved. In fact we have in this compressible unsteady case a *four-region* structure, for large Reynolds number, relative to time and the coordinate normal to the wall.

2) The behaviour of acoustic waves, generated at the initial time, in a closed domain with a deformable (in time) wall, for a slightly compressible

(low-Mach number), viscous fluid with a vanishing viscosity (high-Reynolds number), which is considered in §8.4 of Chapter 8, but only for an inviscid Eulerian fluid when the Reynolds number is infinite. For very *large Reynolds* (slightly viscous) flow the *damping* phenomena of the acoustic oscillations (which are generated by initial impulse of the wall) appears to be a difficult problem and raises many questions.

3) The three-region *unsteady compressible* Stokes-Oseen problem for a vanishing Reynolds number. The incompressible case being analyzed for the flow past a sphere in §9.3 of Chapter 9.

4) The *low-Mach* number asymptotics for the atmospheric flows (considered in the Chapter 11) without the classical Boussinesq hypothesis. Concerning the asymptotic Boussinesq inviscid atmospheric equations see §4.7 of Chapter 4.

5) The model of through-flow in a turbomachine for a *vanishing viscosity.* In the framework of an inviscid and incompressible fluid the through-flow model is considered in §6.6 of Chapter 6.

6) The *unsteady triple-deck* model (significant near the initial time!) and the analysis of the associated unsteady adjustment process to steady, Neiland -Stewartson and Williams, classical triple-deck structure considered in Chapter 12.

7) The *initialization* of model equations (for instance, the KS equation) for the thin film problems (considered in Chapter 10), which are derived via a *long-wave approximation.* Such an approximation being, unfortunately, invalid close to initial time.

8) A 'speculation' about local *Boussinesq-like flows embedded* within more general low-Mach number flows; this problem is obviously strongly related to problem 4).

9) The *asymptotically consistent approach to the modelling of turbulent phenomenon.* In particular it is possible, actually, to derive in a consistent way a pair of closed equations for the kinetic energy of the turbulence (as a matter of fact of fluctuations) per unit mass and the *rate of turbulence dissipation,* which are local quantities which change from time-point to time-point. This pair of equations results from a strategy which consists in applying $\boldsymbol{k}$-$\boldsymbol{\varepsilon}$ type modelling to a set of equations which is not the usual set

for the fluctuations around the means but rather fluctuations, $u'_i(t,x_i)$ and $p'(t,x_i)$, appearing in the following incompressible (ρ_0 = *const*) system of equations:

$$\frac{\partial u_i'}{\partial t}+U_j\frac{\partial u_i'}{\partial x_j}+u_j'\frac{\partial U_i}{\partial x_j}+\frac{1}{\rho_0}\frac{\partial p'}{\partial x_i}$$

$$+\frac{1}{\rho_0}\frac{\partial}{\partial x_i}\left[\tau_{ij}+\rho_0 u_i'u_j'\right]-\nu_0\Delta u_i'=0$$

$$\frac{\partial u_i'}{\partial x_i}=0$$

$u_i' = 0$, on any wall; $u_i' = 0$, at initial time,

and U_j, τ_{ij} are assumed known. This contribution arose from discussion with J.P. Guiraud on attempts to push forward our last co-signed paper (1986) and the main idea is to put a stochastic structure on fluctuations and to identify the large eddies with a part of the probability space. The Reynolds stresses are derived from a kind of Monte-Carlo process on equations for fluctuations. Those are themselves modelled against a k-ε technique, using the Guiraud and Zeytounian (1986). The scheme consists in a set of k-ε like equations, considered as random, because they mimic the large eddy fluctuations. The Reynolds stresses are got from stochastic averaging over a family of their solutions. Asymptotics underlies the scheme, but in a rather loose hidden way. We explain this in relation with homogenization-localization processes (described within the §3.4 of Chapter 3). Of course the mathematical well posedness of the scheme is not known and the numerics would be formidable! Whether this attempt will inspire researchers in the field of highly complex turbulent flows is not foreseeable and we have hope that the idea will prove useful. Obviously the modelling of turbulence deserves further careful investigation and I hope to return to this problem very soon!

Actually, various technological, complex fluid flow problems are resolved via massive computations and often *ad hoc* arguments to lead to their results, little attempt being made justify the assumptions on mathematical grounds and it seems of great importance that a rational, consistent, approach be adopted to make sure that if in the equations a term is neglected, then it is really much smaller than those retained. Obviously, until this is done, the value of the results of such a computation is questionable. In fact, it is necessary to understand that actually both the numerics and asymptotics both are useful and complementary. As has is recently been noticed by Paul Germain, actually a "new" mechanics of fluids has emerged, namely:

"Fluid Mechanics inspired by Asymptotics ".

Unfortunately, in the present book I will have to leave out some important topics which are can also be explain in the spirit of asymptotic modelling.

For example, among other, I mention: The Prandtl lifting line concept (steady and unsteady) and its applications; Riley-Stuart-Pedley theory related to the phenomenon of acoustic streaming (double boundary-layer structure); Asymptotic models in flame propagation theory, in dynamic detonation and combustion; Weakly non-linear internal and solitary waves in shear flows and asymptotic analysis of Ocean Circulation.

1.2. SUMMARY OF CHAPTERS 2 TO 12

Chapter 2:

The classical continuum theory makes it possible to formulate four balances equations, (integral conservation laws), respectively, for: mass, momentum, moment of momentum and energy. In expressing the integral laws, the fluid domain $\boldsymbol{\Omega}$ and its boundary $\partial\boldsymbol{\Omega}$ will be considered fixed with respect to the system of coordinates chosen, hence in general not fixed with respect to the fluid particles. By letting the volume of the domain $\boldsymbol{\Omega}$ tend to zero, one may derive two types of relations. The first type occurs when $\boldsymbol{\Omega}$ encloses a surface of discontinuity. One then obtains jump conditions such as the shock wave relations in the theory of inviscid (nonviscous) fluids considered in Chapter 6. If, on the other hand, one assumes that the functions occurring in the integral relations have sufficiently many derivatives in $\boldsymbol{\Omega}$ one obtains local forms of these balances as differential equations. More precisely: the equation of continuity, the equation of motion (Cauchy's second law), the fact that the stress tensor $\boldsymbol{T}$ is symmetric $(T_{ij} \equiv T_{ji})$, and the energy equation. Then the so-called, Navier-Stokes-Fourier (NS-F) equations are derived when we write: (1) the constitutive relations with a Fourier law for a viscous, compressible and heat conducting Newtonian fluid flow (which is a particular case of a Stokesian fluid flow) and also, (2) the equations of state for the pressure and specific internal energy, when we assume that the Newtonian fluid is a thermally perfect gas with constant specific heats.

Our intent in Chapter 2 is not to condense all of the knowledge about continuum mechanics into a few pages. Rather, we present the material we will use in later Chapters. Namely: in §2.1, we give the precise statement of the conservation laws (balance equations) and the fundamental concept of Newtonian fluids. Then we write the constitutive relations, Fourier's law, dissipation function and equations of state for a thermally perfect gas and an expansible liquid. Next, from the associated (local) partial differential equations, we give the formulation of the NS-F equations. In the §2.2 we formulate initial and boundary conditions. Some preliminary information concerning the dimensionless analysis and the corresponding main dimensionless parameters (Reynolds, Mach, Strouhal, Prandtl, Froude, Rossby...) are given in §2.3.

Chapter 3:

Formerly (in the years 1960-1980), asymptotic analysis was used in order to obtain closed-form solutions, but this aspect of applied mathematics has been rendered somewhat obsolete by the tremendous increase in capacities of numerical simulations. A number of fluid flow problems may yet be solved by closed form solutions to highly idealized mathematical models, but most of interesting/important problems are either unsolvable or only amenable to numerical simulation after some appropriate asymptotic modelling. Obviously, for some time the growth in capabilities of numerical simulations will be depend on, or at least, be related to the development of asymptotic modelling. The asymptotic modelling is expected to generate simpler adequate and significant models amenable to numerical simulation and is very efficient for the *stiff* local problems. Here we note only that the perturbation (asymptotic) theory is based on the concept of an *asymptotic solution.* If the fluid dynamical equations describing a precise flow problem can be expressed such that one of the parameters or variables is small (or very large) then the full equations can be approximated by letting the perturbation quantity approach its limit and an approximate solution can be found in terms of this perturbation quantity. Such a solution approaches a limit as the perturbation quantity approaches zero (or infinity) and is thus an asymptotic solution. The result can often be improved by expanding in a series of successive approximations, the first term of which is the limiting solution. One then has an asymptotic series or expansion. Thus, we perturb the limiting solution by parameter or coordinate. One is then concerned with the asymptotic expansions, generally for a small parameter such as the Mach number or for a large parameter such as the Reynolds number, of the solutions of the NS-F, compressible, viscous baroclinic and thermally conducting, equations. In Chapter 3, first, in the short §3.1, we discuss briefly the role of 'asymptotics' and the various facets of the asymptotic modelling. The §3.2 is devoted to the idea of Method of Matched Asymptotic Expansions (MMAE), where I give a simple, but efficient, matching principle. The Multiple Scale Method (MSM) is considered briefly in §3.3, where some indications concerning the 'elimination of secular terms' are given. Finally, in the last very short §3.4, the reader can find a brief comment on the homogenization method and averaging process, which is an important step for modelling complex flow or turbulent flows. For the sake of simplicity, the case of several small (or large) parameters has not been included in the presentation of Chapter 3, although it will come up many times in what follows [for instance, when we consider a slightly viscous (large Reynolds number), quasi-incompressible (low Mach number), flow]. In any case, we can always return to the case considered by taking, for instance, ε as the main small parameter and considering the others as functions thereof via the *similarity relations*. We feel that this short discussion presented in §3.2 and §3.3, should give the reader a sufficient idea of two main asymptotic techniques: MMAE and MSM, for understanding the applications which will be presented in the coming Chapters 4 to 12.

Chapter 4:

In this short chapter, from the dimensionless NS-F equations, of §2.3, in a very naive approach, various particularly usual forms of fluid flow equations are derived in §4.1 to 4.7. Namely: Navier-Stokes (isentropic and viscous), Euler (nonviscous), Navier (viscous and incompressible), acoustics, Stokes-Oseen (for low-Reynolds number), Prandtl (boundary layer, for large Reynolds number), isochoric (for a conservative density along the trajectories), for a liquid (expansible) and Boussinesq (inviscid, with the buoyancy force) equations.

Chapter 5:

The equation of motion of a viscous but homogeneous and incompressible (with $\rho = \rho_0 = const$) fluid flow were first obtained by Navier (1821) and Poisson (1831), assuming a molecular model. This Navier equation takes the following rather simple (dimensional) form:

$$\frac{D\boldsymbol{u}}{Dt} + \frac{1}{\rho_0}\nabla p + \boldsymbol{g} = \nu_0 \nabla^2 \boldsymbol{u}\,,$$ where ν_0 is the kinematic viscosity.

This Navier equation for a divergenceless velocity $\boldsymbol{u}$ but with, in place of p, a pseudo-pressure perturbation π and $\rho_0 = 1$, in the case of aero-hydrodynamics (without the gravity g) is derived asymptotically, in §5.1, from the full NS-F equations for a compressible and heat conducting fluid flow, when the characteristic Mach number tends to zero. It is important to note (again!) that the passage from compressible to incompressible fluid flow (from NS-F to Navier equations), which filters out the fast acoustic waves, is a singular limit.

In particular, a very interesting problem is the "initialization" problem (initial condition for the incompressible Navier equation, at $t = 0$), when the initial data are given for the full compressible, viscous and heat conducting (NS-F) equations. This problem is considered in §5.2 and §5.3 is devoted to the derivation of the associated Fourier equation for the perturbation of the temperature and also the influence of a weak compressibility. In §5.4 we give some information concerning the influence of a weak compressibility on the Navier-Fourier model (Navier equations for the divergenceless velocity vector $\boldsymbol{u}$ and pseudo-pressure perturbation π, and associated Fourier equation for the temperature perturbation, with initial and boundary conditions).

It is important to observe that the gradient of pressure, ∇p, in the Navier equation for an incompressible but viscous fluid, is not an unknown quantity of the initial-boundary value problem. In fact, $\nabla(p/\rho_0)$, is the force term acting on the particles of fluid allowing them to move as freely as possible, but in a way compatible with the incompressibility constraint, $\nabla \cdot \boldsymbol{u} = 0$. As a consequence of the above, it is sufficient to consider the incompressible viscous Navier equation in terms of vorticity (we assume that $\boldsymbol{g}$ is conservative):

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \nu_0 \Delta \boldsymbol{\omega}, \text{ with } \nabla \cdot \boldsymbol{u} = 0, \text{ where } \boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}.$$

Obviously, when we assume that $\boldsymbol{g}$ is conservative, any potential flow trivially satisfies the Navier vorticity equation! However, to obtain a well-posed boundary value problem for fixed $\nu_0 > 0$, one must also (following Stokes) replace the *slip boundary condition* for Eulerian velocity, by the more stringent boundary condition of *no-slip.*

Chapter 6:

Fluid Dynamics was first envisaged as a systematic Mathematical-Physics Science in Johann Bernoulli's "Hydraulics" (in 1737), in Daniel Bernoulli's

"Hydrodynamics" (in 1738) and also in d'Alembert's "Traité de l'équilibre et du mouvement des fluides" (in 1744). But the fundamental ideas expounded in these three books were formulated mathematically as partial differential equations in a groundbreaking paper by Euler (in 1755) and actually, it is firmly established that Euler is indeed the founder of rational fluid dynamics. Nevertheless, Euler considers only nonviscous (inviscid) fluid flows with pressure a function only of density (a so-called "barotropic" fluid flow).

Indeed, Euler was referring, first of all, to what today called the equation of motion (momentum equation) of a nonviscous (inviscid) fluid:

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{g} - \nabla p \text{, where } \boldsymbol{u} \text{ is the velocity vector,}$$

with the gravitational force $\boldsymbol{g}$, per unit mass, the pressure p and the density ρ.

The (Cartesian) components of the nabla (∇) operator are the $\partial/\partial x_i$, $i = 1, 2, 3$ while: $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$, denotes the material (or substantial) derivatives (Euler rule of differentiation). Time is noticed by t and $\boldsymbol{x} = (x_1, x_2, x_3)$ is the position vector [$(t, \boldsymbol{x})$ are the so-called Eulerian time-space variables]. The above Euler equation is a direct consequence of Newton's Second Law (1687); namely:

"A body moves in such a way that at each moment the product of its acceleration vector by the density is equal to the sum of certain other vectors called forces, which can determined from the motion which takes place" - i.e.,

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{g} + \textit{internal force per unit volume.}$$

An inviscid (nonviscous) fluid is one in which it is assumed that the internal force acting on any surface element dS, at which two elements of the fluid are in contact, acts in a direction normal to the surface element. At each point P (with coordinates x_i, $i = 1, 2, 3$) the stress, or internal force per unit area, is independent of the orientation (direction of the normal) of dS and the value of this stress is called pressure, p, at the point P. Therefore the internal force per unit volume, appearing in Newton's equation, has x_i-component $(-\partial p/\partial x_i)$, $i = 1, 2, 3$. As a consequence, for an inviscid fluid we recover, from Newton's equation, the classical Euler equation.

A second part of Newton's Principles is related to the conservation of mass, namely:

"To each small solid body can be assigned a positive number m, invariant in time called its mass" - i.e.,

$$\frac{D}{Dt}\left(\int \rho dV\right)=0\,,$$

where dV is a volume element in the neighborhood of the point P and to this volume element will be assigned a mass ρdV.

In order to express this conservation of mass in the form of a differential equation, the differentiation indicated in this conservation equation could be carried out by transforming the integral suitably. In this case, we derive the so-called equation of continuity [this derivation is, in fact, due to Euler (1755)]:

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\boldsymbol{u}=0\,.$$

The (compressible) equation of continuity remains unaltered when viscosity is admitted. The above derived equations, which express Newton's Principles for the motion of an inviscid fluid, are usually referred to as (compressible) Eulerian fluid flows equations, and include one vector equation and one scalar equation for the velocity vector $\boldsymbol{u}$, the density ρ and the pressure (five unknowns). It follows that one more equation is needed in order that a solution of the system of equations be uniquely determined for given initial and boundary conditions. According to Euler:

"If we add to these equations, the following specifying equation, $p = p(\rho)$, *which gives the relation between the pressure and the density, we shall have five equations (a closed system) which include all the theory of the motion of fluids".*

By this formulation, Euler believed that he had reduced fluid dynamics in principle to a *mathematical-physics science.* But, it is important to note that, in fact, the relation: $p = p(\rho)$, between p and ρ, is not an equation of state, but specifies only the particular type of motion (so-called "barotropic") under consideration and in this case the fluid is called an elastic fluid.

Observe that, for Eulerian incompressible fluid flow, for instance, the equation $Du/Dt = 0$, admits solutions violating the condition: $\nabla\cdot\boldsymbol{u} = 0$ for $t > 0$, even if the divergence is vanishes at time zero. In the present case the pressure can be determined when we have found the velocity field $\boldsymbol{u}$ (which is the solution of the incompressible Euler equation). Indeed, taking the divergence of this equation we obtain a (elliptic) Poisson equation:

$$\Delta p = -\rho_0\{\nabla\cdot[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}] + \nabla\cdot\boldsymbol{g}\},$$

and, knowing $\boldsymbol{u}$ (and the external force $\boldsymbol{g}$), we can find p by solving this above elliptic equation with the following Neumann boundary condition:

$$\frac{\partial p}{\partial n} = -\rho_0[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \boldsymbol{g}]\cdot\boldsymbol{n},$$

obtained by projecting the incompressible Euler equation on the outward normal for a bounded domain. In Chapter 6, we study some features of inviscid (nonviscous) adiabatic (no heat output or input occurs for any particle) fluid flows and consider also some hydro-aerodynamics inviscid model problems derived from the Euler equations rationally in a consistent asymptotic way.

In §6.1, first, we consider the high Reynolds number NS-F fluid flow and the corresponding Euler model, and then, in §6.2, the initial, boundary conditions and uniqueness problem for this Euler model. §6.3 is devoted to linearization and associated wave phenomena. In §6.4 we derive some nonlinear 2D wave equations (in particular, for isochoric and Boussinesq fluid flows). The nonlinear long-surface-waves on water (potential theory) are considered in §6.5 and in §6.6, the incompressible rotational turbomachinery flows - when the blades within a row are very closed spaced - is discussed from an asymptotic point of view. The model problems for transonic (Mach number near unity) and hypersonic (high Mach number) gasdynamics flows are derived in §6.7 and 6.8. Finally, an asymptotic theory for rolled-up vortex sheets - when the turns of the sheet are very closely spaced (tightly wound vortex sheet) - is given in §6.9.

The reader can find in our recent book "Theory and Applications of Nonviscous Fluid Flows" (Springer-Verlag, 2001, in press) a very complete and detailed discussion of the various facets of Eulerian fluid flows.

Chapter 7:

Another facet of high/large Reynolds number flows related to the so-called boundary-layer (BL) concept which makes possible to model fluid flow, with a vanishing viscosity, near the wall of a solid body. Mathematically the inviscid solution cannot be uniformly valid when the viscosity tends to zero or $Re \to \infty$ [$Re = U_\infty L/\nu_0$, where U_∞ is the velocity in uniform motion, at infinite distance from the body (with L as characteristic length), in the direction defined by the unit vector $\boldsymbol{i}$ of the horizontal axis x], because it fails to satisfy the no-slip condition on the body, and this non-uniformity can only be removed by introducing a (thin) boundary layer near the body where viscosity matters.

In fact, for $\nu_0 = 0$, the Navier equation reduces to the Euler equation for an incompressible homogeneous and nonviscous fluid flow, namely:

$$\frac{D\boldsymbol{u}}{Dt} + \frac{1}{\rho_0}\nabla p + \boldsymbol{g} = 0,$$

but, because ν_0 multiplies the derivative of highest order, it cannot be inferred that the solutions of the Navier equation, for very small values of ν_0, tend uniformly to a solution of the incompressible Euler equation as $\nu_0 \downarrow 0$. For $\rho = \rho_0 = const$, we recover again, from the compressible Euler equation, the above Euler incompressible equation. For this reason, the Navier equation (for an incompressible but viscous fluid flow) is called a singular perturbation of the incompressible Euler equation. The theory of singular perturbations of Navier system [Navier equation with the incompressibility constraint $\nabla \cdot \boldsymbol{u} = 0$], for vanishing viscosity (large Reynolds numbers) is actually very well understood by the fluid dynamicists, thanks to Prandtl's (1904) boundary-layer (BL) concept and the asymptotic (inner-matching-outer) theory initiated by S. Kaplun (1954, 1957) and Kaplun and Lagerstrom (1957). But the mathematically rigorous theory (existence and uniqueness) is very difficult!

After a short discussion concerning the 'vanishing viscosity' singular problem, in §7.1, the reader can find a consistent derivation of 'dominant equations' from the steady NS-F equations, valid near of the wall of a 3D solid body, which is the first step in the derivation of the steady Prandtl BL equations. The process of matching with the Euler equations is discussed in §7.2 and the second order BL equations, for steady 2D flows, and their influence on the inviscid outer fluid flow, are considered in §7.3.

But, curiously, the BL equations are singular in the vicinity of initial time $t = 0$ (since a characteristic property of the boundary layer is the constancy of the pressure across the thickness of the BL) and, in §7.4, this initialization problem via the so-called Rayleigh-Howarth equations is discussed. In the theory of the BL the more familiar problem is the basic Blasius problem related to a steady (incompressible) flow past a solid flat plate oriented in the direction of a uniform stream, and in §7.5 we consider this Blasius problem, but for a slightly compressible fluid flow. The next section, §7.6 is devoted to Navier flow with variable viscosity, which leads to a three-layer asymptotic model Finally, the modelling of fluid flow within the Taylor shock layer is briefly considered in §7.7.

Chapter 8:

Some Aspects of Nonlinear Acoustics are presented in Chapter 8, where, again, the low Mach (M) number asymptotics plays a significant role. In particular, in §8.4, we consider "The low-Mach number flow affected by acoustic effects in a confined gas over a long time", which is an interesting model problem for the flow in a modern gas-turbine combustor. In fact, the problem relates to the consistent derivation of an asymptotic model, within a *closed container,* the walls of which deform (with time) very slowly in comparison to the speed of sound. In this case, unfortunately, the matching with the initial acoustic stage of motion is not (always) possible because of the persistence of acoustic oscillations for (dimensionless) time $t = O(1)$ and the 'divergence' of the corresponding unsteady adjustment problem when the motion of the wall is started impulsively, or accelered to a finite velocity, from the rest, in a time $O(M)$. Even a double-time scale analysis (as recently, in 1998, attempted by Müller) is insufficient and it is necessary to 'imagine' a more complicated multiple-scale technique - the gas motion within the closed container being considered as a superposition of acoustic oscillations depending on an (enumerable) infinity of fast times, and of an averaged flow depending on the slow time t. Because of the persistence of acoustic oscillations, a difficult problem is the long lifetime of these oscillations generated within the closed container by the starting process and how one can predict their evolution when the container has been deformed a great deal from its original shape.

For the case of an inviscid perfect gas, this problem has been posed and solved by Zeytounian and Guiraud in 1980. If we deal with a slightly viscous flow, we must start from the Navier-Stokes (at least compressible and isentropic) equations, in place of the inviscid adiabatic, compressible Euler equations and bring into the analysis a second small parameter ($1/Re$), the inverse of a characteristic Reynolds number. Then we must expect that the oscillations are *damped* out after a sufficient long time, but a precise analysis of this damping phenomenon appears to be a difficult problem and raises many questions.

Nevertheless, the case (apparently more realistic for the applications) when the parameter $\alpha = Re\ M >> 1$, is resolvable, via a boundary-layer analysis.

But, first, in §8.1 we derive the famous Burgers model equation and in §8.2 the so-called KZK equation. §8.3 is devoted to a discussion related to the emergence of the acoustics at large distance and its relationship to aerodynamic sound generation, and, via the Steichen equation for a compressible barotropic potential flow, we investigate the singular nature of the far-field in low Mach number asymptotics. Indeed, for the external aero-hydrodynamics, the Mach number plays also a peculiar role far from the body and, for any finite time, the process of going to infinity cannot be exchanged without caution with the case of letting the Mach number go to zero. In this case, again, an acoustic field approximates the flow and this is a basic process in the pioneering work by Lighthill on the generation of sound by turbulence. But the singular nature of the Mach number expansion which requires two matched asymptotic expansions was recognized, first, sixteen years later by Lauvstad, and more thoroughly discussed by Crow, Viviand and Obermeier; matching with the Navier incompressible and viscous limit model is a necessary step for the obtaining a 'three-region' significant asymptotic model. Further comments on the far flow may be found in Zeytounian and Guiraud (1984) and also in the recent thesis by Sery-Baye. The consistency of the MMAE (in relation to the far field) was extensively investigated by Leppington and Levine and by Tracey, and pushed up to $O(M^6)$ in the inner region, checking consistency with the outer one, ,in the thesis by Sery-Baye. As a matter of fact, this problem of the far flow is not a purely academic one, since any numerical simulation, in external aero - hydrodynamics, must work on a bounded computation grid, and is faced with the problem of choosing appropriate boundary conditions to be enforced on the external boundary of this grid. Finally, in the short §8.5, we discuss briefly some problems relating to computational aeroacoustics.

Chapter 9:

Some aspects of the low-Reynolds number flows are explained in §9.1, namely the Stokes and Oseen models for steady and unsteady incompressible fluid flows and also for steady compressible fluid flow. Two limiting processes play a fundamental role, according to P.A.Lagerstrom, in the asymptotic study of low- Reynolds number flow, namely the Stokes limit (inner limit) and the Oseen limit (outer limit). But for a compressible flow, when $Re \to 0$, considered in §9.1, it is necessary to specify the role of the Mach number, *M*! In fact, it is necessary to pose the problem concerning the behaviour of solutions of NS-F equations when simultaneously $Re \to 0$ and $M \to 0$.

Naturally, for the validity of the NS-F equations, it is obvious that it is assumed that the limiting compressible flow, at low Reynolds and low Mach numbers, remains a continuous medium and this implies that the so-called (dimensionless) Knudsen number, $Kn = M/Re$ is also a small parameter, $Kn << 1$.

As a consequence, the above double limiting process must be made with the following similarity relation:

$$M = R^*(Re)^{1+a}\text{, with } R^* = O(1) \text{ and } a > 0 \text{ when } Re \to 0.$$

The case of the incompressible (low Reynolds number) unsteady Stokes problem in a bounded domain is investigated in §9.2. But, again, in the case of the exterior problem (for example, flow past a sphere) low Reynolds number flows are singular in the vicinity of the initial time and at infinity, and it is necessary to consider a three-region (initial, Stokes and Oseen) unsteady asymptotic theory to obtain a consistent model, as explained in §9.3 for the case of the flow past a sphere. In §9.4 a model equation for hydrodynamic lubrication theory is derived. Finally in §9.5 the reader can find some complementary remarks.

Chapter 10:

The Bénard thermal convection problem is considered in various situations in the §10.1 to §10.5. In §10.2 we give, first, a mathematical formulation of the full Bénard thermal convection problem, taking into account the temperature dependent free surface tension and the deformation of the free surface.

Then, an asymptotic derivation of the so-called Oberbeck-Boussinesq equations for the shallow convection is given in §10.3 and the 'modified' Rayleigh-Bénard (R-B) problem is formulated. The case of a so-called, 'deep convection' problem, is also considered in §10.4. In §10.5 the Marangoni effect (related to the temperature dependent free-surface tension), for a deformable free surface, is taken into account and the so-called Bénard-Marangoni (B-M) problem is formulated. A lubrication evolution equation for the thickness of the film is derived. The K-S and KdV-KS model equations for long waves in thin layers are derived and analyzed in the framework of the B-M problem in §10.6. §10.7 is devoted to the interaction between short-scale Marangoni convection and long-scale deformational instability. Finally, the reader can find in §10.8 various concluding remarks and a discussion of recent publications.

Chapter 11:

In Chapter 11, we will restrict our discussion to atmospheric flows for which the horizontal scale $L°$ is much smaller than the mean radius a_0 ($\cong$ 6367 km) of the earth. Based on this hypothesis, since the parameter: $\delta = L°/a_0 << 1$, one can describe to a very good approximation, atmospheric flows in a system of Cartesian coordinates associated with the plane normal to the gravitational acceleration $\mathbf{g}$ (of magnitude g) resulting from the Newtonian gravitational attraction (the true gravitational acceleration owing to the pull of the earth, on the surface) and the centrifugal force per unit mass, due to the earth's rotation.

First, in § 11.2 we derive the NS-F atmospheric equations and in § 11.3 the large-scale hydrostatic non-tangent model equations and various simplified forms of these equations for atmospheric motions are considered. The so-called "Kibel meteorological primitive" (tangent with β-effect) equations are considered in §11.4. From these Kibel equations, in §11.5, we derive the quasi-geostrophic model, when the Rossby/Kibel number is a small parameter. §11.6 is devoted, again, to low-Mach number asymptotics and, in particular, to discussion of the so-called quasi-nondivergent model. In §11.7, the influence of a local relief is investigated and we consider, asymptotically, lee-waves and primitive problems as inner and outer expansions. In §11.8 a system of model equations for the lee-waves in the troposphere is derived - the so-called "deep atmospheric convection model", when the Boussinesq approximation is not adequate for the lee waves problem considered. Finally, some complementary references are given in §11.9.

Chapter 12:

Partly by making use of observation, Prandtl conjectured that the main stream leaves the surface when the skin friction coefficient on the body wall:

$$\tau(x) = \left(\frac{\partial u}{\partial y}\right)_{y=0}, \text{ vanishes,}$$

and that this occurs in an adverse pressure gradient: $\partial p_e/\partial x < 0$. In some sense this condition defines the position of the detachment point. Although this was a definite step forward and, in another context, of importance for the aeronautical industry among others, it leaves open the questions of why the vanishing of $\tau(x)$ is critical and of the mechanism by which the inviscid free streamline leaves the neighborhood of the body! In fact, when $\tau(x) = 0$

there is a failure of the notion that the BL has only a small effect on the inviscid flow outside.

At least, the inviscid flow must be modified in a significant way near the body at:

$$x = x_s,\ y = 0, \text{ with } \tau(x_s) = 0.$$

Prandtl pointed out that experimentally the inviscid flow in fact leaves the neighborhood of the body at this point and introduced the term "separation" to describe the phenomenon. Goldstein showed that the solution of the Prandtl BL equations could not be continued downstream of separation, if a singularity occurs, and thus we have a complete breakdown of the hierarchical system.

On the other hand, the weak-interaction principle - "noninteracting" coupling - outer (Euler) and inner (Prandtl) limits together give a complete picture of the flow as $Re \to \infty$, meant to postulate that the external flow adheres to the aerofoil without breakaways and tends to a uniform flow at large distance from the body. The condition of zero of the normal velocity component v_e at the body surface then suffices to determine it uniquely. In particular, the values of pressure p_e and horizontal velocity component u_e along the body surface are thereby determined. Breakaway involves a reversal of flow direction near the aerofoil surface and hence a negative 'wall shear' $\partial u/\partial y$, and before the solution of the BL problem can reach such a point, weak interaction appears to fail decisively.

For a pertinent short "qualitative" introduction to the asymptotics of the triple-deck (TD) theory, see Guiraud (1995, pp. 262-271). For a comprehensive review of the progress made in using this TD concept, see the papers by Nayfeh (1991) and Meyer (1983).

Indeed, there are many ways of breaking the (Prandtl) asymptotic structure, but one is especially significant, because it leads to a new asymptotic structure which seems to be very rich and it is this one which leads to the TD asymptotic model.

The first breaking is to place oneself near a particular position, say x_a, and set:

$$x - x_a = O\left(Re^{-3/8}\right),$$

and this specific order $Re^{-3/8}$, is obtained by inspection, through trial and error, but it comes also from examination of the instability theory, or from Lighthill's (1953) solution to the upstream influence paradox. We note, that recently Mauss (1995) has indicated a logical way to proceed, in order to obtain it deductively (see §12.5). Once the asymptotic order of the longitudinal extent is obtained, one finds that a three-tier structure is needed normal to the wall.

There are different equations depending on the order of y, namely:

$$y = O\left(Re^{-5/8}\right) \text{ for the "lower deck",}$$

$$y = O\left(Re^{-1/2}\right) \text{ for the "main deck",}$$

$$y = O\left(Re^{-3/8}\right) \text{ for the "upper deck".}$$

Then, the classical Prandtl (Blasius) BL divides into two parts, usually referred to as the lower and main decks respectively. The third deck of the triplet is the region of the external flow field, which is most significantly affected by the rapid changes in the BL - it is called the upper deck and lies just above the other two decks. The classical Prandtl BL equations hold to leading order in the lower deck, but their (asymptotic) derivation requires the application of different scaling laws (for the velocity components and pressure) which we shall not attempt to specify here. In Meyer (1983) an introduction to TD theory for steady, two-dimensional BL in low-speed flow is presented. It aims to clarify how the rational structure of the theory can rest on a few premises and to make its new ideas and challenges more widely accessible - in particular, Meyer introduces the notions of: mass-flow bound, penetration, localization and upstream condition. The above mentioned references are cited in the list of references at the end of the Chapter 12.

In §12.1 the various facets of breakdown of the Prandtl BL theory are considered: separation, Oleinik stability paradox of the BL, and d'Alembert paradox, with a discussion of the Kutta-Joukowsky (and Villat) condition; a phenomenological approach to the TD theory is also given. §12.2 is devoted to a 'clarification' of the classical TD structure and in §12.3 the definition of the steady canonical problem is presented (for a compressible and heat conducting fluid flow). Sychev's proposal for laminar separation is the subject of §12.4. In §12.5, according to recent results of Mauss *et al.*, we explain how the TD is a distinguished asymptotic structure arising from local perturbation to an Euler-Prandtl BL structure. Finally, some applications of the TD asymptotic model are discussed in §12.6

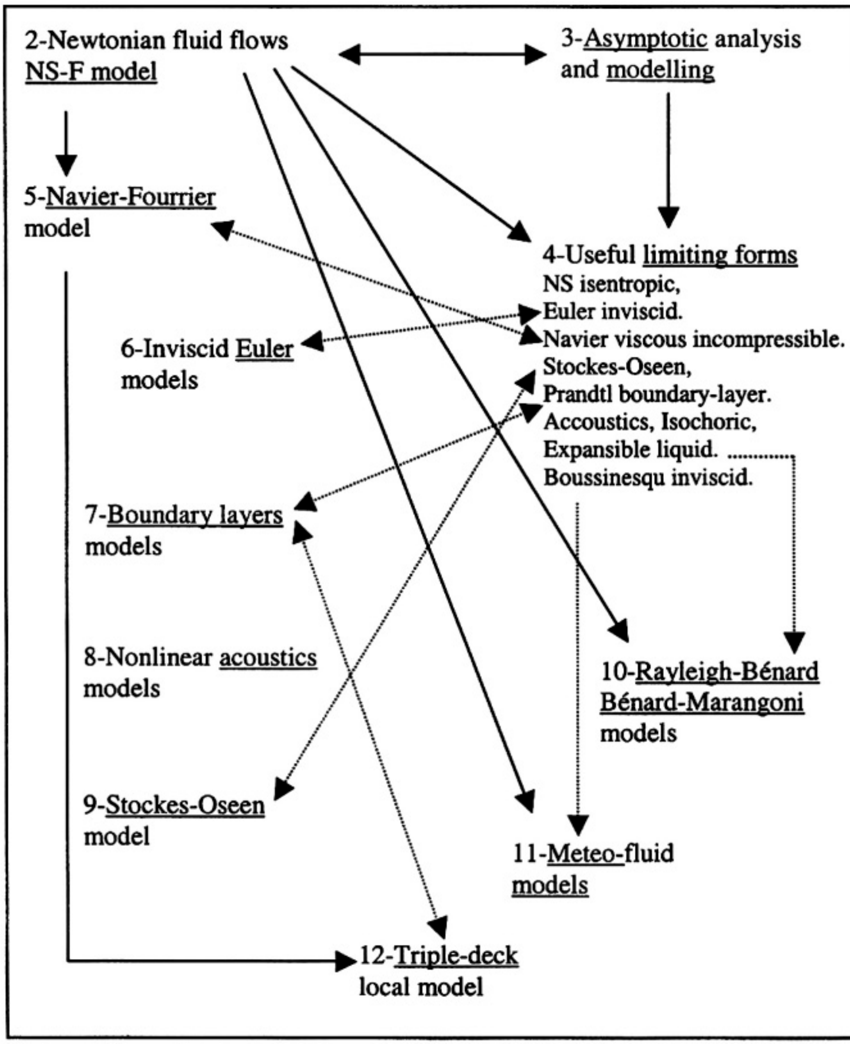


Fig. 1.1. Interconnection between Chapters 2 to 12

CHAPTER 2

NEWTONIAN FLUID FLOW: EQUATIONS AND CONDITIONS

2.1. FROM THE CLASSICAL CONTINUUM THEORY TO THE NAVIER-STOKES-FOURIER EQUATIONS

Generally speaking [see, for example, the review paper by Germain (1972)], a conservation law is one which says that:

For a given material (fluid) within a domain Ω, bounded by a closed surface $\partial\Omega$, which we follow during its motion, the rate of what is furnished by exterior (its volume density is specified by the components F_{ij}) is equal to the rate of what flows outside $\partial\Omega$ (its surface density is specified by a_{ijk}) plus the rate of what is "located" inside Ω (its volume density is specified by A_{ij}).

In the above formulation we use orthogonal Cartesian coordinates and we assume that the quantities (for which the balance (1.1) is written) F_{ij} and A_{ij} are the components of second - order tensors, while a_{ijk} are the components of a third - order tensor. The precise statement of the conservation law (balance equations) is:

$$\frac{D}{Dt}\left[\iiint_{\Omega} A_{ij}\,dv\right] + \iint_{\partial\Omega} a_{ijk} n_k\,ds = \iiint_{\Omega} F_{ij}\,dv, \tag{2.1}$$

if $\boldsymbol{n}$ (with the Cartesian components n_k) is the exterior normal unit vector and D/Dt the material derivative operator.

We assume, now, that our fixed domain Ω is, in fact, the sum of two subdomains, Ω_+ and Ω_-, adjacent to a common boundary G, located inside Ω, having a continuously varying tangent plane at each of its point P. Call $\boldsymbol{W}(t, \boldsymbol{x})$ the velocity field of the points of G, $\boldsymbol{N}$ the normal unit vector at P, to the surface G, pointing inside Ω_- and $\boldsymbol{U}(t, \boldsymbol{x})$, with the Cartesian components U_i, the velocity of the particle located at P at time t. In such a case,

$$V(t,x) = U - W$$

is then the relative velocity of the medium with respect to *G*.

Finally, *[f]* is the discontinuity of *f*, when crossing *G* in the direction *N*.

Now, [Germain (1972, p. 147)], the following Theorem can be proven. In any domain *D* where A_{ij}, a_{ijk}, F_{ij} and U_i are functions with piecewise continuous and bounded derivatives, which satisfy the associated PDEquation:

$$\frac{\partial}{\partial t}\left(A_{ij}\right) + \frac{\partial}{\partial x_k}\left(A_{ij}U_k\right) + \frac{\partial}{\partial x_k}\left(a_{ijk}\right) = F_{ij} \tag{2.2}$$

at any point of continuity, and

$$\left(A_{ij}v + a_{ijk}N_k\right) = 0 \text{, where } v = V \cdot N \text{ ,} \tag{2.3}$$

on every surface of discontinuity *G*.

The conservation law (2.2) is valid for every subdomain Ω of *D*, and if viscosity and heat conduction are neglected, the jump conditions for a perfect (inviscid) thermally non-conducting fluid are:

$$[\rho v] = 0,\ [pN + \rho v U] = 0, \tag{2.4a}$$

$$\left[\rho\left(E + \frac{1}{2}U^2\right) + p(v + w)\right] = 0, \tag{2.4b}$$

where $w = N \cdot W$, ρ is the density, *p* the pressure and *E* the internal energy. Thus: $m = \rho v$ is continuous across *G*, and if $\underline{m = 0}$, *G* is called a contact discontinuity, and: *p* is continuous on G, but ρ and temperature *T* may be discontinuous on *G*. When $\underline{m \neq 0}$, the surface *G* is called a shock wave and relations (2.4a, b) may be written in the following form (shock relations):

$$[\rho v] = 0,\ \left[p + \rho v^2\right] = 0,\ \left[V_T\right] = 0,\ \left[h + \frac{1}{2}v^2\right] = 0, \tag{2.5}$$

where V_T is the tangential component of the vector *V* and *h* the enthalpy.

We note that the shock relations (2.5) can be written with the relative velocity $\boldsymbol{V}$ only (as consequence of Galilean relativity - the laws of mechanics must have the same expression in any inertial frame).

2.1.1. Local differential equations in continuum theory

In this book we consider only te Newtonian fluids, and for a rigorous definition of a Newtonian fluid it is necessary to introduce two second order tensors: the rate of strain (deformation) tensor $\boldsymbol{D}$ and the stress tensor $\boldsymbol{T}$. In a rectangular Cartesian coordinates system, we have three rectangular coordinate axes and we call these axes, x_1, x_2, and x_3-**axes.** Let the velocity vector components be designated by u_1, u_2 and u_3, which are functions of x_1, x_2, x_3 and time t, $\mathbf{u}(t, \mathbf{x})$ represents the velocity vector, and the components of $\boldsymbol{x}$ are the x_i, $i = 1, 2, 3$.

By definition, the (Cartesian) components of $\boldsymbol{D}$ are:

$$d_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right], \tag{2.6}$$

and we can represent the tensor $\boldsymbol{D}$ as a symmetric matrix. By definition the quantity $\boldsymbol{T}\cdot\boldsymbol{n}$ is called the stress across $\partial\Omega$.

Indeed, in mechanics of continua [see, for example, the book of Prager (1961)], in place of the general PDEquation. (2.2), from the conservation of mass, we derive first the classical equation of continuity:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0. \tag{2.7}$$

But, by definition of the material or convective derivative:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho, \tag{2.8}$$

and as consequence

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0, \tag{2.9}$$

which is the spatial form or Eulerian form of the conservation of mass.

The Cartesian coordinates of the gradient vector ∇ are the $\partial/\partial x_i$, with $i = 1, 2, 3$.

By analogy, from the balance of momentum we derive (as the local form) the equation of motion (Cauchy's equation):

$$\rho \frac{D\boldsymbol{u}}{Dt} = \rho \boldsymbol{f} + \nabla \cdot \boldsymbol{T}, \tag{2.10}$$

where $\boldsymbol{f}$ is the body force per unit mass.

This form of the momentum equation (which is an equation of motion for $\boldsymbol{u}$) is an expression of the idea that "the product of mass and acceleration is balanced by applied forced" for a "particle". The Eulerian, or spatial form of the equation of balance of momentum is:

$$\rho\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] = \rho \boldsymbol{f} + \nabla \cdot \boldsymbol{T}. \tag{2.11}$$

Moment of momentum is also a quantity that is balanced for material bodies, and the local form of the balance of moment of momentum is:

$$\boldsymbol{T} = \boldsymbol{T}^T \tag{2.12}$$

and as a consequence the stress tensor is a symmetric tensor: $T_{ij} = T_{ji}$.

It is also necessary to consider the balance of energy for a body. The local form of the energy equation in material coordinates is

$$\rho \frac{DE}{Dt} = \boldsymbol{T} : \boldsymbol{D} - \nabla \cdot \boldsymbol{Q} + \rho r, \tag{2.13}$$

where r, is the heat supply per unit mass per unit time in the body - often the heat supply is due to radiation, and this term is often called the "radiation".

Using the material derivative leads to the energy equation in spatial coordinates:

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E \boldsymbol{u}) = -\nabla \cdot \boldsymbol{Q} + \boldsymbol{T} : \nabla \boldsymbol{u} + \rho r. \tag{2.14}$$

A third form for the energy equation can be obtained by applying the equation for the decomposition of the stress tensor (see the relation (2.18) below) into a pressure p, and a viscous shear stress $\boldsymbol{S}$, and by introducing the enthalpy, $h = E + p/\rho$. This produces the enthalpy equation:

$$\frac{\partial \rho h}{\partial t} + \nabla \cdot \rho h \boldsymbol{u} = -\nabla \cdot \boldsymbol{Q} + \frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \nabla p + \boldsymbol{S} : \nabla \boldsymbol{u} + \rho r\,, \tag{2.15}$$

Finally, it is possible also to write the energy equation with the help of entropy S, namely:

$$T\rho \frac{DS}{Dt} = -\nabla \cdot \boldsymbol{Q} + \boldsymbol{\Phi}\,, \tag{2.16}$$

where the entropy S per unit mass is a state variable whose dependence on the other state variables (T, E, p and ρ) is such that: $T\,dS = dE + p\,d(1/\rho)$, and $\boldsymbol{\Phi}$ is the dissipation function.

The classical kinetic theory of dilute monoatomic gases indicates, that the three equations of balance - those for mass (2.7), momentum (2.10) and energy (2.13) - are all that can be expected.

2.1.2. Constitutive relations and equations of state

In general, for the so-called Stokes fluid the reduced (or viscous) stress (shear) tensor

$$\boldsymbol{S} = \boldsymbol{T} + p\boldsymbol{I}\,, \tag{2.17}$$

is a function only of the deformation tensor $\boldsymbol{D}$ and we note that, when a fluid is at rest with zero rate of strain, there are normal stress components $(-p)$ which are the same in all directions. In fact, the Newtonian fluid is a particular case of a Stokes fluid when in the relation:

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}\,, \tag{2.18}$$

We have:

$$\boldsymbol{S} = 2\mu \boldsymbol{D} + \lambda(div\,\boldsymbol{u})\boldsymbol{I}\,, \tag{2.19}$$

where, $\boldsymbol{I} = (\delta_{ij})$ and $\delta_{ii} = 1$, $\delta_{ij} = 0$ for $j \neq i$.

We note that: $div\,\boldsymbol{u} = d_{kk}$ and the following relation is satisfied, when $div\,\boldsymbol{u} \neq 0$,

$$p - p^* = \left[\lambda + \frac{2}{3}\mu\right] d_{kk}, \quad (2.20a)$$

where:

$$-p^* = \frac{1}{3}Tr(\boldsymbol{T}) = \frac{1}{3}[T_{11} + T_{22} + T_{33}]; \; \boldsymbol{T} = (T_{ij}), \quad (2.20b)$$

and p^* may be interpreted as the mechanical pressure of the fluid and the term $[T_{11} + T_{22} + T_{33}]$ is the "trace" of the stress tensor $\boldsymbol{T}$. In the constitutive relation (2.19), for a Newtonian fluid, the scalar λ is called the second coefficient of viscosity and:

$$\mu_v = \lambda + \frac{2}{3}\mu$$

is called the bulk viscosity of the fluid.

For an incompressible fluid, $div\ \boldsymbol{u} = d_{kk} = 0$ and therefore the scalar p in (2.18) is simply the mechanical pressure, being either the hydrostatic pressure when the fluid is at rest or the hydrodynamic pressure when the fluid is in motion.

For a monatomic gas:

$$\mu_v = 0 \text{, hence: } \lambda = -\frac{2}{3}\mu,$$

which is known as *Stokes' relation;* so there is only one independent viscosity for a monatomic gas - furthermore, the kinetic theory of gases indicates that this Stokes relationship should hold. For polyatomic gases and for liquids the departure of μ_v from zero is frequently small and in such cases a good approximation is $p = p^*$. In any case, for liquids, it is immaterial whether μ_v is or is not zero, since $p = p^*$ always holds.

The Newtonian fluid hypothesis causes λ and μ to be independent of the rate of strain tensor and this is observed to be satisfied by a wide range of commonly encountered fluids such as water and air. It is necessary to note that the (so-called 'Lamé') viscosity coefficients λ and μ, are functions of ρ and T, in view that the Stokes (or classical) fluid is a continuum where the constitutive law for the viscous stress tensor $\boldsymbol{S}$ is in fact a function of $\boldsymbol{D}(\boldsymbol{u})$, but also of ρ and T, through the Lamé coefficients. For a Newtonian fluid we assume also that, in the energy equation (written for the internal energy E per unit mass), the heat flux (vector) per unit area of the volume fluid, $\boldsymbol{Q}$, is related to the temperature T according to Fourier's law of heat conduction:

$$\boldsymbol{Q} = -k\nabla T\,, \tag{2.21}$$

with k being the thermal conductivity of the fluid. For a Newtonian fluid, in the energy equation the term: $T_{ij}\ (\partial u_i/\partial x_j) = -\ p\ (div\ \boldsymbol{u}) + \Phi$, where the dissipation function, Φ, is a measure of the rate at which mechanical energy is being converted into thermal energy. For Φ we have the following relation:

$$\Phi = 2\mu d_{ij} d_{ji} + \lambda (div\,\boldsymbol{u})^2\,, \tag{2.22}$$

which shows it to represent the rate of work of the viscous stresses per unit volume. Since the coefficient $\lambda + (2/3)\mu = \mu_v$ of bulk viscosity, like μ, is positive, Φ is actually a positive definite form in $\boldsymbol{d}_{ij}$, as may be seen writing it out in components as:

$$\begin{aligned}\Phi = 4\mu\left((d_{12})^2 + (d_{23})^2 + (d_{31})^2\right)\\ +\left(\lambda + \frac{2}{3}\mu\right)[d_{11} + d_{22} + d_{33}]^2\\ +\frac{2}{3}\mu\left[(d_{11} - d_{22})^2 + (d_{22} - d_{33})^2 + (d_{33} - d_{11})^2\right].\end{aligned} \tag{2.23}$$

This shows explicitly which part of the dissipation is due to the off-diagonal, or shear stresses, and which to the diagonal, or normal viscous stress components.

In the above-derived three main partial differential equations (2.7), (2.10) and (2.13) we have as unknowns: pressure p, density ρ, internal energy E, the stress tensor $\boldsymbol{T}$, the heat flux vector $\boldsymbol{Q}$ and the velocity vector $\boldsymbol{u}$. The viscosity coefficients λ and μ and the thermal conductivity k are assumed to be known a priori from experimental data; they may be constants or more generally specified functions of T (and eventually of ρ). The body force $\boldsymbol{f}$ per unit mass and the the heat supply $\boldsymbol{r}$, per unit mass per unit time in the body, are also known functions of the position vector $\boldsymbol{x}$ and of the time t. For a Newtonian fluid, when we have the constitutive relations (2.19) and (2.21), the continuity equation (direct consequence of the conservation of mass) and the vectorial equation of motion (direct consequence of the conservation of linear momentum) then provide four equations for p (or ρ) and the three velocity components u_i. For E we have the energy equation (as consequence of the first law of thermodynamics since the conservation of the energy is essentially an application of this law to an element of fluid in motion). For the derivation of a closed system of equations (the so-called Navier-Stokes-Fourier equations) for our seven unknowns (p, ρ, E, T and $\boldsymbol{u} = (u_i)$) it is necessary to add two equations of state. In the present book we consider mainly the perfect gas (air) and the expansible liquid (water). For a perfect gas the two equations of state are:

$$p = R\rho T \text{ and } E = C_v T, \tag{2.24}$$

$R = C_p - C_v$ being a constant while C_v and C_p are the specific heats at constant volume and constant pressure. On the other hand, γ denotes the ratio C_p/C_v.

For an expansible homogeneous liquid we assume as equations of state:

$$\rho = \rho(T) \text{ and } E = E(T), \tag{2.25}$$

and in this case as specific heat we have:

$$\frac{dE}{dT} = C(T), \tag{2.26}$$

and obviously: $\lambda = \lambda(T)$, $\mu = \mu(T)$ and $k = k(T)$. It must be noticed that if the pressure p and the coefficients μ and μ_v $(= \lambda + (2/3)\mu)$ do not depend of the temperature T, then in this case the equation of state is simply:

$$p = P(\rho), \tag{2.27}$$

and we will refer to this situation as the *barotropic* case.

But, for a perfect gas (with constant specific heats C_p and C_v) we can rewrite the equation of state ($p = R\rho T$) in the following form:

$$\frac{p}{\rho^\gamma} = exp\left(\frac{S}{C_v}\right), \text{ with } \gamma = \frac{C_p}{C_v}, \tag{2.28}$$

if we introduce the corresponding (specific) entropy S for the perfect gas. Now, if the entropy is constant in the flow motion: $S = S_0 = const$, then in place of (2.28) we obtain an isentropic fluid flow with:

$$p = \kappa_0 \rho^\gamma, \tag{2.29}$$

where $\kappa_0 = exp(S_0/C_v)$.

2.1.3. NS-F equations

First, if we take into account the relations (2.6), (2.18), (2.19), (2.21) and (2.24), then for a *thermally perfect gas* we derive from the above equations (2.10) and (2.13) a system of four equations for six unknows p, ρ, T and u_i, namely:

$$\rho\frac{Du_i}{Dt}+\frac{\partial p}{\partial x_i}=\rho f_i+\frac{\partial}{\partial x_i}\left[\lambda\frac{\partial u_k}{\partial x_k}\right]$$
$$+\frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right], \tag{2.30a}$$

$$\rho C_p\frac{DT}{Dt}=\frac{Dp}{Dt}+\Phi+\frac{\partial}{\partial x_i}\left[k\frac{\partial T}{\partial x_i}\right]+\rho r, \tag{2.30b}$$

where C_p is the specific heat at constant pressure and for Φ we have the relation (2.23) with (2.6). Now, if we add to above equations (2.30a, b) the equation of continuity (2.7), written in the indecial form:

$$\frac{D\rho}{Dt}+\rho\frac{\partial u_k}{\partial x_k}=0, \tag{2.30c}$$

and the equation of state (the first of equations (2.24))

$$p = R\rho T, \tag{2.30d}$$

we obtain for a perfect gas with constant specific heats, the NS-F equations, which are a closed system for the unknows: u_i $(i = 1, 2, 3)$, the three Cartesian components of the velocity vector $\boldsymbol{u}$, the pressure p, the density ρ and the temperature T.

2.2. INITIAL AND VARIOUS BOUNDARY CONDITIONS

2.2.1. Initial conditions

In the above NS-F equations (2.30a) - (2.30d) we have three time-derivatives for the velocity vector $\boldsymbol{u}$, density ρ and temperature T, since the pressure $p = R\rho T$. These equations are, in fact, evolution equations. As consequence, if we want to resolve a Cauchy initial data problem, it is necessary to have a complete set of initial conditions for $\boldsymbol{u}$, ρ and T:

$$t = 0: \ \boldsymbol{u} = \boldsymbol{u}^0(\boldsymbol{x}), \quad \rho = \rho^0(\boldsymbol{x}), \quad T = T^0(\boldsymbol{x}), \tag{2.31}$$

where $\rho^0(\boldsymbol{x}) > 0$ and $T^0(\boldsymbol{x}) > 0$. Moreover, when considering the free - surface problem (as in §6.5 of Chapter 6, or in §10.1 of Chapter 10) an initial condition for the (moving) boundary $\partial\Omega(t)$ has to be specified.

2.2.2. Boundary conditions

Several boundary conditions could be considered with respect to different physical situations. If we consider, as simple example, the motion of the fluid in a rigid container Ω (with $\partial\Omega$ independent of time t), a bounded connected open subset of $\boldsymbol{R}^d$ (where $d \geq 1$ is the physical dimension), the different structure of the equations leads to the necessity of distinguishing between viscous and inviscid fluids.

Viscous fluids: $\mu > 0$ and $\mu_v > 0$

In this case, the physical effects due to the presence of the shear (dynamic) viscosity coefficient μ yield the validity of the no-slip steady condition:

$$\boldsymbol{u} = 0 \ \text{ on } \partial\Omega. \tag{2.32}$$

Bulk - viscous fluids: $\mu = 0$ and $\mu_v > 0$

Since only the bulk viscosity coefficient $\mu_v (\equiv \lambda)$ is different from zero, in this situation the slip steady boundary condition:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \ \text{ on } \partial\Omega \tag{2.33}$$

is assumed, where $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ denotes the unit outward normal vector to $\partial\Omega$.

Inviscid fluids: $\mu = 0$ and $\mu_v = 0$

Also in this case the steady slip boundary condition (2.33) is assumed.

Concerning the (absolute) temperature T, the boundary condition takes a different form in the two alternative cases: $k > 0$ and $k = 0$.

Conductive fluids: $k > 0$

Several boundary conditions have a physical meaning. Just to limit ourselves to the most common cases, we can require:

$$T = T_w \text{ on } \partial\Omega \text{ (Dirichlet),} \tag{2.34}$$

or

$$k\frac{\partial T}{\partial n} = \Theta \text{ on } \partial\Omega \text{ (Neumann),} \tag{2.35}$$

or else

$$k\frac{\partial T}{\partial n} + h(T - T_o) = \Theta \text{ on } \partial\Omega \text{ (Third type),} \tag{2.36}$$

where $T_w > 0$ and Θ are known functions and $h > 0$ is a given constant.
We can also, in place of (2.34)-(2.36), write the following condition for the temperature T:

$$\text{on } \partial\Omega\text{: } T = T_c + \Delta T_0\, \Xi(t, P), P \in \partial\Omega, \tag{2.37}$$

where ΔT_0 is a temperature fluctuation associated with the function $\Xi(t, P)$ which is assumed to be a given temperature field which governs the temperature distribution at the surface $\partial\Omega$ and T_c is a reference constant temperature.

Non conductive fluids: $k = 0$

No boundary condition has to be imposed on temperature T, if (2.32) or (2.33) are satisfied, since in these cases the temperature is not subject to conductive transport phenomena through the boundary $\partial\Omega$.

2.2.3. Conditions at infinity

When the domain of fluid motion is infinite (unbounded) the following simple conditions (in various cases) have to be added for stationary flow past a finite body (exterior problem):

$$\boldsymbol{u}(\infty)=\boldsymbol{U}_\infty,\ p(\infty)=p_\infty,\ \rho(\infty)=\rho_\infty,\ T(\infty)=T_\infty, \tag{2.38}$$

where $\boldsymbol{U}_\infty$, p_∞, ρ_∞ and T_∞ are all constants.

2.2.4. Other types of boundary conditions

We will mainly focus on the velocity field and the density, since the conditions (2.34)-(2.37) are general enough for the absolute temperature T.

First of all, in many situations (inflow-outflow problems) the velocity cannot be assumed to vanish on $\partial\Omega$. This is the case, for instance, for the flow around an aerofoil, where an inflow region is naturally present upstream (and an outflow region appears in the wake), or the flow near a rigid body, where the velocity can be assumed to vanish only on the boundary of the body. In these cases, several different boundary conditions may be prescribed.

Let us start by considering the viscous case. Concerning the velocity field, a non-zero Dirichlet boundary condition can be imposed everywhere, or, alternatively, only in the inflow region: i.e., the subset of $\partial\Omega$ where:

$$\boldsymbol{u}\cdot\boldsymbol{n}<0, \tag{2.39}$$

whereas, in the remaining part of the boundary, the conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n}=U^+\geq 0 \text{ and } (\boldsymbol{n}\cdot\boldsymbol{D})\cdot\boldsymbol{t}=0, \tag{2.40}$$

have to be prescribed [here $\boldsymbol{t}$ is a unit tangent on $\partial\Omega$ and $\boldsymbol{D}$ is the strain tensor].

Let us moreover remark that the conditions:

$$\boldsymbol{u}\cdot\boldsymbol{n}=0 \text{ and } (\boldsymbol{n}\cdot\boldsymbol{D})\cdot\boldsymbol{t}=0, \tag{2.41}$$

could also be considered on the whole of $\partial\Omega$; in this case however, no inflow or outflow regions would be present. More important is to analyse the boundary conditions for the density ρ, since now it turns out that it is necessary to prescribe it on the inflow region. In fact, the first order hyperbolic equation of continuity (2.9) can be solved by means of the theory of characteristics, and the boundary datum for ρ on the inflow region is indeed a (necessary) Cauchy datum for the density on a non-characteristic surface.

Let us notice that if the heat conductivity coefficient k is vanishing, the same type of Dirichlet - inflow boundary condition has to be imposed on the temperature T, since in that case also equation (2.30b) is of hyperbolic type for T (the terms Φ and Dp/Dt being assumed known).

More complicated is the situation when the inviscid (non-viscous) case (λ, μ and k vanishing) is considered, In fact, in this case the NS-F system (2.30a, b, c), with (2.30d), is, in fact, a first order (Euler) hyperbolic one, and the number of boundary conditions is different:

If the flow is *subsonic* ($|\boldsymbol{u}| < a$) or *supersonic* ($|\boldsymbol{u}| > a$), where $a = [\gamma RT]^{1/2}$ is the *local sound speed* for a perfect inviscid gas.

Take, for example, $d = 3$, then an analysis of the sign of the eigenvalues of the associated characteristic matrix yields the conclusion that the number of boundary conditions must be five or four on an inflow boundary, depending if the flow is supersonic or subsonic, and zero or one on an outflow boundary, again depending if the flow is supersonic or subsonic. Further information on inflow-outflow boundary-value problems for compressible NS-F and inviscid (Euler) equations can be found in two pertinent papers due to Gustafsson and Sundström (1978) and Oliger and Sundström(1978).

Another interesting set of boundary conditions appears when we consider the free-boundary problem, i.e., a problem for which the fluid is not contained in a given domain but can move freely. In this case the vector:

$$\boldsymbol{n} \cdot \boldsymbol{T} \quad \text{is prescribed on (interface) } S, \tag{2.42}$$

where $\boldsymbol{T}$ is the stress tensor; moreover $\boldsymbol{u} \cdot \boldsymbol{n}$ is required to be zero (stationary case) or equal to the normal velocity of the interface itself (unsteady case).

The value of $\boldsymbol{n} \cdot \boldsymbol{T}$ can be zero (free expansion of a fluid in the vacuum) or equal to:

$$-p_e\boldsymbol{n} + 2\sigma K\boldsymbol{n} + \nabla_s\sigma, \text{ on interface } S, \tag{2.43}$$

where p_e is the external pressure, σ is the surface tension (which depends on temperature when the fluid is an expansible liquid; see, for instance, the § 10.1 of Chapter 10), K is the mean interface curvature and the operator: $\nabla_s = \nabla - \boldsymbol{n}\,(\boldsymbol{n} \cdot \nabla)$, is the surface (projected) gradient at the interface.

But, in this case it is necessary to write also a heat transfer condition across the interface:

$$k_s \frac{\partial T}{\partial n} + h_s T = prescribed\, function \tag{2.44}$$

where the heat-transfer (constant) coefficient h_s is sometimes called Biot coefficient. Finally, we assume also the continuity of temperature across the interface separating the region occupied by the fluid from the exterior region. Naturally, we are now imposing one more condition on the interface *S*, since it is an unknown of the problem; in the unsteady case, an initial condition for the interface has to be added too.

It is also important to note that if $\boldsymbol{U}_w\,(t,\, \boldsymbol{P})$ is the velocity of the point of the solid body in motion, then the relative velocity, $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{U}_w$, where $\boldsymbol{u}$ is the fluid velocity vector, satisfies, for an impermeable solid moving wall $\partial\Omega$ the general slip condition:

$$\boldsymbol{n} \cdot \boldsymbol{v} \equiv \boldsymbol{n} \cdot (\boldsymbol{u} - \boldsymbol{U}_w) = 0 \text{ on } \partial\Omega, \tag{2.45}$$

and on the other hand, from the kinetic theory of gases, we can write on the moving wall the following condition, when the Knudsen number, $Kn = M/Re$, tends to zero (see, the §2.3 below for the definition of the Mach number M and the Reynolds number Re):

$$\boldsymbol{n} \wedge (\boldsymbol{u} - \boldsymbol{U}_w) = 0 \text{ on } \partial\Omega. \tag{2.46}$$

As consequence of the above two conditions (2.45) and (2.46), we deduce the no-slip condition for a moving solid impermeable wall, namely:

$$\boldsymbol{u} - \boldsymbol{U}_w = 0 \text{ on } \partial\Omega, \tag{2.47}$$

which is the so-called weak form of the no-slip condition on the moving wall.

Concerning the temperature T on the wall, again from the kinetic theory of gases and when the Knudsen number $Kn \downarrow 0$, we have the following boundary condition on the wall:

$$T = T_w - \alpha(\boldsymbol{Q} \cdot \boldsymbol{n}), \tag{2.48}$$

where T_w is the reference (adiabatic) temperature and α a constant. This condition (2.48) is very similar to condition (2.35) or (2.44), if we take into account Fourier's Law (2.21).

In the sequel we will comment again one some of these alternative sets of boundary conditions, but for the moment we stop our discussion concerning the boundary conditions and consider below, in §2.3, an important step in asymptotic modelling, namely: the nondimensional form of the NS-F (2.30a)-(2.30c) equations, with (2.30d), and the emergence of the main dimensionless parameters.

2.3. DIMENSIONLESS PARAMETERS AND THE NONDIMENSIONAL FORM OF THE NS-F EQUATIONS AND BOUNDARY CONDITIONS

2.3.1. Dimensionless parameters

Our analysis which follows will be mainly formal, resting on limiting processes and asymptotic expansions applied to the NS-F equations. This requires that all is, at the outset, written in dimensionless form. For a perfect viscous and thermally conducting gas, in the basic NS-F equations (2.30a)-(2.30d) appears the following main dimensionless parameters:

$$Re = U_c L_c/(\mu_c/\rho_c);\ M = U_c/a_c;\ \ S = L_c/t_c U_c;\ Pr = C_p\mu_c/k_c, \tag{2.49}$$

which are well known and are, according to the order of the writing: Reynolds, Mach, Strouhal, and Prandtl numbers. Except for the constant specific heats C_p and C_v and also $R = C_p - C_v$, all the quantities occurring in (2.49) are indexed by stands "*c*" which holds for "characteristic value" of the indexed quantity.

If the body force $\boldsymbol{f}$ (per unit mass) is assimiled with the gravity force (as in meteorological problems) then we have also the following parameter (a so-called 'Boussinesq number', according to Zeytounian (1990)):

$$Bo = \frac{gL_c}{RT_c}, \tag{2.50}$$

where g, is the magnitude of the acceleration due to gravity. Indeed, we have the following relation

$$Bo = \frac{\varepsilon\gamma M^2}{Fr_{Hc}} \tag{2.51}$$

where Fr_{Hc} is the Froude number based on the vertical length H_c:

$$Fr_{Hc} = \frac{U_c}{(gH_c)^{1/2}}, \tag{2.52}$$

and in this case

$$\varepsilon = \frac{L_c}{H_c}, \tag{2.53}$$

is the 'long wave or hydrostatic' parameter.

In the case of atmospheric motions it is necessary to take into account also the so-called Rossby number, which characterizes the effect of the earth's rotation on the atmospheric motions:

$$Ro = \frac{1}{f^\circ}\frac{U_c}{L_c}, \tag{2.54}$$

with $f^\circ = 2\Omega^\circ \sin\phi^\circ$, the constant Coriolis parameter, where Ω° is the magnitude of the vector of the earth's rotation and ϕ° a constant value of the algebraic latitude. In the Chapter 11, in place of *Ro*, we consider the so-called Kibel number:

$$Ki = S\,Ro = \frac{1}{f^\circ t_c}, \tag{2.55}$$

Now, we precise that all space coordinates are made dimensionless by dividing their true measure by the (horizontal) length L_c, while t_c replaces L_c when time is considered. Any material velocity, in the fluid flow, is made dimensionless by dividing its measure by U_c. The reader should consider that *a*, the local speed of sound, rendered dimensionless through the characteristic value a_c, is not a material velocity, in the above mentioned meaning.

Four other characteristic quantities appear in (2.49), namely ρ_c for the density, T_c for the temperature, μ_c for both coefficients of viscosity and k_c for the coefficient of thermal conduction. Below, we conserve (by simplicity) for the dimensionless quantities the former notations.

2.3.2. Dimensionless aerodynamical NS-F equations

Part of the discussion that follows rests on the dimensionless aerodynamical NS-F equations for a viscous, thermally conducting perfect gas:

$$\left[S\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right]\rho+\rho\nabla\cdot\boldsymbol{u}=0; \tag{2.56a}$$

$$\rho\left[S\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right]\boldsymbol{u}+\left(\frac{1}{\gamma M^2}\right)\nabla p+\frac{Bo}{\gamma M^2}\rho\boldsymbol{k}$$
$$=\frac{1}{Re}\nabla\cdot\left[2\mu\boldsymbol{D}+\lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{I}\right] \tag{2.56b}$$

$$\rho\left[S\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla\right]T+(\gamma-1)p\nabla\cdot\boldsymbol{u}=\frac{\gamma}{PrRe}\nabla\cdot[k\nabla T]$$
$$+\frac{\gamma M^2(\gamma-1)}{Re}\left[2\mu Tr(\boldsymbol{D}^2)+\lambda(\nabla\cdot\boldsymbol{u})^2\right] \tag{2.56c}$$

which form a *closed set* of evolution equations, provided one adds the dimensionless equation of state for a perfect gas:

$$p=\rho T, \tag{2.56d}$$

and an expression for the way in which both coefficients of shear viscosity $\mu(T)$ and bulk viscosity, $\mu_v(T) = \lambda(T) + (2/3)\mu(T)$, depend on the temperature T, as might be obtained from the kinetic theory of dilute, non-dissociating, gases.

For the pressure, the characteristic value is p_c and we remind the reader that the true equation of state reads, for characteristic values, $p_c = R\rho_c T_c$, while the characteristic speed of sound satisfies: $(a_c)^2 = \gamma p_c/\rho_c$.

We go on (again) with notations: $\boldsymbol{u}$ is the dimensionless material velocity, and, from it, we define the rate of strain tensor, $\boldsymbol{D} = (1/2)[(\nabla\boldsymbol{u}) + (\nabla\boldsymbol{u})^T]$, where the superscript '^{T}' stands for the transpose of a second rank tensor. We adopt the convention that, when writing $(\nabla\boldsymbol{u})$, the first index (in component notation) refers to ∇ while the second one refers to $\boldsymbol{u}$. A dot $(\cdot)$ stands for the Euclidian product and the $Tr(\cdot)$ for the *trace operator*. Let us add that $\boldsymbol{x}$ is the dimensionless vector of spacial coordinates and that ∇ is the dimensionless gradient with respect to $\boldsymbol{x}$, both rendered dimensionless through appropriate use of L_c.

2.3.3. Other dimensionless parameters

First, if we consider the temperature boundary condition (2.37), then the following temperature parameter appears, namely:

$$\tau = \frac{\Delta T_c}{T_c}, \tag{2.57a}$$

In this case, in place of (2.37) we write the following dimensionless boundary condition for the temperature T:

$$T = 1 + \tau \Xi(t, \boldsymbol{P}), \boldsymbol{P} \in \partial\Omega, \tag{2.57b}$$

Next, when we consider fluid flow around a profile which has a characteristic horizontal length l_c and a vertical characteristic length h_c, then in the steady two- dimensional case we write the following dimensionless equation for the profile:

$$y = \delta h(\alpha x), \tag{2.58a}$$

where

$$\delta = \frac{h_c}{H_c} \text{ and } \alpha = \frac{L_c}{l_c}, \tag{2.58b}$$

and usually in gas-dynamics problems (see §6.7 and §6.8 of Chapter 6) the dimensionless parameter $\delta(<< 1)$ is a small parameter. Such a small parameter (denoted by $\eta << 1$) appears also in §6.3 of Chapter 6 in the linearization method. For instance, in the water-wave theory (see §6.5) a classical solution for the free surface is

$$\varsigma = a^\circ sech^2\left[\frac{(x-ct)}{l^\circ}\right], \tag{2.59a}$$

and, in fact,

$$\frac{a^\circ}{h^\circ} = \varepsilon << 1 \text{ and } \left(\frac{h^\circ}{l^\circ}\right)^2 = \Delta^2 = O(\varepsilon), \tag{2.59b}$$

for a flat bottom simulated by the equation: $z = - h^\circ = const.$ In (2.59a), a° is the characteristic amplitude for the initial elevation of a free surface characterized by the function $\zeta^\circ(x/l^\circ)$ and l° is the characteristic wavelength in the horizontal x-direction. In the water wave theory l° is determined by the Ursell (1953) criterion, which is, in fact, a similarity relation between the two small parameters ε and Δ:

$$Ur = \frac{3\varepsilon}{4\Delta^2} = 1, \tag{2.60}$$

In the turbo-machinery flow (considered in §6.6), the asymptotic theory for an axial compressor relies on inviscid, incompressible ($\rho = const \Rightarrow \nabla \cdot \boldsymbol{u} = 0$) flow equations and slip boundary conditions on the blades and is derived from the assumption that the number of blades per row, N, is much greather than one. The thickness of a row divided by the length of the axial compressor it is assumed to be small, as is the ratio of the mean blade to blade distance divided by the mean diameter of the row. The basic assumption is that the ratio of the mean blade to blade distance divided by the thickness of a row, which is related to the pitch of the cascade configuration, is near one or smaller and this means that one small parameter:

$$\frac{1}{N} = \kappa << 1, \tag{2.61}$$

controls the limiting flow. On the other hand, the so-called 'actuator-disk' theory is derived when the pitch of the cascade configuration is near unity while the ratio of the distance between two rows (fixed-rotating) to the thickness of the row is large. In fact, when the ratio of the mean blade to blade distance divided by the thickness of a row, which is related to the pitch of cascade configuration, is small, we approach the through-flow theory, taking account of the forces exerted on the fluid by the blades - this the main situation that we shall consider in §6.6 of Chapter 6.

In nonlinear acoustics theory (see Chapter 8) the derivation of Burgers equation (in §8.1) is related with the high Strouhal and Reynolds numbers when we consider low Mach number flow, such that:

$$S\,M = 1 \text{ and } M\,Re = Re^* = O(1), \tag{2.62}$$

where Re^* is a similitude parameter. When $1/Re^*$ tends to zero, then the dissipative coefficient (so-called the "Stokes number") in the Burgers equation:

$$\nu^{\circ} = \frac{1}{2}\frac{1}{Re^*}\left[\left(\frac{4}{3} + \frac{\mu_v{}^{\circ}}{\mu^{\circ}}\right) + \frac{(\gamma - 1)}{Pr}\right], \tag{2.63}$$

tends also to zero.

Finally, in the free-surface problem for a thin film layer (the so-called 'Bénard-Marangoni' problem, considered in §10.4 of Chapter 10) we have the competition between, Weber (We), Biot (Bi) and Marangoni (Ma) numbers:

$$We = \frac{\sigma^{\circ} h^{\circ}}{\rho^{\circ} \nu^{\circ 2}}; \; Bi = \frac{h_s h^{\circ}}{k^{\circ}}; \; Ma = \frac{s^{\circ} h^{\circ} \Delta T^{\circ}}{\rho^{\circ} \nu^{\circ 2}}, \tag{2.64}$$

where σ° is a constant reference value (at T°) of the surface tension $\sigma(T)$, h° is the thickness of the film layer,

$$s^{\circ} = -\left.\frac{d\sigma(T)}{dT}\right|_{T^{\circ}}, \; \nu^{\circ} = \frac{\mu^{\circ}}{\rho^{\circ}} \text{ and } T_w = T^{\circ} + \Delta T^{\circ}$$

is the temperature of the lower plane in the framework of the Bénard thermal convection problem (see §10.1 in Chapter 10); μ°, ρ° and k° are the value of μ, ρ and k, at $T = T^{\circ}$. In Zeytounian (1998) the reader can find a review of the problems encountered in the Bénard-Marangoni thermocapillary-instability.

CHAPTER 3

SOME BASIC ASPECTS OF ASYMPTOTIC ANALYSIS AND MODELLING

3.1. THE CONCEPT OF ASYMPTOTIC MODELLING AND FLUID FLOW PROBLEMS

3.1.1. Why asymptotic?

Actually, the word "asymptotics" is often used in place of "asymptotic methods or analysis". Indeed, Asymptotics (as "Numerics") plays an important role in various areas of scientific activities and is related to various technological problems. In Guiraud (1995, pp. 257-262) paper, the reader can find some considerations concerning the place of asymptotics in scientific activities. In particular, concerning Numerics as a substitute for Asymptotics, I think (as does J. P. Guiraud) that one should not consider them as opposing, bur rather, as complimentary tools for attacking either different problems or different aspects of the same problem. Often, the complete solution of quite a complex problem begins with asymptotics and ends with a numerical solution to some mathematical model issuing from, previously worked out, asymptotic analysis (or more precisely 'asymptotic modelling').

Personally, I prefer to speak about *asymptotic modelling rather than asymptotics*!

3.1.2. Asymptotic modelling!

The main goal of asymptotic modelling is to derive a simplified (approximate) set of equations, but also, initial and boundary conditions, which can be solved with less numerical effort (as a "model" problem) than the original full NS-F equations, with the corresponding initial and boundary conditions, related to a specific physical problem.

In this way, my purpose, in the present book, is to initiate a process which does not seem to have sufficiently attracted the attention of scientists.

This process involves the use of methods of formal asymptotic analysis for building approximate models based on various complex physical situations.

We do not, of course, assert that this is the only way, or even the most efficient one, for deriving such models.

We do, however, feel that when such a procedure is feasible it should be undertaken. As a matter of fact, the application of this approach implies that the approximate, asymptotic-limit, model is associated with an asymptotic expansion procedure which, in principle, makes it possible to improve the approximation obtained with the model used by progressing through the hierarchy of approximations - going to higher-order terms in the asymptotic expansion. This is the rational and consistent basis of asymptotic modelling. It must be kept in mind that, at the present time, modelling, i.e., the translation of a complex physical situation into correctly expressed mathematics, has become very important for numericists who are confronted with various technological problems. For this, it seem obvious that an improvement in numerical simulation of fluid flows depends largely on obtaining more 'efficient' models and not only on the development of numerical techniques of analysis and calculation as is thought by certain specialists in the field of numerical simulation.

We feel, in particular, that developing accurate asymptotic models, no longer of first order but of second order (for example, in low Mach number fluid flows), as well as taking into account more systematically (by matching) various uniformly valid asymptotic representations, should provide us with rational models for the numerical simulation of complex fluid flow problems.

I insist that the main goal is modelling:

A scientific activity which consists of deriving technological fluid model problems in such a rational way that they become amenable to mathematical analysis and to numerical simulation.

Very often asymptotic modelling will be quite closely tied to the preparation of the numerical simulation, especially when the computation involves simultaneously dominant and negligible effects in so-called "stiff problems".

In the survey paper by Guiraud and Zeytounian (1986a) the reader can find a discussion concerning our (with J.-P. Guiraud) interpretation of the role of asymptotic modelling in research on Newtonian fluid flows.

3.1.3. Asymptotic modelling and fluid flow problems

Asymptotic modelling necessitates that one or several parameters occur in the mathematical formulation of the fluid flow problem issuing from the physical phenomenon to be modeled. The simplest situation is when the small (or large) parameter is directly built into the mathematical model, either in the equations or in the boundary conditions. The obvious example is the Reynolds number which occurs in the dimensionless NS-F equations and which leads to the inviscid and boundary-layer models when it is large or to the Stokes and Oseen models when it is small. A second example illustrates the occurrence of the small parameter in the boundary conditions, it concerns high aspect ratio wings, the small parameter being the inverse of the aspect ratio, and it occurs in the model when one goes into the details of writing the no-slip condition on the wing. A more subtle occurrence of small parameters concerns the fluid flow in thin films and in Chapter 10 the reader can find some examples of modelling in the long-wave approximation.

The second type of situation corresponds to the case when the small parameter is built into the particular solution one looks for without being directly apparent into the formulation of the problem. This is the case in the theory of small amplitude waves of any kind, surface waves for example. Here the small parameter is a ratio of lengths, namely the amplitude divided by the wavelength. Of course the parameter may be built into the model, in order to carry over expansions, but it is basically a property of the solution considered, rather than a parameter existing beforehand. Another situation occurs when two models are considered for the same physical phenomenon and the coupling between the two models involves a small parameter. A very broad field of application of the idea of asymptotic modelling may be included under this heading and in the paper by Guiraud and Zeytounian (1986a) the reader can find a discussion concerning various examples.

3.1.4. General main models

My thesis is that, very often, various Chapters in a Fluid Dynamics Course may be organised through models which are best obtained by asymptotic modelling. Below we give a number of examples but we make no reference to the literature because such reference is both unnecessary and arbitrary.

Inviscid Euler flows which are often considered as models, used from the outset, need to be embedded in the more general main model of *slightly viscous* (laminar) or *slightly frictional* (turbulent) flow, to which asymptotic modelling is applied.

Creeping flows, with numerous applications to thin films (lubrication), microhydrodynamics,... should be considered as flows at *low Reynolds* *number.* Entire books are devoted to creeping flows, the role of the low Reynolds number as main small parameter being ignored except in the very few expository pages.

Incompressible flows are seldom considered as flows at *small Mach number.* This can become almost nonsensical as when one deals with incompressible aerodynamics.

Such phenomena as *sound* produced by quite low speed flow cannot be understood other than as *low Mach* number (*hyposonic*) aerodynamics. *Environmental flows* which dominate the various recent applications to terrestrial phenomena are typically low Mach number flows.

Rapidly rotating flows which dominate applications of both industrial and geophysical nature, are indeed asymptotic models of flow at *low Rossby* (or *Kibel*) number.

Large-scale models of flows, of current use in simulations of *meteorological* or *oceanographic* applications, are extrated from asymptotic modelling which explains the role of the *hydrostatic balance.*
A number of models, for flow in porous media or flow with *suspensions* or *turbulent* flow should be considered as models obtained through some kind of *homogenization.*

3.1.5. Local models

A number of models used for the understanding of flows issue from asymptotic modelling applied in order to *elucidate the behavior* in some *localized region.* The examples are numerous and dispersed widely in all the fields of fluid dynamics. I list a few of these, again without reference to any literature.

In *wing theory,* the flow in the *vicinity* of either the *leading,* the *trailing,* the *side* or the *round-edge* of the *planform* may be considered as understandable only on the grounds of asymptotic modelling.

Laminar separation, a challenge to fluid mechanicists for seven decades, has been understood only recently as an application of the *triple deck* (with Sychev's proposal) asymptotic model.

Interfaces endowed with material properties or simply too thick to be considered as pure discontinuities are best understood when considered as *thin layers embedded within a large-scale flow.*

Tightly wound rolled vortex sheets may be grasped, independently of the question of their persistence against disintegration, by an application of *multiple scaling.*

The so-called *Riemann problem* in gas dynamics, especially when used in a numerical scheme like *Godounov*'s, is actually an asymptotic model for small increases in time, after a definite time, where some information is given, including more or less accurately a departure from smoothness.

Flows containing *particles in suspension,* and their embedding in a large-scale flow, are to be considered as local asymptotic models.

Cores of vortex filaments and derivations of algorithms for the motion of the filaments are asymptotic models

I end this list with *entrance flows* and *subgrid scale modeling of turbulence* this last one being indeed an open field of research.

3.1.6. Global specific models

It occurs sometimes that the flow under consideration, as a whole, may be grasped via asymptotic modeling. I give a few examples.

Slender bodies and the flows over them correspond to asymptotic models which have been or are widely used in aerodynamics or in ship hydrodynamics.

High aspect ratio wings or *propellers* give quite a useful application of the concept of asymptotic modelling capable of grasping the whole of a flow.

The *flow within a rotating engine (turbomachinery flows)* may be reduced to gluing together, via some asymptotic machinery, two apparently widely different models, namely *cascade flow* and *through-flow* theories.

Even the so-called *actuator disc* model may be considered as part of the whole.

Thin film coating of a wire by a very viscous liquid may be attacked, as a whole, through asymptotic modeling.

Flow which occurs within a *closed cavity,* the walls of which deform with motion very slowly in comparison to the speed of sound (*low Mach number internal flow*) is a very interesting singular asymptotic problem related to acoustic phenomena.

In this case we see the *impossibility of matching an approximation of the starting process with the approximation resulting from the* (*outer*) *classical approach, as a consequence of the persistence of acoustic oscillations in the deformable cavity* and the application of a Multiple Scale Method (MSM) is necessary, with an "homogenization" procedure.

3.2. METHOD OF MATCHED ASYMPTOTIC EXPANSIONS (MMAE) AND MATCHING

The MMAE achieves uniform validity by supplementing the regular (main) perturbation expansion, which is now called the *outer expansion,* with an (local) *inner expansion* in which the independent variable is stretched out such that it describes the behaviour in the region where the main (outer) expansion breaks down. A *uniformly valid* solution is finally derived by *matching* the outer and the inner expansions according (for example) to Van Dyke's (1975, Chapter V) or some other matching principle. Formally, this means that:

The outer solution, when expressed in the inner variables and expanded for small ε*, must agree with the inner solution, when expressed in the outer variables and expanded for small* ε*.*

In fact, we impose a "limit matching rule" and this rule states that:

The asymptotic behaviour of the outer solution when extended into the inner region must be the same as that of the inner solution when it is extended into the outer region.

More precisely, in any perturbation problem, namely:

$$E\left(\frac{\partial}{\partial x}, y(x), \varepsilon\right)=0,\ C(y(x),\varepsilon)=0 \text{ on } \partial D, \tag{3.1}$$

involving a small positive parameter $\varepsilon << 1$ it is natural to seek an approximate solution of the form:

$$y^{\circ}(x) \approx a_0(x)\delta_0(\varepsilon)+a_1(x)\delta_1(\varepsilon)+a_2(x)\delta_2(\varepsilon)+\dots,$$
$$\text{as } \varepsilon \to 0, \text{ with } x \text{ fixed,} \tag{3.2}$$

where x ranges over some (usually bounded) domain D and $[\delta_j(\varepsilon)]$ is an asymptotic sequence, often the power sequence $[\varepsilon^j]$, which tends to zero as $\varepsilon \to 0$ cannot be valid uniformly in x. For example, this approximate solution may fail to satisfy all boundary conditions (moreover, in applications, physical considerations will often indicate which boundary conditions are so omitted). However, the expansion (3.2), as: $\varepsilon \to 0$, *with* x *fixed,* will generally be satisfactory in the "outer region" away from (part of) the boundary ∂D of D. It will be called an *outer asymptotic expansion* (or outer approximate solution). Now, in order to investigate regions of nonuniform convergence, one introduce one or more stretching transformations, namely :

$$\xi = \phi(x;\ \varepsilon) \tag{3.3}$$

which "blow up" a region of nonuniformity (near a part of ∂D with neglected boundary conditions, for example). Thus

$$\xi = \frac{x}{\varepsilon^{\alpha}},\ \alpha > 0, \tag{3.4}$$

might be used for non-uniform convergence at $x = 0$. Then: if ξ is fixed and $\varepsilon \to 0$: $x \to 0$, while if $x > 0$ is fixed and $\varepsilon \to 0$, then: $\xi \to \infty$.

Selection of correct stretching transformations is an art sometimes motivated by physical considerations, or mathematically. In terms of the stretched variable ξ, one might seek an asymptotic solution of the form:

$$y^{i}(\xi) \approx b_0(\xi)\lambda_0(\varepsilon) + b_1(\xi)\lambda_1(\varepsilon) + b_2(\xi)\lambda_2(\varepsilon) + \dots,$$
$$\text{as } \varepsilon \to 0, \text{ with } \xi \text{ fixed}, \tag{3.5}$$

where the sequence $[\lambda_j(\varepsilon)]$ is asymptotic as $\varepsilon \to 0$ and valid for values of ξ in some "inner region". This will be called an *inner asymptotic expansion* and this inner expansion often accounts for boundary conditions neglected by the outer expansion. The inner region will generally shrink completely as $\varepsilon \to 0$, when expressed in terms of the outer variable x. Hence, the inner expansion is a local one. In most problem, it is impossible to determine both the outer and the inner expansions $y^{\circ}(x)$ and $y^{i}(\xi)$ completely by straightforward expansion procedures.

As a consequence, it is necessary to match them!

3.2.1. Matching

Since both expansions should represent the solution of the original problem (3.1) asymptotically in different regions, one might attempt *to match* them, i.e., to formally relate the outer expansion $y^{\circ}(x)$ in the inner region: $[y^{\circ}(x)]^{i}$ and the inner expansion $y^{i}(\xi)$ in the outer region: $[y^{i}(\phi(x;\ \varepsilon))]^{\circ}$, through use of the stretching (3.3), as $\varepsilon \to 0$. It is important to note that (fundamental ideas enunciated by Saul Kaplun in the mid-fifties):

Matching is possible only when the relevant expansions have a domain of overlap and matching is thus in essence intermediate matching.

Below, if we have in view (3.4), then we can consider only one power sequence $[\varepsilon^j]$, in place of asymptotic sequences $[\delta_j(\varepsilon)]$ and $[\lambda_j(\varepsilon)]$. Intuitively (!), it is clear that if the MMAE is used, then we can assume that the limiting values of $b_j(\xi)$ in (3.5) for $\xi \to \infty$, $b_j(\infty)$, exists!

On the other hand, we can suppose that the values of $a_j(x)$, in (3.2), in the neighbourhood of $x = 0$ are well defined (the outer expansion (3.2) remains valid near $x = 0$!).

In this case we can write a Taylor-type expansion near $x = 0$ and we obtain, in place of (3.2), the following outer limiting expansion:

$$y^\circ(x) \approx a_0(0) + x\left(\frac{da_0}{dx}\right)_{x=0} + \frac{x^2}{2}\left(\frac{d^2a_0}{dx^2}\right)_{x=0} + \dots$$
$$+ \varepsilon a_1(0) + \varepsilon x\left(\frac{da_1}{dx}\right)_{x=0} + \varepsilon^2 a_2(0) + \dots \tag{3.6}$$

As a consequence, written in terms of the inner variable, $\xi = x/\varepsilon$, the matching is expressed by the following relation:

$$a_0(0) + \varepsilon\left[\xi\left(\frac{da_0}{dx}\right)_{x=0} + a_1(0)\right]$$
$$+ \varepsilon^2\left[\left(\frac{\xi^2}{2}\right)\left(\frac{d^2a_0}{dx^2}\right)_{x=0} + \xi\left(\frac{da_1}{dx}\right)_{x=0} + a_2(0)\right] + \dots$$
$$= b_0(\infty) + \varepsilon b_1(\infty) + \varepsilon^2 b_2(\infty) + \dots \tag{3.7}$$

Finally, we obtain from (3.7) a hierarchy of matching relations; namely:

$$a_0(0) = b_0(\infty); \tag{3.8a}$$

$$\xi\left(\frac{da_0}{dx}\right)_{x=0} + a_1(0) = b_1(\infty); \tag{3.8b}$$

$$\left(\frac{\xi^2}{2}\right)\left(\frac{d^2 a_0}{dx^2}\right)_{x=0} + \xi\left(\frac{da_1}{dx}\right)_{x=0} + a_2(0) = b_2(\infty) \qquad (3.8c)$$

....

....

It is clear [according to Shivamoggi (1978)] that the above matching principle is simply a rational generalization of the basic matching principle due to Prandtl (1928):

$$lim_{x\to 0}\, y^{\circ}(x) = lim_{\xi\to\infty}\, y^{i}(\xi), \qquad (3.9a)$$

or

$$y^{\circ}(0) = y^{i}(\infty). \qquad (3.9b)$$

Note that the above principle (3.7) puts a less stringent restriction on the domain of validity of the outer solution in that the latter is required to extend merely to the neighbourhood of the inner boundary whereas the basic principle due to Prandtl requires the domain of validity of the outer solution to extend right up to the inner boundary - a probable source of the difficulties the latter method develops of higher order problems. This may also be the reason why the present principle, due to Shivamoggi, succeeds where Prandtl's principle fails. It is also important to understand that the MMAE is applicable only if the behaviour (at infinity) of the inner solution (valid near $x = 0$) is "good" - such that the relations (3.8) remains valid!

As is noted by Wiktor Eckhaus (see his book edited in 1979):

The asymptotic matching principle (according to Van Dyke), when written out formally bears no resemblance to matching relations in intermediate variable (originated by Kaplun and Lagerstrom).

Although Van Dyke, when formulating the principle, was guided by Kaplun's idea (see Van Dyke (1975, Note 4)), a justification of the asymptotic matching has been achieved on the basis of an entirely different hypothesis, employing detailed assumptions on the structure of uniformly valid approximations (see, Fraenkel (1969)).

Indeed, the relationship between the two methods of matching has been investigated by Eckhaus and the result is that:

For a large class of functions overlap (Kaplun) implies the validity of a generalized asymptotic matching principle (Van Dyke).

The generalization (with respect to Van Dyke's formulation) consists of a precise prescription for truncating asymptotic series. The class of functions under consideration is defined by mild conditions, usually satisfied in practical applications. In fact, in this class of functions,

overlap implies the existence of a uniformly valid composite expansion

(any series that reduces to the outer expansion when expanded asymptotically for $\varepsilon \to 0$ in the outer variable, and to the inner expansion in the inner variable) of a type commonly used in applications. But, it is necessary to bear in mind that, however,

overlap is a sufficient but not a necessary condition for the validity of an asymptotic matching principle!

3.2.2. Some historical notes

The ideas underlying the MMAE have grown through the years. It was already being used in the 1950's by K.O. Friedrichs (1953) and his students. Then systematically developed and applied to viscous fluid flows at the California Institute of Technology (as noted by Milton Van Dyke (1975, p. 77)). The important basic idea of regarding perturbation problems systematically in terms of limits has been emphasized and applied by Paco A. Lagerstrom. The idea of matching expansions valid in different regions is an old one, employed for special cases by A.A. Dorodnitsyn (1947) and K. O. Friedrichs. However, the late Saul Kaplun, a student of Lagerstrom's, made deep contributions to the theory of matching asymptotic expansions. Kaplun (1954) introduced the formal inner and outer limit processes for boundary-layer theory, and the corresponding inner and outer asymptotic expansions. The special role of the subcharacteristics and the notion of a composite expansion also appear in the California Institute of Technology, Ph. D. thesis of Gordon Latta (1951). Later, in studying flows at low Reynolds number, Kaplun and Lagerstrom (1957) made a penetrating analysis of the matching process (see also Lagerstrom (1957)). Kaplun (1957) used those ideas to gain deeper insight into the resolution of the Stokes paradox for a plane flow at low-Reynolds number. Lagerstrom and Julian D. Cole (1955) evaluated the method in comparaison with new exact solutions of Navier equations for a sliding and expanding circular cylinder and D. Coles (1957) applied it to some special solutions for the compressible boundary-layer. Proudman and Pearson (1957) applied this expansion method to treat flow past a sphere and circular cylinder at low

Reynolds number. Goldstein (1956, 1960) and Imai (1957) gave the first correct extension of Blasius' boundary-layer solution for the semi-infinite flat plate. Following this development period, the MMAE was applied to a variety of problems in fluid mechanics.

As noted by Van Dyke (1995, p. 6): "it was evidently from them that Jean-Pierre Guiraud (at that time working in the O.N.E.R.A., Chatillon, France) learned of the MMAE, for he cited a paper of Kaplun and Lagerstrom (1957) when, as early as 1958 - before he spent six months in California - he first applied the MMAE to hypersonic flow past a blunt-nosed plate (Guiraud (1958))".

On the other hand, Paul Germain and J. P. Guiraud (1960, 1962) calculated the effect of the curvature upon the structure of a shock wave.
In the Soviet Union M. I. Vichik and L. A. Liusternik (1957) have rigorously studied the boundary-layer concept for the differential equations, and Bulakh (1961) use the correct linearized supersonic conical flow and its higher approximations in the vicinity of the bow shock wave.

In England, Fraenkel and Watson (1964) have attacked the "pseudotransonic" flow past a triangular wing that occurs when the bow wave lies close to the leading edge and Whitham and Lighthill (from Manchester University) have studied the technique for removing nonuniformities from perturbation solutions of nonlinear problems. In Lighthill (1949), such a technique for rendering approximate solutions to physical problems uniformly valid is considered, and according to Lighthill:

"Higher approximations shall be no more singular than the first".

For this Lighthill uses the so-called "method of strained coordinates", the straining of coordinates being initially unknown, and determined term-by-term as the solution progresses. Giving credit to the contibutions of Poincaré (1886, 1892), Lighthill (1949), and later Kuo (1953, 1956)), the name PLK method was coined by Tsien (1956). But this method fails of higher order (see, for instance, Crocco (1972)) and uniformly valid results generally require the application of the MSM (see the next §3.3).

Finally, it is necessary to note a fundamental application of the MMAE to 'breaking' the classical Prandtl boundary-layer (BL) structure. There are many ways of breaking the Prandtl BL structure, but one is especially significant, because it leads to a new asymptotic structure, which seems to be very rich and it is this one which leads to the so-called *triple-deck*. The triple deck concept (see, for instance, Stewartson (1974)) is the Keith Stewartson's greatest contribution to applied mathematics and theoretical fluid mechanics. Without doubt (see, J. T. Stuart (1986, p.7)); "It has been immensely influential, and perhaps might be compared in importance to Prandtl's original idea of the boundary layer as a structure". In the Chapter 12 the reader can find a presentation of the singular coupling concept and the associated triple-deck model. In Lagerstrom's (1988) book the reader can find a very pertinent discussion of the ideas and techniques of MMAE.

New trends in application of asymptotic techniques to mechanical problems, is presented in the recent review paper by Andrianov and Awrejcewicz (2001, with 340 references). These authors consider the various methods which allows for the possibility of extending the perturbation series application space and hence omiting their local character; therefore, an idea of constructing a single solution valid for a whole interval of parameter ε changes is very attractive! For this, in particular, the Padé approximate may be very useful.

3.3. MULTIPLE SCALE METHOD (MSM) AND ELIMINATION OF SECULAR TERMS

Thus, in order for the inner-outer matched asymptotic expansions method to operate it is necessary that at least the matching principle (3.8a) holds. This reduces to the existence and boundedness of the limit for large values of the boundary-layer variable $\xi = x/\varepsilon$ of the first term, $b_0(\xi)$, of the inner expansion (3.5) and to the existence and boundedness of the limit for small values of the independent variable x of the first term, $a_0(x)$, of the outer expansion (3.2). When $a_0(0)$ and/or $b_0(\infty)$ do not exist or are not bounded, recourse is made to the MMAE. This is the reason why the MSM was developed, which replaced the matching from the inner-outer expansion method by the *elimination of the secular terms* (which are responsible for a cumulative effect), brought about by the introduction of one or more new independent variables.

In fact, the fundamental characteristic of the MSM can be summarized as follows (according to Germain 1977; pp. 73):

When the data of a problem show that the small parameter ε is the ratio of two scales (of time or space), the MSM consists in first introducing two variables constructed with these scales - one of them possibly being "distorted" (two-variable expansion). Next, the formal expansion of the solution in ε is considered, each coefficient of the expansion being a function of the two variables introduced (e.g., time) which are considered as independent during the entire calculation. In order to completely determine a coefficient of this expansion, it is not enough to solve the equation where it appears for the first time. The indeterminants which necessarily remain are *chosen by making sure that the equation in which the next term appears will lead to a solution which does not destroy, but on the contrary, best guarantees the validity of the approximation which is sought.*

From a certain *a priori* knowledge of the solution $U(t, x)$ of the fluid dynamical problem, we generally assume that U depends on the variables t and x so that a rapid variation having a repetitive character analogous to an oscillatory phenomenon is made to appear. This variation is itself modulated from one moment to another and from one point to another. Such a structure is described by:

$$U(t, x) = U^*(t, x; \gamma(t, x))), \tag{3.10}$$

where $\gamma(t, x)$ is a so-called fast intermediary variable because the function γ varies rapidly as a function of t and x. If only dimensionless variables are used in the exact fluid dynamical problem, then this property can be characterized by writing:

$$\frac{\partial \gamma}{\partial t} = -\frac{\omega}{\varepsilon} \text{ and } \nabla\gamma = \frac{\lambda}{\varepsilon}, \tag{3.11}$$

where ω and $|\lambda|$ are of order unity. From (3.11) we deduce two relations, for the time and spacial derivatives, namely:

$$\frac{\partial U}{\partial t} = -\left(\frac{\omega}{\varepsilon}\right)\frac{\partial U^*}{\partial \gamma} + \frac{\partial U^*}{\partial t}, \tag{3.12}$$

and

$$\nabla U = -\frac{1}{\varepsilon}\left[\frac{\partial U^*}{\partial \gamma}\right]\lambda + \nabla U^*, \tag{3.13}$$

with analogous formulae for the higher order derivatives. By substituting these expresssions for the derivatives into the basic equations of the fluid dynamical problem, the small parameter ε is introduced into the latter (even if it did not appear initially). We are thus led to such an approximation of U^* via an *uniformly valid expansion,* namely:

$$U^* = U^*_0 + \varepsilon U^*_1 + \ldots;\ \gamma = \gamma_0 + \varepsilon\gamma_1 + \ldots. \tag{3.14}$$

By substitution within the equations (where expressions of the type (3.12) and (3.13) have previously been introduced) and by setting to zero the terms proportional to the successive powers of ε, a *hierarchy* of systems of equations is obtained for U^*_0, U^*_1, ... The first system in this hierarchy determines at best U^*_0 in its dependence with respect to (fast leading variable) γ_0, but not with respect to t and x! It is usual while seeking to determine U^*_1 - or even other higher order terms - that the dependence of U^*_0 with respect to t and x is prescribed by the *cancelling of the secular terms,* i.e., of terms in U^*_1 which do not remain bounded when γ_0 increases indefinitely! Indeed, if we want (3.14) to cover an interval of $O(1)$ in variation of t and x, then because of (3.11), this corresponds to a variation of γ_0 which is $O(1/\varepsilon)$.

Basically, the multiple scale method was invented, probably by Henri Poincaré (see the book by Lebon (1912)), in order to deal with phenomena which occur recursively, almost perodically, over a great number of cycles and which evolve slowly from one cycle to another. This MSM which was originally devised for predicting the evolution of celestial trajectories over secular periods of time, has since been applied to a variety of phenomena.

The recent book by Kevorkian and Cole (1996), which is a revised and updated version of the well-known book by the same authors, *Perturbation methods in applied mathematics,* is indeed a new, completely rewritten, edition and contains a chapter on multiple-scale expansions for partial differential equations. A comprehensive account with various applications of the MSM is also given in Nayfeh's (1981) book.

Finally, in Zeytounian (1994, in French), the §5.3 to §5.6, in Chapter V, are devoted to MSM and its various applications.

3.4. THE HOMOGENIZATION METHOD AND AVERAGING PROCESS

It is also interesting to note that one variant of the MSM, used to study problems with perodic microstruture, is the *homogenization method.* In fact in fluid mechanics one must often know the effective properties of inhomogeneous (or composite) flow.

When the heterogeneity spreads over a large number of regions, a detailed analytical or even numerical approach becomes unfeasible [Mei, Auriault and Ng (1996)]. A natural idea is to gloss over the rapid variations of the heterogeneities and replace

the composite flow by an equivalent homogeneous fictitious flow whose behaviour over a macroscopic scale represents the averaged behaviour of the composite real flow,

and a basic route to achieve this goal is a rational *process of averaging.* The Asymptotic Method of Homogenization (AMH) is characterized by the mathematical techniques of multiple scales and is especially used for flows with periodic microstructure. In the homogenized (macro) equations, after averaging over the microscale, memory terms appear which parameterize (are the trace of) the microstruture present in the exact formulation of the original inhomogeneous fluid flow problem. The use of the MSM as a systematic averaging tool for problems other than wave propagation can be traced to the earlier works by Sanchez - Palencia (1974) in France.

There now exist several mathematical treatises of the method [see, for example, the book by: Bakhvalov and Panasenko (1989) and the book edited by the "Direction des Etudes et Recherches d'Electricité de France", CEA-EDF-INRIA, Ecole d'Eté d'Analyse Numérique (1985)], but the level of mathematics used in these books may appear forbidding to many fluid dynamicst readers.

Although differing in technical details, the basic idea of the theory of homogenization has been employed for a long time [see, for instance, in Chapter 6, §6.6, our theory of axial-flow machinery, with J.P. Guiraud, (1971a, b, 1974) - clearly the long scale is that of the mean (macro) flow, and the short (micro) scale that of the individual blades, the small parameter being the reciprocal of the number of blades on a row or of stages of rows].

In the theory of wave propagation in slowly varying media, the familiar ray theory (geometrical optics approximation) is also one such example. There one employs the MSM, to average over the locally periodic waves and find the slow variation of the wave envelope - but again this procedure is rarely known as homogenization, however!

In Zeytounian and Guiraud (1985), a mixture of immiscible continuous media is considered, where the phases are distinguishable only at the microscopic level but intimately mixed macroscopically. Balance laws at the macroscopic level rely on an averaging process, which is formalized according to a methodology, which may be applied to a variety of situations. This allows us to find the microscopic interpretation of terms, which correspond to exchange between phases. A homogenized equation, with an (averaged) exchange term between the phases, is also derived.

The problem of sound propagation through a liquid populated sparsely by bubbles is another interesting application treated by Caflisch *et al.* (1985). The main objective is to find an effective equation for the propagation of sound, whose wavelength is much greater than the bubble spacing, which is, in turn, much greater than the bubble radius. In particular, it is known that the presence of a small volume fraction of bubbles in water can drastically reduce the speed of sound.

In B. Mohammadi and O. Pironneau (1993) the curious reader can find a very interesting application of the AMH to modelling of turbulent flows (the so-called 'K-Epsilon' turbulence model).

Concerning this very difficult problem see also the two short Notes by Zeytounian and Guiraud (1985) and Guiraud and Zeytounian (1986b) and also the paper by Deriat and Guiraud (1986).

3.4.1. The averaging in space-time

The averaging process in space and time is a fundamental step in the asymptotic modelling of the turbulence but also for a wide class of complex physical phenomena with the various scales.

They occur in space-time with respect to a Galilean frame having dimensionless coordinates x_i, $i = 1, 2, 3$, and dimensionless time t. We glue all the four into the notation x_α, $\alpha = 0$ *to* 4, $x_0 = t$. Our physical phenomenon is assumed to be accurately described by an element U belonging to an N dimensional vector space. For turbulence $N = 4$ and U stands for u_i, the three components of velocity vector and pressure p. For the time being we do not care about the equations which rule the phenomenon except that when dealing with turbulence they will be the Navier (incompressible, viscous with a constant density) ones.

We start our discussion by referring to several physical situations for the purpose of motivating our analytical machinery. It seems fair to start with flow within porous media which, under the periodicity assumption, leads to a mathematically consistent theory initiated by Sanchez-Palencia (1970) who gives credit, in an Appendix, to Tartar for a basic argument from the mathematical aspect of the theory. Periodicity allows to put the argument on a strong mathematical setting but it is successful under much wider conditions (Dewest (1965), Brinckman (1947)). The argument is that the flow occurs at two different scales: a macroscopic one which carries the main interest while ignoring the details of the porous material, and a microscopic one which allows those details to be taken care of in order to derive constitutive laws suited for the study of flow at the macroscopic scale. The success of the analysis relies on the existence of a small parameter ε, which is the ratio of the microscopic scale to the macroscopic one. One accounts for this by stating, as an "ansatz", that any physical quantity U may be represented as:

$$U \Rightarrow U\left(t, x, \frac{x}{\varepsilon}; \varepsilon\right). \tag{3.15}$$

The dependency on x/ε allows to take into account the details of the porous matrix, while the dependency on x is the one, with t, of main interest. A fast time is not necessary mainly because the details of the matrix do not change with the time and one is not interested in the way macroscopic motion starts from rest.

A similar situation occurs in solid mechanics involving strong heterogeneity and may be found in many references (for, instance: Hill (1967), Hashin (1983), Marigo, Mialon, Michel and Suquet (1987)), and, again (3.15) is an essential tool.

3.4.1a. From $U(t, x, x/\varepsilon; \varepsilon)$ to $U(x_\alpha, \xi_\alpha^{(1)}, \xi_\alpha^{(2)}, ..., \xi_\alpha^{(n)})$

Indeed, the fractal structure of the turbulence suggests a more complex structure than (3.15) with x/ε^2, x/ε^3, ... (a very simple fractal is the Koch's curve).

But the chaotic structure with respect to time suggests also to introduce fast time variables. We may proceed step-wise. Let us begin by using just two scales for each one of the four space-time coordinates and write: $U = U(x_\alpha, \xi_\alpha(x_\beta))$ with

$$\frac{\partial \xi_\alpha}{\partial x_\beta} = \lambda_{\alpha\beta}(x_\gamma) \text{ and } |\lambda_{\alpha\beta}| >> 1. \tag{3.16}$$

Let us draw a first conclusion by assuming that, under (3.16), only the $\lambda_{\alpha\beta}$ come into the determination of U, then we may argue that the most general solution to (3.16) is

$$\xi_\alpha(x_\beta) = \xi^P{}_\alpha + \xi^*{}_\alpha \tag{3.17}$$

where ξ_α^P is a particular solution dependent of the x_β, while ξ_α^* is arbitrary and independent on the x_β. Then, the very process of representing analytically the physical phenomenon, and relying on (3.17), has generated a function from a 2 x 4 dimensional space-time to the N dimensional vector space within which the values of U are embedded. This has been exploited by Deriat and Guiraud (1986) assuming that a boundary layer has, locally, the structure of a parallel shear flow of infinite extent in the plane tangent to the wall but slowly varying. The idea was carried over up to the writing of asymptotic set of equations: the Reynolds equations, equations for the fluctuations and the mean kinetic energy of fluctuations, and also for the dissipation of turbulence.

Now we consider, in place of $U = U(x_\alpha, \xi_\alpha(x_\beta))$,

$$U = U\left(x_\alpha, \xi^{(1)}{}_\alpha(x_\beta), \xi^{(2)}{}_\alpha\left(\xi^{(1)}{}_\beta\right), \ldots\ldots\ldots, \xi^{(n)}{}_\alpha\left(\xi^{(n-1)}{}_\beta\right)\right) \tag{3.18}$$

with

$$\lambda^{(n)}{}_{\alpha\beta} = \frac{\partial \xi^{(n)}{}_\alpha}{\partial \xi^{(n-1)}{}_\beta} = \lambda^{(n)}{}_{\alpha\beta}\left(\xi^{(n-1)}{}_\gamma\right) \text{ and } |\lambda^{(n)}{}_{\alpha\beta}| >> 1. \tag{3.19}$$

Homogeneity is the property that $\lambda^{(n)}{}_{\alpha\beta}$ is constant, self-similarity occurs when $\lambda^{(n)}{}_{\alpha\beta}$ no more depends on n. The same argument as the one used in (3.17) may be repeated at each step, so that the representation has generated a function having its arguments in a $4(n+1)$ dimensional space-time if we stop at the nth step, namely

$$U = U\left(x_\alpha, \xi_\alpha^{(1)}, \xi_\alpha^{(2)}, \ldots\ldots\ldots, \xi_\alpha^{(n)}\right) \tag{3.20}$$

each one of the x_α, $\xi_\alpha^{(p)}$, $p = 1, 2, \ldots, n$, being independent of all the others.

It is only when we need a specific representation that we substitute the arguments appearing in (3.18). We observe that such a technique occurs when applying the MSM as may be found in Kevorkian and Cole (1996).

The process which generates (3.17) at each step is fairly conspicuous in examples of applications of MSM: the conditions (3. 16), (3.19) allow to generate as much set of equations, analogous to the initial one for U, as one has introduced new fast variables. Usually one needs boundary conditions; a substitute at each step is the asymptotic consistency expressed by the 'elimination of secular terms'.

3.4.1b. Averaging process in space-time

We need now explain how means are extrated and how equations for those means are got. We start with the rather simple situation, namely:

$$U = U\left(x_\alpha, \xi_\beta; \varepsilon\right) \text{ with } \frac{\partial \xi_\beta}{\partial x_\gamma} = \varepsilon^{-1}\lambda_{\beta\gamma}\left(x_\delta\right), \tag{3.21}$$

but at this step we need not care about the way in which U depends, asymptotically, on ε. The first step is the one of so-called homogenization discussed in the framework of a mathematically consistent theory in

Sanchez-Palencia (1980), Bensoussan, Lions and Papanicolaou (1978). There are various ways to work it out and we use the one in Zeytounian and Guiraud (1985), which is not mathematically founded contrary to one in the above mentioned two books, but is much more flexible. The basic tool is averaging in space-time.

Let us consider a four-dimensional domain D in the space-time of the ξ_β, and put:

$$LIM_H\left\{\int_D U(x_\alpha,\xi_\beta;\varepsilon)d\xi_0 d\xi_1 d\xi_2 d\xi_3\right\} = << U >> (x_\alpha). \quad (3.22)$$

where

$$LIM_H = lim\{|D|\to\infty, \varepsilon^4|D|\to 0\}, \quad (3.23)$$

and $|D|$ is the four-dimensional Lebesgue measure of the domain D.

The main goal to be achieved with homogenization is to eliminate the dependency with respect to the rapidly varying space-time ξ_β, leaving an homogenized $<<U>>$ which depends only on the slowly varying space-time x_α. Of course, what is needed is equations allowing to make prediction about the $<<U>> (x_\alpha)$. Usually one has to face the following typical situation. The equations for U contain quantities like $Q(x_\alpha, U(x_\beta))$, $Q(x_\alpha, U(x_\beta), D^\alpha U(x_\beta))$,..., and we consider the simplest case when we need

$$<< Q >> (x_\alpha, << U >> (\cdot))$$
$$= LIM_H\left\{\int Q(x_\alpha, U(x_{\beta,}\xi_\beta;\varepsilon))d\xi_0 d\xi_1 d\xi_2 d\xi_3\right\} \quad (3.24)$$

and the formidable difficulty we have to cope with is that, even if we know very well how Q depends of U, there is no reason why we should be able to figure out, simply, how $<<Q>>$ depends, functionnally, on $<<U>>$!

This functional dependency is stressed into our writing $<<Q>>(x_\alpha, <<U>> (\cdot))$.

We borrow one example to the so-called "Large Eddy Simulation" (LES) of turbulence. Let u_i, $i = 1, 2, 3$ be the Cartesian components of the turbulent velocity field. Various kinds of homogenization procedure lead to $<<u_i>>(x_\alpha)$ but the Navier equations, when homogenized by the same procedure, require knowledge of

$$<< u_i u_j >> = << u_i >><< u_j >> + \tau_{ij}, \quad (3.25)$$

the $\rho_0\tau_{ij}$ (with ρ_0 = *const*, for an incompressible fluid) being usually called Reynolds stresses because they occur through $\partial\tau_{ij}/\partial x_j$ like $\partial T_{ij}/\partial x_j$ where T_{ij} are the components of the Newtonian molecular stress tensor $\boldsymbol{T}$.

A thoroughly used model originally considered by Smagorinsky (1963) uses as a mathematical model:

$$\tau_{ij} = \nu_T << d_{ij} >>, \tag{3.26a}$$

with

$$\nu_T = cl^2\{<< d_{ij} >><< d_{ij} >>\}^{1/2}, \tag{3.26b}$$

and

$$<< d_{ij} >> \equiv \frac{\partial << u_i >>}{\partial x_j} + \frac{\partial << u_j >>}{\partial x_i}, \tag{3.26c}$$

into which summation over repeated indices is understood. This is one of the simplest functional dependency on $<<u_i>>$ through derivation and algebra.

Now, the main problem is, leaving aside that specific example, to go on at the general level for a derivation of a couple of closed equations for the kinetic energy of the turbulence (as a matter of fact of fluctuations) per unit of mass and the *rate of turbulence dissipation,* which are local quantities changing from time-point to time-point!

CHAPTER 4

USEFUL LIMITING FORMS OF THE NS-F EQUATIONS

In this short chapter, from the dimensionless NS-F equations, of the §2.3, in a very naive approach, various particular usual forms of fluid flows equations are derived in §4.1 to §4.7.

Namely: Navier-Stokes (isentropic and viscous), Euler (nonviscous), Navier (viscous and incompressible), acoustics, Stokes-Oseen (for low-Reynolds numbers), Prandtl (boundary layer, for large Reynolds number), isochoric (for a conservative density along the trajectories), for a liquid (expansible) and Boussinesq (inviscid, with the buoyancy force) equations. These various models are skeched in figure 4.1, which is also useful in explaining the ensuing derivation from the full (exact) NS-F equations when, respectively, the Reynolds numbers are high or low and the Mach number is low.

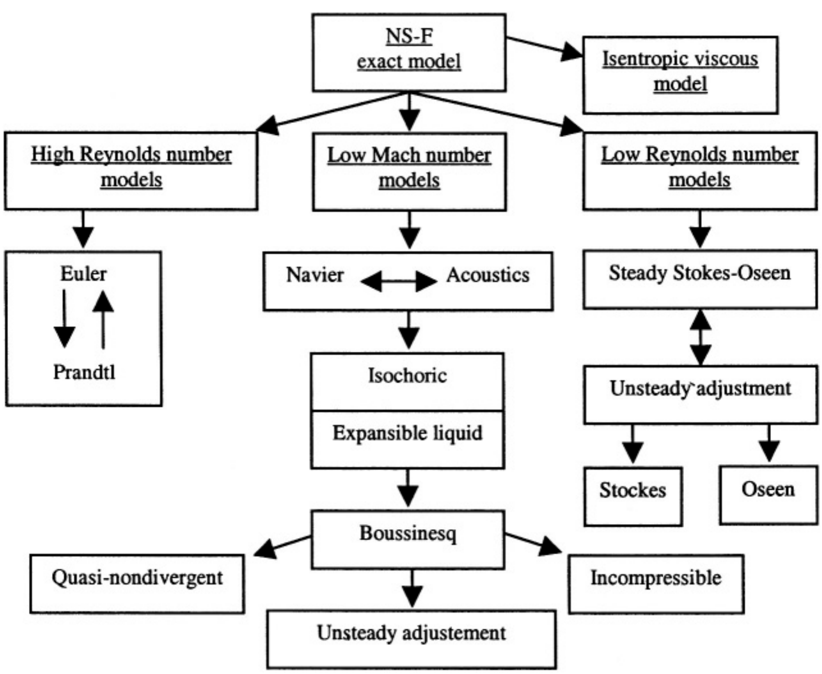


Fig. 4.1 From NS-F exact equations to useful limiting forms

4.1. THE NAVIER-STOKES (NS) ISENTROPIC VISCOUS EQUATIONS

We consider the first two dimensionless equations (2.56a) and (2.56b) and assume that the Stokes relation ($\mu_v \equiv 0$) is satisfied for μ and λ both constant. In this case we obtain for $\boldsymbol{u}$, p and ρ the following two equations:

$$\rho S \frac{D\boldsymbol{u}}{Dt} + \frac{1}{\gamma M^2}\nabla p = \frac{1}{Re}\left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u})\right], \tag{4.1}$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0, \tag{4.2}$$

when we assume that $Bo = 0$. Since we do not expect shocks, we assume also that the specific entropy: $S \equiv S_0 = const$, and we can write the following dimensionless *specifying relation* between p and ρ (polytropic gas with $\gamma > 1$):

$$p = p(\rho) = \rho^\gamma, \tag{4.3}$$

which is *not* an equation of state.

The equations (4.1), (4.2), with (4.3), form a closed system for $\boldsymbol{u}$, ρ and p, which is called the Navier-Stokes (NS) equations for a *viscous compressible isentropic fluid flow.* In fact, we have two evolution equations for $\boldsymbol{u}$ and ρ, namely:

$$S\frac{D\rho}{Dt} + \rho \nabla \cdot \boldsymbol{u} = 0, \tag{4.4a}$$

$$S\frac{D\boldsymbol{u}}{Dt} + \frac{1}{M^2}\rho^{\gamma-2}\nabla\rho = \frac{1}{Re}\frac{1}{\rho}\left[\nabla^2 \boldsymbol{u} + \frac{1}{3}\nabla(\nabla \cdot \boldsymbol{u})\right], \tag{4.4b}$$

with, $S\, D/Dt = S\, \partial/\partial t + \boldsymbol{u} \cdot \nabla$, where S is the Strouhal number.

Unfortunately, the system of two evolution equations (4.4a, b) is not a significant degeneracy of the full (exact!) NS-F equations, in the spirit of asymptotic modelling - the above NS system is only an 'ad hoc' approximate isentropic model. In the case of low Mach numbers ($M \ll 1$), when viscous dissipation and thermal conduction are negligible (!), the above two equations (4.4a, b) are an appropriate starting point for various mathematical analyses. Namely, the recent rigorous investigations of applied mathematicians, devoted to the "incompressible limit" - in fact, low-Mach asymptotics - are performed mainly with the above NS viscous, isentropic, equations (4.4a, b) and the reader can consult, for instance, the more recent paper by Desjardin, Grenier, P.-L. Lions and Masmoudi (1999).

But, obviously, if we consider the entropy equation in an *adiabatic fluid flow*:

$$S\frac{DS(\boldsymbol{x},t)}{Dt}=0\text{, with } S(\boldsymbol{x},t)=S_0=const \text{ at } t=0, \tag{4.5}$$

then we deduce $S(\boldsymbol{x}, t) \equiv S_0$ (we do not expect shocks since λ, $\mu > 0$). But for this it is necessary, in equation (2.16), for $S(\boldsymbol{x}, t)$, to assume that, first, the term with $\boldsymbol{Q} \equiv 0$ (heat flux is neglected) and then that the heating due to viscous dissipation $\Phi \equiv 0$ (an approximation which is reasonable except for hypersonic motions for large Mach number ($M >> 1$).

4.2. THE EULER, NAVIER AND ACOUSTICS EQUATIONS

4.2.1. The Euler equations

The Euler equations are derived from the dimensionless NS-F equations (2.56a, b, c), when we consider the *high Reynolds numbers.* More precisely, it is necessary to consider the following *Euler limit:*

$$\begin{gathered} lim^E = [Re \uparrow \infty \text{ with } \boldsymbol{x} \text{ and } t \text{ fixed,} \\ \text{and for } S, M, \gamma, Bo \text{ and } Pr, \text{ all } O(1)]. \end{gathered} \tag{4.6}$$

In this case for the limit functions, $(\boldsymbol{u}_E, p_E, \rho_E, T_E) = lim^E(\boldsymbol{u}, p, \rho, T)$, we derive the Euler compressible adiabatic equations:

$$\begin{gathered} S\frac{D\rho_E}{Dt}+\rho_E\nabla\cdot\boldsymbol{u}_E=0; \\ \rho_E S\frac{D\boldsymbol{u}_E}{Dt}+\frac{1}{\gamma M^2}\nabla p_E+\frac{Bo}{\gamma M^2}\rho_E\boldsymbol{k}=0, \\ \rho_E S\frac{DT_E}{Dt}+(\gamma-1)p_E\nabla\cdot\boldsymbol{u}_E=0, \end{gathered} \tag{4.7}$$

with

$$p_E = \rho_E T_E. \tag{4.8}$$

In place of the last equation of (4.7) we can write an equation for the entropy $S_E(\boldsymbol{x}, t)$:

$$S\frac{DS_E(\boldsymbol{x},t)}{Dt}=0\,. \tag{4.9}$$

For the above Euler equations, according to inviscid theory, we assume on the solid wall the slip condition. Indeed, this condition is a matching condition, which emerges automatically in an outer-inner asymptotic theory for the high Reynolds numbers (see, for instance in Chapter 7, §7.2).

If the solid surface in Cartesian coordinates (x, y, z) is known at time t, in (say) the dimensionless form: $F(t, x, y, z) = 0$, then the slip condition (on $F = 0$) is written in the following form (S is the Strouhal number):

$$\text{on } F = 0\text{:}\ S\frac{DF}{Dt}=0\,. \tag{4.10}$$

4.2.2. Navier equations

The Navier equations govern incompressible viscous fluid flows. These equations are derived formally from the full NS-F equations (2.56a, b), when the Mach number M tends to zero. More precisely, it is necessary to consider the following *Navier limit* (we consider only the case when $Bo = 0$):

$$lim^N = [M \downarrow 0, \text{ with } \boldsymbol{x} \text{ and } t \text{ fixed, and for } S, Re, \gamma \text{ and } Pr, \text{ all } O(1)]. \tag{4.11}$$

In particular, from (2.56b), in a very naïve way, we assume the following asymptotic expansion for the pressure:

$$p = p_O(t) + M^2\, p_N(\boldsymbol{x}, t) + o(M^2), \tag{4.12}$$

and we enquire how one can get information concerning the function $p_O(t)$? Scrutinizing the equations and trying more sophisticated expansion processes proves to be of no help. However, a way out appears if (4.12) holds in the whole of a domain of space where p is known to leading order, with respect to the smallness of M, as far as its dependence on $\boldsymbol{x}$ is concerned.

This occurs when (4.12) holds in a neighbourhood of infinity where the pressure is constant (see, for example, (2.38)), and this leads, then, to $Po(t) = 1$. As a matter of fact, we get any constant in place of one, but this is only a matter of a convenient choice for p_c. This occurs in what is called *external aerodynamics,* a field of intensive studies.

On the other hand, let us assume that the gas is contained in a *container* Ω bounded by an impermeable but, eventually, *deformable wall,* so that the volume occupied by the gas be a given function of time, namely *V(t).* An obvious way to proceed is to assume that the density and temperature go to definite limits $\rho_O(x,t)$, $T_O(x,t)$, according to (4.11). It is a very easy matter to get, from (2.56c) an equation satisfied by T_O. Such an equation involves the unknown function $p_O(t)$, and it has an obvious solution $T_O = T_O(t)$, which holds provided that the two unknown functions, $\rho_O(t)$ and $T_O(t)$, meet the requirement that: $T_O/(\rho_O)^{\gamma-1}$ be independent of time, and, by simply choosing the reference constant values, ρ_c and T_c, we may asume that this constant is equal to unity. It is then very easy to reach the conclusion that:

$$\rho_O(t) = [p_O(t)]^{1/\gamma} \tag{4.13a}$$

and, *conservation of the mass for the whole of the gas contained within the container,* gives

$$p_O(t) = [V(t)]^{-\gamma}, \tag{4.13b}$$

so that we have found our way out of the indeterminacy concerning the leading term in (4.12). Of course, our argument relies on $T_O(t)$ being independent of space and we have to discuss the adequacy of that. It is, obviously, a matter of conduction of heat within the gas. Such a phenomenon might have two origins: one is dissipation of energy within the gas, but consideration of equation (2.56c) tells use that this enters into account at a rate of $O(M^2)$ and is negligible as far as $T_O(t)$ is concerned, a second origin for the conduction of heat is through variations of temperature on the wall or from heat transfer through it.

Let us come back to (2.56a) and (2.56b) and assume that u goes to $u_N(x, t)$ according to (4.11). We put:

$$\frac{p_N}{\gamma p_O(t)} = lim^N \left\{ \frac{p - p_O(t)}{\gamma M^2 \rho_O(t)} \right\} = \pi , \tag{4.14}$$

a *fictitious* pressure perturbation, and we get the following *quasi-incompressible* equations for the velocity vector u_N and the pressure perturbation π:

$$\nabla \cdot \boldsymbol{u}_N = -\frac{S}{\rho_O(t)}\frac{d\rho_O}{dt}; \tag{4.15a}$$

$$\left[S\frac{\partial}{\partial t} + \boldsymbol{u}_N \cdot \nabla\right]\boldsymbol{u}_N + \nabla\pi = \frac{\mu(T_O)}{Re\rho_O}\nabla^2\boldsymbol{u}_N, \tag{4.15b}$$

with

$$\rho_O(t) = \frac{1}{V(t)};\ p_O(t) = [V(t)]^{-\gamma};\ T_O(t) = [V(t)]^{1-\gamma}. \tag{4.15c}$$

The above equations (4.15a, b), with (4.15c), are rather, a slight variant of the Navier equations which consists of:

A fluid with time dependent viscosity, but also, rather than a divergenceless motion, one with constant in space, variable in time, divergence.

The usual set of Navier equations is, obviously, obtained for a *constant volume container,* and in this case we can write in place of (4.15a), (4.15b) the following *classical Navier equations* for a divergenceless velocity $\boldsymbol{u}_N$ and *fictitious* pressure perturbation $\boldsymbol{\pi}$:

$$\nabla \cdot \boldsymbol{u}_N = 0,\ \left[S\frac{\partial}{\partial t} + \boldsymbol{u}_N \cdot \nabla\right]\boldsymbol{u}_N + \nabla\pi = \frac{1}{Re}\nabla^2\boldsymbol{u}_N. \tag{4.16}$$

4.2.3. Acoustics equations

For the NS-F full compressible (aerodynamics, again with $Bo = 0$) unsteady equations (2.56a)-(2.56c), with (2.56d), if we want to resolve a Cauchy problem, it is necessary to impose a complete set of initial conditions for $\boldsymbol{u}$, ρ and T, as in (2.31).

On the other hand, when considering the above Navier limit (hydrodynamics) equations (4.16), we must give only the initial value for $\boldsymbol{u}_N$ and this initial condition $(\boldsymbol{u}_N)^0$ is such that:

$$\nabla \cdot (\boldsymbol{u}_N)^0 = 0. \tag{4.17}$$

Obviously, the Navier incompressible limit (outer) system (4.16) *is not valid* near $t = 0$, and as a consequence it is necessary to consider an initial (inner) region with a 'short time':

$$\tau = \frac{t}{M^{\alpha}} = O(1), \text{ when } M \downarrow 0, \tag{4.18}$$

where $\alpha > 0$ is a scalar.

This short time τ is suited for studying this transient behavior. In fact, for the case of external aerodynamics, we must deduce from full NS-F equations (2.56.a, b, c), with (2.56d), an inner-local (near-initial time) system of equations. For this we consider with (4.19) the following asymptotic expansions:

$$\boldsymbol{u} = \boldsymbol{u}_a + \ldots, \; p = 1 + M^a p_a + \ldots, \; \rho = 1 + M^b \rho_b + \ldots, \\ T = 1 + M^c T_c + \ldots, \tag{4.19}$$

and consider the following local (inner-acoustics) limiting process:

$$lim^a = [M \downarrow 0 \text{ with } \tau \text{ and } \boldsymbol{x} \text{ fixed, and for } S, Re, \gamma \text{ and } Pr, \text{ all } O(1)]. \tag{4.20}$$

Indeed, the asymptotic local expansion (4.19) is consistent with the situation corresponding to an unbounded fluid flow, outside of a solid bounded body $\boldsymbol{\Omega}$, starting in motion impulsively (mimicking a catapulting process).

A "poor" justification, for working this way, is that this is a classical problem in inviscid incompressible fluid dynamics and that it is worthwhile to try to elucidate the transient behaviour (in fact, this transient behaviour is essentially characterized by the weak compressibility) in the NS-F equations, when we consider the local (in time) limit process (4.20) with (4.19).

First, with (4.18) and (4.19), from the dimensionless continuity equation (2.56a), we derive (at leading order) the least "degenerated" limit (acoustics) equations when $\alpha = 1$ and $b = 1$, and then it is easy to show that the more consistent (dimensionless) limit system, which is derived from the full NS-F compressible unsteady equations by the acoustic limiting process (4.20), is the *linear acoustics system* of equations, if: $a = c = 1$. Namely,

$$S\frac{\partial \rho_a}{\partial \tau}+\nabla\cdot \boldsymbol{u}_a=0,$$

$$S\frac{\partial \boldsymbol{u}_a}{\partial \tau}+\frac{1}{\gamma}\nabla p_a=0, \tag{4.21}$$

$$S\frac{\partial T_a}{\partial \tau}+(1-\gamma)\nabla\cdot \boldsymbol{u}_a=0,$$

$$p_a=\rho_a+T_a,$$

and now we can write for the above linear homogeneous equations, (4.21), of acoustics, the full initial conditions, for $\boldsymbol{u}_a$, ρ_a, and T_a, corresponding to the external aerodynamics problem; namely:

$$t=0:\ \boldsymbol{u}_a=0,\ \rho_a=0,\ T_a=0, \tag{4.22}$$

when we consider the transient behaviour of a NS-F flow set into motion *from rest* by the displacement of a solid body in an unbounded medium.

For the determination of the (unknown!) initial condition $(\boldsymbol{u}_N)^0$, for the Navier incompressible but viscous system (4.16), it is necessary to match the acoustic velocity $\boldsymbol{u}_a$ (when $\tau\to\infty$) with the Navier incompressible divergenceless velocity $\boldsymbol{u}_N$ (when $t\to 0$):

$$lim_{\tau\to\infty}\boldsymbol{u}_a=lim_{t\to 0}\boldsymbol{u}_N. \tag{4.23}$$

Here, we note only that, in the book by Wilcox (1975) the reader can find a scattering theory, which makes it possible to analyse the behaviour of the equations of acoustics (4.21), with (4.22), when $\tau\to\infty$ (unsteady adjustement problem).

Recently, this matching has been accomplished by Zeytounian (2000) and in §5.2 (Chapter 5) we return in detail to this "initialisation" problem for the incompressible and viscous (Navier) limit fluid flow.

4.3. THE STOKES AND OSEEN EQUATIONS

4.3.1. Steady Stokes equation

The Navier equation (4.16) written in dimensionless form is:

$$Re\frac{D\boldsymbol{u}_N}{Dt}+\nabla p_s=\nabla^2\boldsymbol{u}_N, \tag{4.24a}$$

when we assume that the Strouhal number $S \equiv 1$ and define the (Stokes) pressure p_S such that:

$$p_S = Re\ \pi \tag{4.24b}$$

If, now, Re tends to zero with time-space dimensionless variables $(t, \boldsymbol{x})$ and $\boldsymbol{u}_S$, p_S fixed, one obtains the following steady Stokes equation:

$$\nabla^2\boldsymbol{u}_S=\nabla p_s, \tag{4.25a}$$

with

$$\nabla\cdot\boldsymbol{u}_s=0 \text{ and } lim^{St}\,\boldsymbol{u}=\boldsymbol{u}_S(t,\boldsymbol{x}), \tag{4.25b}$$

where

$$lim^{St}=[Re \downarrow 0, \text{ with } p_s, S\equiv 1, t \text{ and } \boldsymbol{x} \text{ fixed}]. \tag{4.26}$$

One would expect the boundary conditions for the steady Stokes equations (4.25) to be same as those for the full Navier equations (4.16)? But it was noted, by Stokes (1851) himself, that solutions *do not* exist for stationary 2D flow past a solid which satisfy *both* conditions, at the solid wall (no-slip) as well as at infinity (uniform flow):

$$\boldsymbol{u}\rightarrow U_\infty \boldsymbol{i}, \text{ at infinity.} \tag{4.27}$$

Indeed, the Stokes flow in an unbounded domain, exterior to a solid body, is only an inner-proximal flow valid in the vicinity of the wall of this body.

4.3.2. Unsteady Oseen equation

Far from the wall, near infinity, when: $Re|\boldsymbol{x}| \equiv |\boldsymbol{x}_O| = O(1)$, it is necessary to derive another consistent [outer - distal, so-called Oseen (1910)] equation, in place of the Stokes equation (4.25a). Using Oseen time-space variables:

$$t_O = Re\, t \text{ and } \boldsymbol{x}_O, \tag{4.28a}$$

and also the Oseen pressure,

$$p_O=\frac{p_S}{Re}, \tag{4.28b}$$

we derive, in place of the unsteady Navier equation (4.16), the following dimensionless Navier equation (again with a Strouhal number $S \equiv 1$):

$$\frac{D_O \boldsymbol{u}}{Dt_O} + \nabla_O p_O = \nabla_O^2 \boldsymbol{u}, \tag{4.29}$$

where

$$\frac{D_O}{Dt_O} = \frac{\partial}{\partial t_O} + \boldsymbol{u} \cdot \nabla_O, \text{ and } \nabla_O = \frac{\partial}{\partial \boldsymbol{x}_O}.$$

But, near the infinity, at the (Oseen limit):

$$lim^{Os} = [Re \downarrow 0, \text{ with } p_O, S = 1, t_O \text{ and } \boldsymbol{x}_O \text{ fixed}], \tag{4.30}$$

a finite body shrinks to a point, which cannot cause a finite disturbance in the viscous fluid.

Thus, in the outer, Oseen region, we can write the following asymptotic expansion:

$$\boldsymbol{u} = \boldsymbol{i} + \delta(Re)\boldsymbol{u}_O(t_O, \boldsymbol{x}_O) + \ldots, \tag{4.31a}$$

when we assume that (with dimensionless quantities):

$$\boldsymbol{u} \rightarrow \boldsymbol{i} \text{ at infinity}. \tag{4.31b}$$

In this case for $\boldsymbol{u}_O$ we derive, from the Navier equation (4.29), written "à la Oseen", the following unsteady Oseen equation:

$$\left[\frac{\partial}{\partial t_O} + \nabla_O \cdot \boldsymbol{i}\right]\boldsymbol{u}_O + \nabla_O p_O = \nabla_O^2 \boldsymbol{u}_O. \tag{4.32}$$

In the above outer (Oseen) asymptotic expansion (4.3la) for $\boldsymbol{u}(t, \boldsymbol{x}_O)$ the gauge function: $\delta(Re) \downarrow 0$, with $Re \downarrow 0$, is arbitrary and only by matching (outer- inner/proximal-distal) we have the possibility to determine this gauge. The matching conditions are somewhat more complicated than in the high Reynolds number case [see, for instance, Lagerstrom (1964, pp. 163-167)]. In Chapter 9 the reader can find a detailed theory of low-Reynolds numbers, but mainly for an incompressible highly viscous fluid flow in the case of steady and unsteady motions.

4.3.3. *Unsteady Stokes and steady Oseen equations*

Indeed, near $t = 0$, it is necessary to take into account, in place of the steady Stokes equation (4.25a), the following unsteady Stokes equation (again with Strouhal number $S \equiv 1$):

$$\frac{\partial \boldsymbol{u}_S^*}{\partial t_S} + \nabla^2 \boldsymbol{u}_S^* = \nabla p_S, \tag{4.33}$$

where

$$t_S = \frac{t}{Re}, \text{ and } lim^{St*} \boldsymbol{u} = \boldsymbol{u}_S^*(t_S, \boldsymbol{x}), \tag{4.34}$$

when the following limit is considered:

$$lim^{St*} = [Re \downarrow 0, \text{ with } p_S, S \equiv 1, t_S \text{ and } \boldsymbol{x} \text{ fixed}]. \tag{4.35}$$

On the other hand, far from $t = 0$ we have a steady Oseen equation, in place of the unsteady Oseen equation (4.32). Namely:

$$(\nabla_O \cdot \boldsymbol{i})\boldsymbol{u}_O^* + \nabla_O p_O = \nabla_O^2 \boldsymbol{u}_O^*, \tag{4.36}$$

with:

$$\boldsymbol{u} = \boldsymbol{i} + \delta^*(Re)\boldsymbol{u}_O^*(t, \boldsymbol{x}_O) + \dots, \tag{4.37}$$

and $\delta^*(Re) \downarrow 0$, with $Re \downarrow 0$.

4.4. THE PRANDTL BOUNDARY-LAYER EQUATIONS

Mathematically the inviscid, Euler, solution cannot be uniformly valid when the viscosity tends to zero (or $Re \uparrow \infty$) because it fails to satisfy the no-slip (steady) condition (2.32) on the body, and this non-uniformity can only be removed by introducing a (thin) boundary layer (BL) near the body wall, where viscosity matters. For this reason, the Navier equation, derived in §4.2:

$$\left[S\frac{\partial}{\partial t} + \boldsymbol{u}_N \cdot \nabla\right]\boldsymbol{u}_N + \nabla \pi = \frac{1}{Re}\nabla^2 \boldsymbol{u}_N,$$

is called a singular perturbation of the incompressible Euler equation:

$$\left[S\frac{\partial}{\partial t}+\boldsymbol{u}_E\cdot\nabla\right]\boldsymbol{u}_E+\nabla\pi_E=0\,, \tag{4.38}$$

with

$$\pi_E=Lim^E\left\{\frac{(p-1)}{\gamma Ma^2}\right\}, \tag{4.39}$$

when we consider an external aerodynamics problem.

The theory of singular perturbations of the Navier system [equations (4.16)], for *vanishing viscosity* (large Reynolds numbers) is actually very well understood thanks to Prandtl's (1904) BL concept and the asymptotic (inner-matching-outer) theory initiated by Kaplun (1954, 1957) and Kaplun and Lagerstrom (1957).

For a pertinent exposition of the so-called "Method of Matched Asymptotic Expansions", (MMAE), which has been briefly discussed in the §3.2 (Chapter 3), see the book by Lagerstrom (1988). In Chapter 7 (see: §7.1 and §7.2), the reader can find an asymptotic derivation of the BL model equations, of the first and second order, relative to viscous small parameter: $\varepsilon=(1/Re)^{1/2}$, according to the MMAE.

4.4.1. Prandtl BL equations

Hopefully, the thickness of this BL is zero in the limit $Re\uparrow\infty$. In its simplest form, for 2D steady (x and y are dimensionless space variables) incompressible slightly viscous flow, this theory assumes that the inviscid flow has been correctly calculated, and gives rise to a slip (horizontal, dimensionless) velocity $u_e(x)$ on the body, and on the body $v_e(x)=0$ which is, in fact, a matching condition. Below, we assume that the body is defined, for our purposes here, by $y^*=0$.

This slip velocity is reduced to zero on the body in a layer of dimensional thickness $O(L/Re^{1/2})$. A set of dimensionless equations for this thin layer is obtained [from the dimensionless 2D steady Navier system (4.16) when: $\nabla=(\partial/\partial x,\partial/\partial y)$ and $\boldsymbol{u}=(u,v)$] by scaling the velocity component v (normal to the body) and y by $Re^{-1/2}$, so that, if (x, y) denote dimensionless distances along and normal to the body and (u, v) the corresponding velocity components, we write the following change of variables and functions (with dimensionless quantities):

$$\eta=\frac{y}{Re^{-1/2}}\,,\ \xi\equiv x\,,\ V=\frac{v}{Re^{-1/2}}\,, \tag{4.40a}$$

$$U \equiv u \text{ and } P = \pi, \tag{4.40b}$$

which are consistent with the boundary-layer theory. In this case for *U, V* and *P,* as functions of ξ and η, the steady (when Strouhal number $S = 0$) 2D dimensionless Navier system (4.16), reduces (under the inner limit (4.42)) to Prandtl (1904) 2D, BL (dimensionless) equations; namely:

$$\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0, \tag{4.41a}$$

$$U\frac{\partial U}{\partial \xi} + V\frac{\partial V}{\partial \eta} = -\frac{dp_e(x)}{dx} + \frac{\partial^2 U}{\partial \eta^2}, \tag{4.41b}$$

with

$$\frac{\partial P}{\partial \eta} = 0 \Rightarrow P = p_e(x), \tag{4.41c}$$

when we consider the following inner limit:

$$lim^{Re} = [Re \uparrow \infty, \text{ with: } \xi, \frac{y}{Re^{-1/2}} \text{ and } \frac{v}{Re^{-1/2}}, \text{ fixed, when } S = 0]. \tag{4.42}$$

Thus: *the pressure is constant across the boundary layer* and given by its value just outside, as in:

$$\frac{dp_e}{dx} = -u_e\frac{du_e}{dx}, \tag{4.41d}$$

which is, in fact, the classical Bernoulli relation, for a potential flow, written in dimensionless form. The corresponding boundary conditions, for the BL equations (4.4la, b) and for an impermeable solid flat wall, are:

$$U = V = 0, \text{ at } \eta = 0, \text{ and } U \to u_e, \text{ at } \eta \to \infty, \tag{4.43}$$

and the behaviour at infinity when $\eta \to \infty$, is also a matching condition with the outer Euler inviscid fluid flow. In (4.41b) and (4.43), p_e and u_e are the value of p_E and u_E for $y = 0$, for the associated Euler steady 2D incompressible 'outer' equations. From a rigorous mathematical point of view it is necessary to assume also that:

$$U = 0, \text{ for } \eta > 0, \text{ at the forward stagnation point } \xi = \xi_0. \tag{4.44}$$

It is now known that under these conditions the solution of the Prandtl BL equations (4.41a, b) *exists* in a region: $\xi_0 < \xi < \xi_1$, if $\partial u_e/\partial x \geq 0$, according to Oleinik (1963), and it is *unique* as long as: $U > 0$, according to Nickel (1973).

Finally, we note, again, that the relation and conditions:

$$\frac{dp_e(x)}{dx} = u_e \frac{du_e}{dx},\ v_e(x) = 0\,,\ U \to u_e\,,\ \text{at } \eta \to \infty\,, \tag{4.45}$$

are, in fact, matching conditions between the Euler (outer) flow and the Prandtl BL (inner) flow.

4.5. THE ISOCHORIC EQUATIONS FOR STRATIFIED NONHOMOGENEOUS FLUID FLOWS

If we perform in the Euler equations (4.7), (4.9), the following (*isochoric*) limit process:

$$\gamma \text{ tends to infinity ("incompressible" limit)}, \quad \text{with } C_p \equiv R = O(1) \text{ such that } C_v \downarrow 0, \tag{4.46}$$

then we derive, first, from the adiabaticity equation (4.9), the following evolution equation (conservation law) for the limit density ρ_i, in place of the conservation of the specific entropy S_E:

$$S\frac{D\rho_i}{Dt} = 0\,, \tag{4.47a}$$

and as a consequence, from the first equation of (4.7), we derive the incompressibility condition, namely:

$$\nabla \cdot \boldsymbol{u}_i = 0\,. \tag{4.47b}$$

Thus, for an inviscid, incompressible but inhomogeneous (a so-called, "isochoric") fluid flow we derive the following system of three equations for the velocity $\boldsymbol{u}_i$, pressure p_i and density ρ_i:

$$\rho_i S\frac{D\boldsymbol{u}_i}{Dt} + \frac{1}{M^*}\nabla p_i + \frac{Bo}{M^*}\rho_i \boldsymbol{k} = 0\,,$$

$$S\frac{D\rho_i}{Dt} = 0, \tag{4.48}$$

$$\nabla \cdot \boldsymbol{u}_i = 0.$$

More precisely, the isochoric approximate equations (4.48) are consistent only in the framework of the 'incompressible limit' (4.46), when the small Mach number is such that the following similarity relation:

$$\gamma M^2 = M^* = O(1), \tag{4.49}$$

holds. For the above system of isochoric inviscid equations (4.48) it is necessary to impose as initial conditions (with $\rho_i°(x) > 0$):

$$t = 0: \quad \boldsymbol{u}_i = \boldsymbol{u}_i°(\boldsymbol{x}) \text{ and } \rho_i = \rho_i°(\boldsymbol{x}). \tag{4.50}$$

It is important to note that for the isochoric, divergence-free flow it is necessary that:

$$\text{on the boundary integral: } \int \boldsymbol{u}_i \cdot \boldsymbol{n} d\Omega \text{ vanish} \tag{4.51a}$$

and

$$\nabla \cdot \boldsymbol{u}_i° = 0. \tag{4.51b}$$

Naturally, this last condition (4.51b) has no analogue for compressible fluid flows because of the occurrence of the term $S\partial\rho/\partial t$ in the continuity equation (2.56a).

We note also that the divergenceless initial data $\boldsymbol{u}_i°(\boldsymbol{x})$, in (4.50), for the isochoric equations (4.48), may be determined in a consistent way only by an *acoustic initial-value adjustment* (inviscid) *problem* valid near $t = 0$, where the full exact initial data (2.31) for the compressible baroclinic Euler equations are imposed.

4.6. THE CASE OF AN EXPANSIBLE LIQUID

For an *expansible/dilatable liquid,* if we take into account the specifying equations (2.25) then, from the NS-F equations (2.30a, b, c), we derive the following three equations for the unknowns $\boldsymbol{u}_i$, $i = 1, 2, 3$, p and T, written with physical (dimensional) quantities:

$$div\,\boldsymbol{u} = -\frac{D[Log\,\rho(T)]}{Dt}, \tag{4.52a}$$

$$\rho(T)\frac{Du_i}{Dt} + \frac{\partial p}{\partial x_i} = -\rho(T)g\delta_{i3} - \frac{\partial}{\partial x_i}\left[\lambda(T)\frac{D(Log\,\rho(T))}{Dt}\right]$$
$$+\frac{\partial}{\partial x_j}\left[\mu(T)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right], \tag{4.52b}$$

$$\rho(T)C(T)\frac{DT}{Dt} - p\frac{D[Log\,\rho(T)]}{Dt} = \frac{\partial}{\partial x_i}\left[k(T)\frac{\partial T}{\partial x_i}\right]$$
$$+\lambda(T)\left[\frac{D[Log\,\rho(T)]}{Dt}\right]^2 + \frac{\mu(T)}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]^2. \tag{4.52c}$$

In Chapter 10, the reader can find a dimensionless form of the above equations (4.52a, b, c), with associated boundary conditions for the Bénard thermal convection problem, when the relation (4.55), for the density, is taken into account as a consequence of (4.53) and (4.54). Frequently encountered form for the density $\rho(T)$ in a liquid is:

$$\rho(T) = \rho(T^\circ)\,[1 - \alpha(T - T^\circ)], \tag{4.53}$$

where α is the constant *coefficient of cubical expansion* and for a typical liquid used in experiments: $\alpha \approx 5/10^4\ K^{-1}$. In the case of the 'Bénard thermal convection problem' for an expansible liquid at rest between two infinite horizontal planes at different constant temperatures:

$$T = T^\circ \text{ on upper plane and } T \equiv T_W = T^\circ + \Delta T^\circ \text{ on lower plane,}$$

considered by Rayleigh, we can introduce the temperature perturbation :

$$\theta = (T - T^\circ)/\Delta T^\circ, \tag{4.54}$$

with $\Delta T^\circ > 0$, and in this case we can write the following relation between ρ/ρ_0 and θ:

$$\frac{\rho}{\rho^{\circ}} = 1 - \beta\theta, \tag{4.55}$$

where $\rho^{\circ} = \rho(T^{\circ})$ and $\beta = \alpha \Delta T^{\circ}$.

When $\beta << 1$, it is possible asymptotically to derive rigorously the so-called *Oberbeck-Boussinesq* approximate equations for an expansible viscous liquid and formulate asymptotically the Rayleigh-Bénard shallow convection problem (see §10.3 in Chapter 10). On the other hand, the so-called 'Bénard- Marangoni' problem (considered in §10.5 of Chapter 10) also emerge asymptotically from the full Bénard convection problem when we take into account the Marangoni effect on the free-surface.

4.7. THE BOUSSINESQ INVISCID ATMOSPHERIC EQUATIONS

4.7.1. The standard atmosphere

For a further analysis, of atmospheric motions, it is very useful to postulate the existence of a so-called 'standard atmosphere' (which is assumed to exist on a day to day) in the form of a basic thermodynamic reference situation (designated by: p^*, ρ^*, T^*) and the latter will be solely a function of the standard altitude, denoted by z^*. In fact, if the relative velocities are small, then the "true" atmospheric pressure will be only slightly disturbed from the basic static value $p^*(z^*)$, defined by the relations (written with dimensions):

$$\frac{dp^*}{dz^*} + g\rho^* = 0 \text{ and } \rho^*(z^*) = \frac{p^*(z^*)}{RT^*(z^*)}, \tag{4.56}$$

where $T^*(z^*)$ is assumed to be known (in the case of the adiabatic motion). More precisely the standard temperature gradient (in the adiabatic case):

$$\Gamma^*(z^*) = -\frac{dT^*}{dz^*}, \tag{4.57}$$

which characterizes, with (4.56), the standard atmosphere, is assumed to be known. The basic standard state is assumed to be known, although its determination from first principles of the thermodynamics requires the consideration of the mechanism of the radiative transfer in the atmosphere [see, for instance, §1.4 of Kibel's (1963) book for a pertinent discussion of this problem].

4.7.2. The Euler inviscid atmospheric equations

To describe the true atmospheric motions, which represent departures from the static standard state, we introduce the perturbation of pressure π, the perturbation of density ω and the perturbation of temperature θ, defined by the relations:

$$p = p^*(z^*)\,(1 + \pi),\ \rho = \rho^*(z^*)\,(1 + \omega),\ T = T^*(z^*)\,(1 + \theta). \tag{4.58}$$

In this case, we can write (in place of the Euler equations (4.7)) the following dimensionless, *exact Euler inviscid atmospheric equations,* if we neglect the Coriolis acceleration and use the same notation, for simplicity, for the dimensionless velocity components (noted by: *u,* v, *w*) and perturbations θ, π, ω:

$$S\frac{D\omega}{Dt} + (1+\omega)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = (1+\omega)\frac{Bo}{T^*(z^*)}\left[1 - \Gamma^*(z^*)\right]w; \tag{4.59a}$$

$$(1+\omega)S\frac{Du}{Dt} + \left[\frac{T^*(z^*)}{\gamma M^2}\right]\frac{\partial \pi}{\partial x} = 0; \tag{4.59b}$$

$$(1+\omega)S\frac{Dv}{Dt} + \left[\frac{T^*(z^*)}{\gamma M^2}\right]\frac{\partial \pi}{\partial y} = 0; \tag{4.59c}$$

$$(1+\omega)S\frac{Dw}{Dt} + \left[\frac{T^*(z^*)}{\gamma M^2}\right]\frac{\partial \pi}{\partial z} - (1+\omega)\frac{Bo}{\gamma Ma^2}\theta = 0; \tag{4.59d}$$

$$(1+\omega)S\frac{D\theta}{Dt} - \frac{(\gamma-1)}{\gamma}S\frac{D\pi}{Dt} + (1+\pi)\frac{Bo}{T^*(z^*)}\left[\frac{(\gamma-1)}{\gamma} - \Gamma^*(z^*)\right]w = 0 \tag{4.59e}$$

with

$$\pi = \omega + (1 + \omega)\,\theta, \tag{4.59f}$$

according to (4.58) and when we assume that (as dimensional reference thermodynamic functions): $p_c = p^*(0)$, $\rho_c = \rho^*(0)$, $T_c = T^*(0)$, and $L_c \equiv H_c$.

But, we observe that in the above equations (4.59) we have (since we work with dimensionless quantities) $T^*(0) \equiv 1$, but $\Gamma^*(0)$ is different from zero. It is necessary to note also that, $H^* = RT^*(0)/g$ is a characteristic length scale for the standard altitude z^*.

As a consequence for the dimensionless (denoted by ') standard altitude $z^{*\prime}$ we can take

$$z^{*\prime} = \frac{z^*}{H^*}. \tag{4.60a}$$

But, for the altitude z of the motion, we write obviously:

$$z' = \frac{z}{H_c}, \tag{4.60b}$$

and with (4.60a, b) we derive the following dimensionless relation between $z^{*\prime}$ and z' (since clearly: $H^* z^{*\prime} \equiv H_C z'$):

$$z^{*\prime} = \frac{H_c}{H^*} z' \equiv Bo\, z', \tag{4.61}$$

with

$$Bo = \frac{gH_c}{RT^*(0)} \equiv \frac{\gamma M^2}{Fr_{H_c}{}^2}, \tag{4.62}$$

if the Mach number, M, is formed with the characteristic value of the speed of sound: $a_c = (\gamma RT^*(0))^{1/2}$, formed with the reference (dimensional) temperature in the standard atmosphere at $z^* = 0$.

Above, the Boussinesq number [introduced in Zeytounian (1990, Chapter 8)], which emerges in a very natural way in the equations (4.59) and the relation (4.61), plays a fundamental role for an asymptotic derivation of the inviscid Boussinesq model equations.

First, as a consequence of (4.61), we have in dimensionless form the following relation for the derivatives in the altitude direction (without '):

$$\frac{dz^*}{dz} = Bo. \tag{4.63}$$

We stress again that the above dimensionless Euler atmospheric equations (4.59a, b, c, d), with (4.59f), for u, v, w, π, ω and θ, are *exact* equations and this remark is important for a consistent asymptotic derivation of the inviscid Boussinesq model equations (see, the equations (4.68)).

4.7.3. The Boussinesq inviscid equations

Now, we consider the above exact Euler atmospheric equations (4.59a)-(4.59e), with (4.59f) and (4.63), when:

$$M << 1 \text{ and } Bo << 1, \tag{4.64}$$

with the following similarity relation,

$$\frac{Bo}{M} = B^* = O(1), \tag{4.65}$$

while t, x, y and z are fixed.

Then we assume that, under the conditions (4.64) and (4.65), the asymptotic solution of the Euler atmospheric equations (4.59) can be supposed to be of the following form:

$$(u, v, w) = (u_0, v_0, w_0) + \dots , \tag{4.66a}$$

$$(\omega, \theta) = M^{\alpha} (\omega_{\alpha}, \theta_{\alpha}) + \dots, \pi = M^{\beta} \pi_{\beta} + \dots, \tag{4.66b}$$

and, after a simple verification it is obvious that the case:

$$\alpha = 1 \text{ and } \beta = 2, \tag{4.66c}$$

is, in fact, the only very significant case. Indeed, when:

$$lim^{Bo} = [M \text{ tends to zero, with } t, x, y, z, \text{ and } S, \gamma, B^* \text{ fixed}], \tag{4.67}$$

from the exact equations (4.59), with (4.65) and (4.66a, b, c), we obtain easily, for limit functions: u_0, v_0, w_0, ω_1, θ_1 and π_2, the following, so-called, *inviscid Boussinesq equations:*

$$S\frac{D_B u_0}{Dt}+\frac{1}{\gamma}\frac{\partial \pi_2}{\partial x}=0\,; \qquad (4.68a)$$

$$S\frac{D_B v_0}{Dt}+\frac{1}{\gamma}\frac{\partial \pi_2}{\partial y}=0\,; \qquad (4.68b)$$

$$S\frac{D_B w_0}{Dt}+\frac{1}{\gamma}\frac{\partial \pi_2}{\partial z}=\frac{B^*}{\gamma}\theta_1\,, \qquad (4.68c)$$

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}=0\,; \qquad (4.68d)$$

$$S\frac{D_B \theta_1}{Dt}+B^*\left[\frac{\gamma-1}{\gamma}-\Gamma^*(0)\right]w_0=0\,; \qquad (4.68e)$$

with

$$\omega_1 = -\,\theta_1, \qquad (4.68f)$$

where

$$S\frac{D_B}{Dt}=S\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y}+w_0\frac{\partial}{\partial z}\,.$$

We note again that in dimensionless form, we have obviously: $T^*(0)\equiv 1$, but in general $\Gamma^*(0)$ is different to zero.

4.7.4. Validity

The above asymptotic derivation of the inviscid Boussinesq equations, when the compressibility plays a role only through the "buoyancy" term: $(B^*/\gamma)\,\theta_1$, in the equation (4.68c), allows us to obtain not only the classical (inviscid) Boussinesq equations but also to define the limits of the approximation through which these equations are derived. Namely, first, for the characteristic vertical height H_c, in fluid motion, we obtain from the similarity relation (4.65), the following relation (4.69):

$$B^*=\frac{Bo}{M}=O(1)\Rightarrow H_c\approx\frac{U_c}{g}\left[\frac{RT^*(0)}{\gamma}\right]^{1/2}\equiv H_B\,. \qquad (4.69)$$

As a consequence, for the usual meteorological values of U_c and $T^*(0)$ we have for H_B the value 10^3 **m** only! This is a strong restriction for the applications of the above Boussinesq equations (4.68) in the atmospheric motions - in particular, for the "lee waves" problem around and downstream of a mountain.

Concerning the Boussinesq approximation for the atmospheric motions, the reader can find in our book, Zeytounian (1990, Chapter 8) a deep discussion of various features of this approximation.

Below, we give some indications concerning the singular nature of the inviscid Boussines qequations, (4.68) near $t = 0$.

4.7.5. Unsteady adjustment problem

Again, the Boussinesq equations (4.68) are not valid in the vicinity of the initial time $t = 0$ (since the *acoustics waves* are *filtered out* of equations (4.68)). As a consequence, it is necessary to consider an unsteady acoustics problem of adjustment to the inviscid Boussinesq state. In Zeytounian (1991, Chapter V), the reader can find the solution of this problem. But the asymptotic solution of this adjustment problem is valid only when we assume, for the exact Euler atmospheric equations, (4.59), with (4.63), as initial conditions:

$$t = 0\text{: } u = u^0,\ v = v^0,\ w = w^0,\ \pi = M\,\pi^0,\ \omega = M\,\omega^0, \text{ and } \theta = M\,\theta^0, \qquad (4.70)$$

where the initial data are given functions of x, y and z. Now, if, with (4.70), we assume for the initial velocity the following form:

$$\boldsymbol{v}^0 = \left(u^0, v^0, w^0\right) = \nabla\phi^0 + \nabla \wedge \psi^0, \qquad (4.71)$$

then, after a transient phase and matching we get, for the inviscid Boussinesq equations (4.68), as initial conditions:

$$t = 0:\ \boldsymbol{v}_0 = \left(u_0, v_0, w_0\right) = \nabla \wedge \psi^0,\ \theta_1 = \frac{1}{\gamma}\pi^0 - \omega^0. \qquad (4.72)$$

The reader should keep in mind that no initial conditions are required on ω_1 and π_2. More precisely, from the Boussinesq equations (4.68), the initial value of π_2 may be computed from the first (4.68a), second (4.68b) or third (4.68c) equations of the above Boussinesq system (4.68) written at $t = 0$, once the initial values of $\boldsymbol{v}_0$ and θ_1 are known (according to (4.72)) and

$$\omega_1 = \omega^0 - \frac{1}{\gamma}\pi^0 \text{, for } t = 0. \tag{4.73}$$

What happens if the initial values for π, ω and θ (solution of the original exact Euler atmospheric equations (4.59)) are different from the data: $M\ \pi^0$, $M\ \omega^0$, and $M\ \theta^0$, assumed in initial conditions (4.70), is not known actually.

4.7.6. Quasi-nondivergent and incompressible equations

Finally, it is interesting to note that if:

$$B^* \to \infty \Leftrightarrow Ma << Bo : Bo \text{ fixed, } M \to 0 \text{ and } Bo \to 0, \tag{4.74}$$

then, in this case we derive, in place of the inviscid Boussinesq equations (4.68), the following so-called "*quasi-nondivergent*" system of equations, which is very degenerate and was been considered first by Monin (see, in his book (1972), §8):

$$w_0 = 0,\ \frac{\partial \pi_2}{\partial z} = 0,\ \theta_1 = 0; \tag{4.75a}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0; \tag{4.75b}$$

$$S\frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + v_0\frac{\partial u_0}{\partial y} + \frac{1}{\gamma}\frac{\partial \pi_2}{\partial x} = 0; \tag{4.75c}$$

$$S\frac{\partial v_0}{\partial t} + u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} + \frac{1}{\gamma}\frac{\partial \pi_2}{\partial y} = 0. \tag{4.75d}$$

On the other hand, if:

$$B^* \to \infty \Leftrightarrow M >> Bo : M \text{ fixed, } Bo \to 0 \text{ and then } M \to 0, \tag{4.76}$$

then, in this case, we derive again, in place of the Boussinesq equations (4.68), the classical "*incompressible*" model equations for u_0, v_0, w_0 and π_2:

$$S\frac{D\theta_1}{Dt} = 0, \tag{4.77a}$$

and

$$\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z}=0$$

$$S\frac{Du_o}{Dt}+\frac{1}{\gamma}\frac{\partial \pi_2}{\partial x}=0;$$

$$S\frac{Dv_o}{Dt}+\frac{1}{\gamma}\frac{\partial \pi_2}{\partial y}=0; \qquad (4.77b)$$

$$S\frac{Dw_o}{Dt}+\frac{1}{\gamma}\frac{\partial \pi_2}{\partial z}=0;$$

with $\theta_1=-\omega_1$, where

$$S\frac{D}{Dt}=S\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y}+w_0\frac{\partial}{\partial z}.$$

So, in both cases, we derive a *less significant* limiting system, from the exact Euler equations (4.59), than the inviscid Boussinesq system of model equations (4.68).

CHAPTER 5

THE NAVIER-FOURIER VISCOUS INCOMPRESSIBLE MODEL

Below, in the figure 5.1, we have schematically sketched the various sub-models which compose the "Navier-Fourier" incompressible global model considered in this Chapter 5.

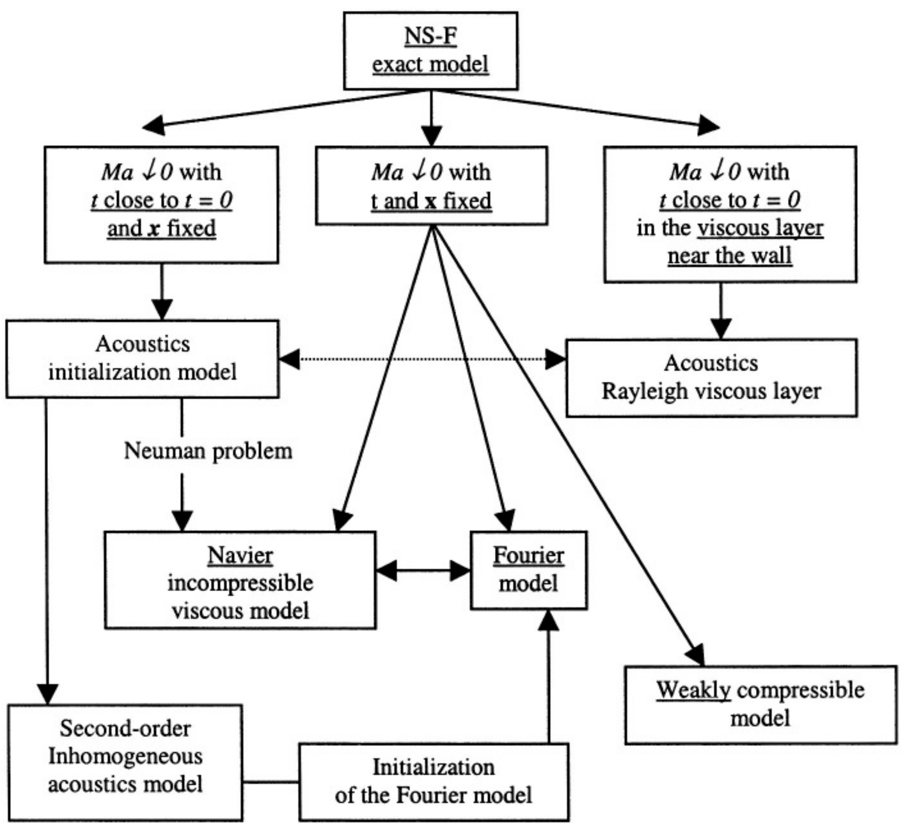


Fig. 5.1 Sketch of the various sub-models which compose the "Navier-Fourier" incompressible global model.

5.1. LOW-MACH NUMBER FLUID FLOWS AND THE NAVIER MODEL

5.1.1. Low-Mach number asymptotics

A typical problem in *external aero/hydrodynamics* is the classical impulsive [or accelered from rest to a finite velocity in a time $O(M)$] motion of a bounded body in an unbounded medium. If the Coriolis force and gravitational acceleration are both neglected, then the main challenge is the derivation of a well-posed initial-boundary value *Navier unsteady main model problem* for an incompressible and viscous fluid flow. For this it is necessary to resolve the problem of the "*initialization*" of the Navier velocity vector (to find an initial condition at $t = 0$). This problem is strongly related to the behaviour (in time) of the solution of associated linear, homogeneous, local equations of acoustics (valid near $t = 0$), when the short acoustic time: $\tau = t/M$ tends to infinity. In other words, it is necessary, in the framework of matched asymptotic 'outer-inner models', to resolve a so-called '*unsteady acoustic adjustment problem'*. Thanks to Wilcox's (1975) scattering theory, for the d'Alembert (acoustic wave) equation in an exterior domain, this above-mentioned unsteady acoustic adjustment problem has recently been resolved by Zeytounian (2000) and a *Neumann problem* for a Laplace equation has been derived, which makes it possible to determine this initial condition for the Navier velocity vector.

An another important, 'companion problem' (for various applications), is the elucidation of the role of the temperature and the *weak* compressibility, for low Mach numbers ($M << 1$). Namely, it is necessary to derive, first, the so-called *Fourier* initial-boundary value problem for the temperature pertubation, which is associated with the Navier model problem, and then take into account the *influence* of this Fourier initial-boundary value problem on the second-order (dynamic) model problem for the velocity (of order $O(M^2)$) and 'pseudo-pressure' (of order $O(M^4)$).

For the external aero/hydrodynamics, the Mach number also plays a peculiar role *far from the body,* for any finite time - indeed, the process of going to infinity cannot be exchanged without caution with the process of letting the Mach number go to zero. In this case, again, the flow is approximated by an acoustic field and this is a basic process in the pionnering work of Lighthill (1952, 1954) on the *generation of sound by turbulence.* But the singular nature of the Mach number expansion which requires *two matched asymptotic expansions* was recognized first, sixteen years later by Lauvstad (1968), and more thoroughly discussed by Crow (1970), Viviand (1970) and Obermeier (1977). As a consequence, matching with the Navier incompressible and viscous limit model is a necessary step for the derivation of a 'three-region' (initial, Navier and external) consistent asymptotic model. Further comments on the far flow may be found in Zeytounian and Guiraud (1984) and also in the recent thesis by Sery-Baye (1994) and in §8.3 of Chapter 8, this emergence of the acoustics at large distances is investigated. The consistency of the MMAE (in relation to the far field) was extensively investigated by Leppington and Levine (1987) and by Tracey (1988), and pushed up to $O(M^6)$ in the inner region, checking consistency with the outer one, in Sery Baye's thesis (1994). As a matter of fact, this problem of the far flow is not a purely academic one, since any numerical simulation, in external aero/hydrodynamics must work on a bounded computational grid and is face the problem of choosing appropriate boundary conditions to be enforced on the external boundary of this grid. The main goal is then to devise the so-called *transparent boundary conditions* which do not pollute the computations and Halpern (1991) has developed a general method for deriving such transparent boundary conditions, and she finds that these conditions have to be non-local in time, of the pseudo-differential (integral) type, which may often be replaced by a local approximation.

Concerning the case of the *internal aero-acoustics,* a very interesting (but difficult) problem, with application to the slow *gas motion into a piston engine,* is related to the consistent derivation of a model flow problem which occurs within a *closed container,* the walls of which deform (with time) very slowly in comparison with the speed of sound. In this case, (see §8.4 in Chapter 8) unfortunately, the usual naive matching with the initial acoustic stage of motion is not (always) possible because of the *persistence* of acoustic oscillations for (dimensionless) time $t = O(1)$ and the 'divergence' of the corresponding unsteady adjustment problem when the motion of the wall is started impulsively, or accelered to a finite velocity in a time $O(M)$, from rest. Even a double-time scale analysis is insufficient, as has recently been assumed by Müller (1998), and it is necessary to 'imagine' a more complicated multiple-scale technique - the gas motion within the closed container being considered as a *superposition of acoustic oscillations, depending on an infinity (but enumerable) of fast times, and of an averaged flow depending on the slow time t.* Because of the persistence of acoustic oscillations, a difficult problem is the *long lifetime* of the oscillations generated within the closed container by the implulsive starting process and how one can predict their evolution when the container has been deformed *a great deal from its original* shape. For the case of an inviscid perfect gas, this analysis has been carried out by Zeytounian and Guiraud in a short Note (1980). If we deal with a slightly viscous flow, we must start from the Navier-Stokes (at least compressible and isentropic) equations, in place of the Euler inviscid, adiabatic, compressible equations (as in Zeytounian and Guiraud (1980)), and bring into the analysis a second small parameter (*1/Re*), the inverse of a characteristic Reynolds number. Then we must expect that the oscillations are *damped* out after a sufficiently long time, but a precise analysis of this damping phenomenon appears to be a difficult problem and raises many questions - nevertheless, the case (apparently the more realistic for the applications) when $Re\ M >> 1$ is resolvable via a boundary layer analysis. In relation to the asymptotic modelling of low-Mach number flow in a closed container, we mention here that Rhadwan and Kassoy (1984) investigate the acoustic response due to boundary heating in a confined (between two infinite parallel planar walls) inert gas. For this purpose a classical asymptotic analysis is employed (via matching) to get a better understanding of the low-Mach number limit of the (compressible) one-dimensional (t, x) Navier-Stokes equations and to derive simplified equations, which account for the net effect of periodic acoustic waves on slow flow over a long time - the results provide an explicit expression for the piston analogy of boundary heat addition. Finally, among various papers devoted to derivation of low-Mach number model equations, we mention first the three recent papers by Zank and Matthaeus (1990, 1991, 1993), and the paper by Bayly, Levermore and Passot (1992) and Ghosh and Matthaeus (1992), which are devoted to so-called "*nearly incompressible hydrodynamics*". As a matter of fact the terminology 'hydrodynamics' is a rather hybrid one which describes the overlap of compressible and incompressible models for fluid flows in the low subsonic regime. Indeed, on the basis of a singular expansion technique, these papers derive modified systems of equations for fluid flows in which the effects of compressibility are taken into account approximately in some terms. On the other hand, for the low-Mach number limit of the Navier-Stokes (compressible) equations, Fedorchenko (1997) provides a number of *exact solutions,* and in Matalon and Metzener (1997), via a low-Mach number expansion, the authors derive a system of approximate equations for the propagation of premixed flames in closed tubes. The properties of a relatively uncommon regime of fluid dynamics: low-Mach number *compressible flow* (!) are investigated by Shajii and Freidberg (1996). In the habilitation thesis of Mûller (1996) and also in recent Doctoral thesis of Viozat (1998) the reader may be find various recent *numerical computations* of compressible low-Mach number flows. The numerical simulation of low-Mach number flows has been investigated in many papers, and in both of the above mentioned thesis, the reader may find a large number of references. Concerning more especially the computational aero-acoustics for low-Mach number flows, we mention the book edited by Hardin and Hussaini (1993). Finally, we note that the paper by Dwyer

(1990) is devoted to calculation of low-Mach number reacting flows. At the end of Chapter 8, the reader can also find various references concening low-Mach number asymptotics, which are related to acoustics.

From a *mathematical point of view,* it is important to note that the filtering of acoustic waves in the compressible fluid flow equations changes drastically the type of these equations and often initially well-posed fluid flow problems become "ill-posed" [see, for instance, the paper by Oliger and Sundström (1978) and the book by Kreiss and Lorenz (1989)]. Concerning the more recent rigorous mathematical results on the singular incompressible and acoustics limits in slightly compressible fluid dynamics, see the papers by: Lions (1993), Beirao DA Veiga (1994), Iguchi (1997), Lions and Masmoudi (1998), Hagstrom and Lorenz (1998) and the more recent paper by Desjardins, Grenier, Lions and Masmoudi (1999). Unfortunately, in these mathematically rigorous papers, the mathematical analysis is carried out almost entirely from the consideration of equations without any boundary and initial conditions corresponding to a physical fluid dynamics problem! But, the experienced reader is aware of the fact that the fluid dynamical differential equations are not sufficient for discussing flow problems, and this seemingly anodyne remark has far-reaching consequences. In particular, Applied Mathematicians do not clearly make the distinction and not properly understand the difference between the external and internal aerodynamics problems. Obviously the behaviour of the acoustic waves generated in the initial stage of the motion in a bounded domain is very complicated and the concept of the initial acoustic layer is not adapted in this case [such a initial-layer has been recently formally considered by Iguchi (1997)]. The low-Mach number asymptotics is used also by Kreiss, Lorenz and Naughton (1991) for the derivation of the Navier incompressible equations, and to prove the convergence of the compressible flow solution to the incompressible-flow solution for $M \downarrow 0$ under certain conditions.

5.1.1a. NS-F governing equations

Below, as, a starting point we consider the dimensionless full NS-F equations (2.56a, b, c), with (2.56d), with $S \equiv 1$ and $Bo \equiv 0$, for the velocity vector $\boldsymbol{u} = (u_i,\ i = 1, 2, 3)$, the density ρ, the pressure p and the temperature T, as function of time t and vector position $\boldsymbol{x} = (x_i,\ i = 1, 2, 3)$. Namely, the following dimensionless equations:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{u}) = 0; \tag{5.1a}$$

$$\rho\frac{D\boldsymbol{u}}{Dt}+\frac{1}{\gamma M^2}\nabla p=\frac{1}{Re}\nabla\cdot\boldsymbol{T}\ ; \qquad (5.1b)$$

$$\rho\frac{DT}{Dt}+(\gamma-1)p(\nabla\cdot\boldsymbol{u})=\frac{\gamma}{Pr\,Re}\nabla\cdot[k(T)\nabla T]+\gamma(\gamma-1)\frac{M^2}{Re}\Phi(\boldsymbol{u}) \qquad (5.1c)$$

where

$$\boldsymbol{T}=\lambda(T)(\nabla\cdot\boldsymbol{u})\boldsymbol{I}+2\mu(T)\boldsymbol{D}, \qquad (5.2a)$$

$$\Phi(\boldsymbol{u})=2\mu(T)Tr\left[\boldsymbol{D}^2\right]+\lambda(T)(\nabla\cdot\boldsymbol{u})^2, \qquad (5.2b)$$

$$Tr\left[\boldsymbol{D}^2\right]\equiv\boldsymbol{D}:\boldsymbol{D}=\frac{1}{4}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]^2, \qquad (5.2c)$$

and

$$p=\rho T. \qquad (5.3)$$

5.1.1b. Dominant NS-F equations for the external problem

Now we are concerned exclusively, with the so-called external problem, for which the thermodynamic functions, p, ρ and T, tend to constant values at infinity, namely: p_∞, ρ_∞, T_∞, and we use those values to obtain dimensionless thermodynamic quantities.

In such a case, we can write:

$$p=1+\pi,\ \rho=1+\omega,\ T=1+\theta. \qquad (5.4)$$

We make the choice that the velocity is zero at infinity, and that the gas is set into motion by the displacement of some body. It is well known from inviscid, external aerodynamics, that the density remains constant up to $O(M^2)$. Consequently, we expect ω to remain small, of order M^2, and, provided θ remain small, π will also remain small. The physics tells us that the temperature of the fluid will change by conduction, with two main sources of heating (or cooling in the second case). The first, which acts always as heating effect, is due to dissipation by friction, and is expected to go like M^2, as an order of magnitude. The second source, which may be heating-like or cooling-like, is due to the temperature of the body immersed within the fluid. We shall be concerned only with situations according to which θ remains small. For both viscosities and heat conductivity we set:

$$\mu = \mu(\theta),\ \mu_V = \mu_V(\theta),\ k = k(\theta) \tag{5.5}$$

and we follow the convention that: $\mu(0) = \mu_v(0) = k(0) = 1$. Of course, the functions $\mu(\theta)$, $\mu_v(\theta)$, $k(\theta)$ are consistent with the (5.4) for the temperature T, and, as a consequence, they are dimensionless, as well as are all their derivatives with respect to θ. We are now ready to state the dominant dimensionless NS-F equations, namely:

$$\frac{\partial \omega}{\partial t} + \boldsymbol{u}\cdot\nabla\omega + \nabla\cdot\boldsymbol{u} + \omega\nabla\cdot\boldsymbol{u} = 0; \tag{5.6}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{\gamma M^2}\nabla\pi + \omega\left[\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right]$$
$$= \frac{1}{Re}\left[\nabla^2\boldsymbol{u} + \left[\frac{1}{3} + \sigma^\circ\right]\nabla(\nabla\cdot\boldsymbol{u}) + \nabla\cdot[N(\boldsymbol{u})\theta]\right] + O(\theta^2); \tag{5.7}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta + (\gamma-1)\nabla\cdot\boldsymbol{u} + \omega\left[\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta\right] + (\gamma-1)\pi(\nabla\cdot\boldsymbol{u})$$
$$= \frac{\gamma}{Pr\,Re}\nabla^2\theta + (\gamma-1)\frac{\gamma M^2}{Re}\left\{\boldsymbol{D}(\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{u}) + \left[\sigma^\circ - \frac{2}{3}\right](\nabla\cdot\boldsymbol{u})^2\right\}$$
$$+ (\gamma-1)\frac{\gamma M^2}{Re}\left[\frac{d\mu}{d\theta} - \sigma^\circ\frac{d\mu_v}{d\theta}\right]_{\theta=0}(\nabla\cdot\boldsymbol{u})^2\theta + O(\theta^2); \tag{5.8}$$

$$\pi = \omega + \theta + \theta\,\omega. \tag{5.9}$$

We emphasize that these above equations are not the exact ones, as indicated by $O(\theta^2)$ occurring in (5.7) and (5.8). As a matter of fact, we are going to investigate the asymptotic structure of these equations, (5.6) to (5.9) when M is small, and, as we shall see, in general, the error goes like $O(M^2)$ so that the neglected terms in (5.7) and (5.8) will have no effect on the analysis to come. But it is necessary to note that these above dominant NS-F equations (5.7) and (5.8) are a direct consequence (for small θ) of the full exact NS-F governing equations for a perfect gas with constant specific heat

(5.1b) and (5.1c), with (5.2a)-(5.2c), when we take into account (5.4) and (5.5). We emphasize that the term involving $N(u)$ arises from the variation of the viscosities with temperature:

$$N(u)=\left[\frac{d\mu}{d\theta}-\sigma^{\circ}\frac{d\mu_v}{d\theta}\right]_{\theta=0}(\nabla\cdot u)I+2\left(\frac{d\mu}{d\theta}\right)_{\theta=0}D(u). \tag{5.10}$$

A similar effect is expected to arise from the variation of the heat conductivity with the temperature but it would be included in the neglected $O(\theta^2)$ term in the equation (5.8) for θ. Taking into account the state equation (5.9), we realize that we face three unknowns: ω, θ and u, for which we hold three equations, (5.6) to (5.8) if we substitute in (5.7) and (5.8), π by: $\omega+\theta+\theta\,\omega$, from which we may solve for $\partial\omega/\partial t$, $\partial\theta/\partial t$, $\partial u/\partial t$, and this suggests, from what is known about partial differential equations, that we should impose one initial condition for each quantity, thus allowing us to integrate forwards in time.

5.1.1c. Initial, boundary and 'at infinity' conditions

Here we make the choice that the fluid starts from a state of rest, at constant density and temperature, so that we get:

$$\omega=0,\ \theta=0 \text{ and } u=0, \text{ at } t=0, \tag{5.11}$$

in the whole of the domain occupied by the fluid. We assume that the fluid is set into motion by the displacement (and eventually the deformation) of a body, the fluid pervading the entire domain Ω, complementary to this body. We set $\Gamma(=\partial\Omega)$ to be the boundary of Ω, and n the unit vector normal to Γ, pointing towards the fluid. The spatial derivative of ω in (5.6) enters only in combination with the derivative $\partial\omega/\partial t$, via $D\omega/Dt$, and the knowledge of partial differential equations suggests that no condition has to be enforced for ω on Γ, except if some mass of fluid were transpiring from the inside of the body through Γ. We rule out this possibility so that we need no condition relative to ω on Γ. On the other hand we need one condition for u and one for θ on Γ. Concerning u we make the usual assumption that the fluid adheres to the wall (Γ) and this amounts to stating that at each point of the wall we attach a dimensionless velocity U_W, depending on the time and on the position P on Γ such that:

$$\boldsymbol{u} = \boldsymbol{U}_W(t, \boldsymbol{P}) \equiv H(t)\, \boldsymbol{u}_W(\boldsymbol{P}), \text{ all along } \Gamma, \tag{5.12}$$

where $H(t)$ is the so-called Heaviside (or unit) function.

This means that we focus our interest on situations when the body starts its motion impulsively. Of course this is not very realistic because it means that one should impart, at time $t = 0$, an infinite impulse to the body, in order to realize such an impulsive motion! A poor justification, for working this way, is that this is a classical problem in inviscid incompressible fluid dynamics and that it is worthwhile trying to elucidate the behaviour of the motion for the NS-F equations. Perhaps a more convincing argument is that, through (5.12) we are somewhat mimicking a catapulting process.

Concerning θ we use the following dimensionless boundary condition (see, for instance, (2.36)):

$$\kappa^{\circ}\frac{\partial \theta}{\partial n} + Bi\,\theta = \theta_W, \text{ all along } \Gamma, \tag{5.13}$$

where Bi and κ° are dimensionless numbers (Bi is a Biot number), while θ_W is a known function of time and position on Γ.

We remind the reader that (5.13) simply means that the heat flux from the fluid through Γ goes inside the body at a rate which is proportional to the difference between the, actual, temperature of the wall and a given temperature. This temperature condition (5.13) is, in fact, a 'third type' condition, which is a mixture of Dirichlet (corresponding to $\kappa^{\circ} \to 0$) and Neumann (when $Bi \to 0$) conditions. The justification for such a condition (5.13) relies on the assumption that heat conduction, within the body is so much faster than within the fluid, that the heat flux on Γ, considered from the inside of the body, may be approximated by such a difference in temperatures. For a more detailed derivation of the condition (5.13), see the book by Joseph (1976).

Being concerned with the motion of the fluid in a domain extending to a neighbourhood of infinity, we need some conditions relative to this limit. It seems obvious that we require:

$$U \equiv (\omega, \theta, \pi, |\boldsymbol{u}|) \to 0, \text{ when } |\boldsymbol{x} - \boldsymbol{x}_0(t)| \to \infty, \tag{5.14}$$

where $\boldsymbol{x}_0(t)$ is some point inside the body. Of course for any finite time we might simply ask that (5.14) holds when simply $|\boldsymbol{x}| \to \infty$. As a matter of fact, (5.14) raises some difficulties which come from the wake trailing behind the body, but we leave aside this peculiarity. We are rather concerned with what arises when the Mach number is small, and we try to solve our problem by an expansion in powers of the Mach number. Then, when we go far from the body, excluding the region within the wake, we know that the effects of viscosity and heat conductivity die out more rapidly than inviscid effects [indeed, the perturbations are expected to decay like $O(|\boldsymbol{x}-\boldsymbol{x}_0(t)|^2)$]. Then again we know, from classical low Mach number aerodynamics, that the Mach number plays a peculiar role when $|\boldsymbol{x}-\boldsymbol{x}_0(t)| \to \infty$. As a matter of fact when

$$[\boldsymbol{x} - \boldsymbol{x}_0(t)]\, M = \boldsymbol{x}^* = O(1), \tag{5.15}$$

the flow approximates an acoustic field, which is a basic process in the generation of sound by turbulence (see the §8.3 in the Chapter 8).

5.1.2. The Navier incompressible viscous model

We start investigating the effect of letting $M \downarrow 0$ in the dimensionless dominant NS-F equations (5.6)-(5.9). Assuming that $Re = O(1)$, it is not difficult to guess that the following "main" expansion is appropriate for the asymptotic investigation of 'dynamic' equations (5.6) and (5.7):

$$\boldsymbol{u} = \boldsymbol{u}_N + M^2\, \boldsymbol{u}' + .., \quad \pi = M^2\, [\pi_N + M^2\, \pi' + ...]. \tag{5.16}$$

As a matter of fact, from (5.7), we see that $\nabla\pi$ is of the order of M^2, and, observing that π has to go to zero at infinity, the only mechanism which could generate a π of an order differing from M^2, is through matching with an outer expansion. We forget about this, but notice that we shall be bound to check that this matching will not change this order of magnitude estimate. From (5.9) we conclude that

$$\omega + \theta = O(M^2), \tag{5.17}$$

but we don't know how ω and θ behave separately when $M \downarrow 0$. From (5.6) alone we may hardly expect to get anything other than some kind of circular logic, so that we need to use both (5.6) and (5.8), and also (5.17). We shall devote the whole of §5.3 to this question but we give here some indication. From (5.17) we may substitute:

$$\omega = -\,\theta + O(M^2) \tag{5.18}$$

into (5.6) to obtain:

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}_N \cdot \nabla \theta - \nabla \cdot \boldsymbol{u}_N = O(|\theta(\nabla \cdot \boldsymbol{u}_N)|) + O(M^2), \tag{5.19}$$

and we substitute the value of $\nabla \cdot \boldsymbol{u}_N$, extrated from (5.19), into the dominant equation (5.8) leading to:

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}_N \cdot \nabla \theta - \frac{1}{PrRe} \nabla^2 \theta = O\left(\frac{M^2}{Re}\right) + O(M^2) + O(\theta^2). \tag{5.20}$$

We shall consider here that the order of magnitude of θ is given by the right hand side of (5.20), but we should examine more deeply this question and this will be done in the §5.3. For the moment, we simply state, provisionaly, the expansions:

$$\theta = M^2 [\theta_N + M^2 \theta' + ...] \text{ and } \omega = M^2 [\omega_N + M^2 \omega' + ...]. \tag{5.21}$$

Of course, the second of (5.21) is a consequence of the first one and of (5.18) and then we get the desired result that:

$$\nabla \cdot \boldsymbol{u}_N = O(M^2). \tag{5.22}$$

Finally we obtain, for the (Navier) couple: $(\boldsymbol{u}_N, \pi_N)$, the well-known set of two equations for an incompressible and viscous fluid flow:

$$\frac{\partial \boldsymbol{u}_N}{\partial t} + \boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N + \frac{1}{\gamma} \nabla \pi_N = \frac{1}{Re} \nabla^2 \boldsymbol{u}_N, \tag{5.23a}$$

$$\nabla \cdot \boldsymbol{u}_N = 0 \tag{5.23b}$$

which are usually [in the mathematical literature, see, for instance, the recent books by Lions (1996) and also by Doering and Gibbon (1995)] referred to as the Navier-Stokes, but which I prefer here to refer to as the Navier equations, to distinguish them from the system (4.1), (4.2). From (5.12) we have, for the Navier model dynamic equation (5.23a) the following boundary condition (since the time $t > 0$ is fixed when $M \downarrow 0$):

$$\boldsymbol{u}_N = \boldsymbol{u}_W(\boldsymbol{P}), \text{ all along } \Gamma, \text{ when } t > 0, \tag{5.24}$$

while, from the conditions at infinity, we may expect that:

$$|\boldsymbol{u}_N| \to 0\,, \text{ as } |\boldsymbol{x}| \to \infty\,, \tag{5.25}$$

provided this conclusion is not invalidated by a different (outer) asymptotic expansion valid near infinity.

Surprisingly enough, we cannot use the initial condition for the velocity vector, (5.11), to set $\boldsymbol{u}_N(0, \boldsymbol{x}) = 0$! Rather, we have to put:

$$\boldsymbol{u}_N(0, x) = \boldsymbol{u}_N{}^{\circ}(\boldsymbol{x}), \tag{5.26a}$$

and, obviously, $\boldsymbol{u}_N{}^{\circ}(\boldsymbol{x})$ should be divergence free:

$$\nabla \cdot \boldsymbol{u}_N^o = 0\,. \tag{5.26b}$$

It is somewhat puzzling that the initial value of $\boldsymbol{u}_N$ is not zero, as one might expect from (5.11), and on the other hand we don't have, from what has been said, any indication of how $\boldsymbol{u}_N{}^{\circ}(\boldsymbol{x})$ might be obtained. Nevertheless, from (5.12) we have the indication that the body is set impulsively into motion. This problem of impulsive motion has long been known, and the reader will find in Lamb (1932, Section 11), a treatment for an inviscid incompressible liquid.

It is shown there that if the motion of the body is changed impulsively at some time t°, then, at the same time, the velocity, in all the space exterior to the body, is changed suddenly by an amount $[|\boldsymbol{u}|]$, such that (ρ is the density of the liquid):

$$[|\boldsymbol{u}|] = \frac{1}{\rho} \nabla \varpi\,, \tag{5.27a}$$

and ϖ is determined by solving a classical Neumann problem:

$$\nabla^2 \varpi = 0\,,\ \frac{1}{\rho} \boldsymbol{n} \cdot \nabla \varpi = \boldsymbol{n} \cdot [|\boldsymbol{u}_W|], \text{ all along } \Gamma. \tag{5.27b}$$

Here $[|\boldsymbol{u}_W|]$ is the discontinuity in the velocity of the wall of the body, at time t°.

Of course, in our case, we have to deal with an asymptotic model and we should go beyond Lamb's treatment, in order to understand how one goes from the initial value $\boldsymbol{u}(0, \boldsymbol{x}) = 0$, for the NS-F problem, to the initial value

$\boldsymbol{u}_N{}^{\circ}(\boldsymbol{x})$, for the asymptotic Navier model equations (5.23a, b), corresponding to the (main-outer, Navier) limit, and obviously, we assume implicitly that the other parameters are fixed and $O(1)$, when

$$M \downarrow 0, \text{ with } t \text{ and } \boldsymbol{x} \text{ fixed.} \tag{5.28}$$

Indeed, the conclusion (5.22) is valid, only if we assume that $\partial\omega/\partial t$ is of the same order as ω, that is, it is small. In the case when a discontinuity in the velocity occurs at $t = 0$, we may suspect that, close to $t = 0$, for $t > 0$, the order of magnitude of $\partial\omega/\partial t$ is not the same as the order of ω itself.

Through the limit process (5.28), the derivative $\partial\omega/\partial t$ is lost, and, from what we know about asymptotic expansions, this a clue that we need a local-inner expansion, in the vicinity of $t = 0$.

The divergence-free character of $\boldsymbol{u}_N$ *is directly tied to this loss of* $\partial\omega/\partial t$.

This omnipotence of the incompressibility constraint, and its relation with the initial condition, which guarantees the well-posedness of the Navier problem (5.23a, b) to (5.26a, b), is thoroughly discussed in the very pertinent review paper by Gresho (1992, pp. 47 to 52), and, scrutinizing what is stated there, we see that there is a close relation with what may be found in Lamb (1932, Section 11).

We shall come back to this matter shortly at the end of §5.2 below.

5.2. THE INITIALIZATION PROBLEM AND EQUATIONS OF ACOUSTICS

Now Our main goal below is to derive a limiting initial boundary value problem, issuing from $M \downarrow 0$ (but for the time near to $t = 0$ and $\boldsymbol{x}$ fixed), such that the time derivatives in (5.6) and (5.8) remain after obtaining the limiting form of the equations. Due to the impulsive character of the motion of the body, we expect that changes occur within a small interval of time after $t = 0$. Although it is not necessarily obvious, we nevertheless may expect that during this short time interval, ω, θ and π remain all small. On the other hand we should not expect such a behaviour for the velocity $\boldsymbol{u}$, because the smallness of such a velocity, with respect to the speed of sound, has already been taken care of within the non-dimensionalization of the original NS-F equations. If we scrutinize (5.6) and (5.8), we find that the leading terms which are expected to remain near to $t = 0$, after the (inner, in time) limiting process are respectively:

$$\frac{\partial \omega}{\partial t} + \nabla \cdot \boldsymbol{u} + ... = 0\,, \tag{5.29a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{\gamma M^2}\nabla \pi + ... = 0\,, \tag{5.29b}$$

$$\frac{\partial \theta}{\partial t} + (\gamma - 1)\nabla \cdot \boldsymbol{u} + ... = 0\,, \tag{5.29c}$$

From inspection, we guess that the following changes:

$$\tau = \frac{t}{M}, \quad \boldsymbol{u} = \boldsymbol{u}_a, \ (\omega, \ \pi, \ \theta) = M\,(\omega_a, \ \pi_a, \theta_a), \tag{5.30}$$

where the "acoustics" functions $\boldsymbol{u}_a$ and ω_a, π_a, θ_a, depend on τ, $\boldsymbol{x}$ and M, will work. As a matter of fact, it is very easy to check that substituting (5.30) into (5.6) to (5.9) gives, neglecting terms which are $O(M^2)$:

$$\frac{\partial \omega_a}{\partial \tau} + \nabla \cdot \boldsymbol{u}_a + M\left[\boldsymbol{u}_a \cdot \nabla \omega_a + \omega_a \nabla \cdot \boldsymbol{u}_a\right] = 0\,, \tag{5.31a}$$

$$\frac{\partial \boldsymbol{u}_a}{\partial \tau} + \nabla\left(\frac{\pi_a}{\gamma}\right) + M\left\{\omega_a \frac{\partial \boldsymbol{u}_a}{\partial \tau} + \boldsymbol{u}_a \cdot \nabla \boldsymbol{u}_a\right\}$$
$$-\frac{M}{Re}\left[\nabla^2 \boldsymbol{u}_a + \left[\frac{1}{3} + \sigma^\circ\right]\nabla(\nabla \cdot \boldsymbol{u}_a)\right] = O(M^2), \tag{5.31b}$$

$$\frac{\partial \theta_a}{\partial \tau} + (\gamma - 1)\nabla \cdot \boldsymbol{u}_a + M\left[\boldsymbol{u}_a \cdot \nabla \theta_a + \omega_a \frac{\partial \theta_a}{\partial \tau} + (\gamma - 1)\pi_a \nabla \cdot \boldsymbol{u}_a\right]$$
$$-\frac{M\gamma}{PrRe}\nabla^2 \theta_a = O(M^2), \tag{5.31c}$$

$$\pi_a - \omega_a - \theta_a - M\theta_a\,\omega_a = 0. \tag{5.31d}$$

As a consequence, through the local-inner-initial limiting process:

$$M \downarrow 0, \text{ with } \tau \text{ and } x \text{ fixed,} \tag{5.32}$$

we derive the following set of leading-order equations:

$$\frac{\partial \omega_{a,0}}{\partial \tau} + \nabla \cdot \boldsymbol{u}_{a,0} = 0, \tag{5.33a}$$

$$\frac{\partial \boldsymbol{u}_{a,0}}{\partial \tau} + \nabla\left(\frac{\pi_{a,0}}{\gamma}\right) = 0, \tag{5.33b}$$

$$\frac{\partial \theta_{a,0}}{\partial \tau} + (\gamma - 1)\nabla \cdot \boldsymbol{u}_{a,0} = 0, \tag{5.33c}$$

$$\pi_{a,0} = \omega_{a,0} + \theta_{a,0}, \tag{5.33d}$$

which has already been derived in the §4.2 of Chapter 4.

The first consequence of this, is that, because none of the time derivatives have been lost in the local-inner-initial limit process (5.32), is that we may apply the initial conditions (5.11) to the system (5.33) and get:

$$\tau = 0: \; \boldsymbol{u}_{a,0} = 0, \; \pi_{a,0} = \omega_{a,0} = \theta_{a,0} = 0, \tag{5.34}$$

everywhere outside the body. Now we run into a problem, this time with the boundary conditions. The equations (5.33) are the dimensionless form of the equations of (linear) acoustics, in a homogeneous gas at rest. We know that, for those equations, the only condition that might be applied on the boundary is one of slip of the gas with respect to the wall. We have to come back to (5.12) and observe that $H(t) = H(\tau)$, provided $\tau > 0$. As a matter of fact, such a statement necessitates a proof, but we may argue physically, and this will be sufficient for our purpose. Then we get the desired boundary condition:

$$\boldsymbol{u}_{a,0} \cdot \boldsymbol{n} = \boldsymbol{u}_W(\boldsymbol{P}) \cdot \boldsymbol{n} \equiv w_w(\boldsymbol{P}), \text{ all along } \Gamma, \tag{5.35}$$

and we observe that $w_w(\boldsymbol{P})$ does not depend on τ. Let us now leave aside the matter of the boundary conditions which have been lost in the process (5.32) and concentrate on the solution of the acoustics problem (5.33) - (5.35). We observe, first, that due to (5.26b), subtracting $\boldsymbol{u}_N{}^o$ from $\boldsymbol{u}_N$ does not change anything in the set of equations (5.33). It is then very easy to check that the following formulae:

$$u_{a,0} = u_N{}^{\circ}(x) + \nabla\phi_{a,0}(\tau, x), \tag{5.36a}$$

$$\omega_{a,0} = -\frac{\partial\phi_{a,0}}{\partial\tau}, \ \pi_{a,0} = -\gamma\frac{\partial\phi_{a,0}}{\partial\tau}, \tag{5.36b}$$

$$\theta_{a,0} = (1-\gamma)\frac{\partial\phi_{a,0}}{\partial\tau}, \tag{5.36c}$$

solve (5.33), provided $\phi_{a,0}$ be a solution for the dimensionless d'Alembert's equation of acoustics, namely:

$$\frac{\partial^2\phi_{a,0}}{\partial\tau^2} - \nabla^2\phi_{a,0} = 0, \tag{5.37}$$

the speed of sound being replaced by unity, due to the choice made in the process of getting the NS-F equations in the dimensionless form (5.6) to (5.9).

From the initial conditions (5.34) we derive:

$$\tau = 0: \ \nabla\phi_{a,0} = -u_N{}^{\circ}(x), \ \frac{\partial\phi_{a,0}}{\partial\tau} = 0, \tag{5.38}$$

everywhere outside the body, while, from the boundary condition (5.35), we get a boundary condition which (because the only restriction put on $u_N{}^{\circ}(x)$ up to now is (5.26b)) is written in the following form:

$$u_N{}^{\circ} \cdot n = w_W(P), \text{ all along } \Gamma, \tag{5.39}$$

so that the boundary condition for $\phi_{a,0}$ on the body wall is:

$$\frac{\partial\phi_{a,0}}{\partial n} = 0, \text{ all along } \Gamma. \tag{5.40}$$

Since (5.26b) and (5.39) are the only restrictions put on $u_N{}^{\circ}$, we are somewhat short of a condition at infinity, for the complete determination of $\phi_{a,0}$.

But we may get rid of this slight difficulty by setting:

$$\boldsymbol{u}_N{}^\circ = \nabla \psi_N{}^\circ, \tag{5.41}$$

which is allowed by (5.36a), and observing that we may then determine $\psi_N{}^\circ$ through the following problem:

$$\nabla^2 \psi_N{}^\circ = 0, \text{ everywhere outside the body,} \tag{5.42a}$$

$$\frac{\partial \psi_N{}^\circ}{\partial n} = w_W(\boldsymbol{P}), \text{ all along } \Gamma, \tag{5.42b}$$

$$\psi_N{}^\circ \to 0, \text{ when } |\boldsymbol{x}| \to \infty, \tag{5.42c}$$

which is a straightforward *Neumann problem for the Laplace equation.*

Then we get, instead of (5.38),

$$\tau = 0: \ \phi_{a,0} = -\psi_N{}^\circ, \ \frac{\partial \phi_{a,0}}{\partial \tau} = 0, \tag{5.43}$$

everywhere outside the body.

Now, the acoustic wave equation (5.37), with (5.40) and (5.43) lead to a well-posed problem for $\phi_{a,0}$ provided we add:

$$\phi_{a,0} \to 0 \text{ as } |\boldsymbol{x}| \to \infty, \tag{5.44}$$

which amounts to added information, namely that no perturbations come from infinity towards the body, and one must consider that such information is of physical rather than mathematical character. We are not actually interested in getting $\phi_{a,0}$, and all that we want to know is that:

$$\tau \to \infty: \ \phi_{a,0} \to 0, \tag{5.45}$$

which is guaranteed by the mathematical theory of the d'Alembert's equation [see, for instance, Wilcox (1975)].

This provides us with the consequence:

$$\tau \to \infty: \ \boldsymbol{u}_{a,0} \to \boldsymbol{u}_N{}^\circ, \tag{5.46}$$

so that, matching the Navier solution with the present (acoustics) one, we get;

$$lim_{t\to+0}\,\boldsymbol{u}_N(t,\boldsymbol{x}) \sim lim_{\tau\to\infty}\,\boldsymbol{u}_{a,0}(\tau,\boldsymbol{x}) \tag{5.47}$$

and, finally, we have found what was missing for achieving a complete Navier initial, boundary value problem, namely: (5.36a), where $\phi_{a,0}$ is the solution of the acoustics problem, (5.37), (5.40), (5.43) and (5.44), and $\boldsymbol{u}_N{}^0$ is completely determined by (5.42a, b, c) with (5.41).

We note that the paper by Ukai (1986) is devoted to a rigorous mathematical analysis of the incompressible limit and the initial layer of the compressible Euler equations. In this last Eulerian case we derive again in the initial-time layer the classical acoustics equations (5.33a, b, c, d).

As a conclusion, we note, first, that it is clear that properties of the initial data influence the type of "nearly incompressible" flow properties obtained asymptotically at low Mach number. Indeed, the flow with small initial density fluctuations (thus one excludes acoustic waves in the initial data) remains nearly incompressible for the duration of the simulation, and in agreement with nearly incompressible theory the density fluctuations remain scaled to the square of the (small) Mach number. There is no suggestion that the nearly incompressible simulations ever evolve toward strongly compressive scalings.

A second remark concerns the case when the body is set in movement rapidly, during an interval of time proportional to M, or else progressively during a period of time $O(1)$.

In the *first case,* the above result [problem (5.42a, b, c) with (5.41)] concerning the initial (Navier) value $\boldsymbol{u}_N{}^0$ remains true, since, in fact, in this case:

$$\boldsymbol{U}_W(t, \boldsymbol{P}) = \boldsymbol{U}_W(\tau, \boldsymbol{P})$$

and according to matching condition, (5.47), with (5.45) and (5.36a), in place of $w_W(\boldsymbol{P})$ in condition (5.42b), we can write: $\boldsymbol{U}_W(\infty, \boldsymbol{P})\cdot\boldsymbol{n} = W_W(\boldsymbol{P})$.

On the contrary in the *second case* the corresponding Neumann problem, (5.42a, b, c), has only the trivial zero solution, since in this case the displacement velocity of a material point **P** of the boundary Γ, of the body Ω,

$$\boldsymbol{U}_W(t, \boldsymbol{P}) = \boldsymbol{U}_W(M\tau, \boldsymbol{P}) \to \boldsymbol{U}_W(0, \boldsymbol{P}) \equiv 0,$$

when $M \downarrow 0^+$ with τ fixed [local-inner limit (5.32)], and as consequence $w_W \equiv 0$, in (5.42b) - in this second case, the acoustic region near the $t = 0$

plays a "passive role" and does not have any influence on the leading-order Navier limit problem.

Finally, a short comment concerning the formulation of a well-posed initial boundary value problem for the Navier equations, according to Gresho (1992, pp. 47-52). According to Gresho (and with the Gresho notations), if the initial ($t = 0$) velocity field $\boldsymbol{u}_0$ for the Navier equations is not divergence-free and if the vertical component of this initial velocity is not equal, on the boundary, to the vertical component specified on the wall of the body at $t = 0$, (denoted by $\boldsymbol{w}_0$), then it this necessary to solve (again!) the following Neumann problem for the unknown scalar λ:

$$\nabla^2\lambda = -\nabla\cdot\boldsymbol{u}_0 \text{ in } \Omega\text{, with } \frac{\partial\lambda}{\partial n} = \boldsymbol{n}\cdot(\boldsymbol{w}_0 - \boldsymbol{u}_0) \text{ on } \Gamma. \tag{5.48}$$

Then, it is necessary to compute $\boldsymbol{v} = \boldsymbol{u}_0 + \nabla\lambda$, such that: $\nabla\cdot\boldsymbol{v} = 0$ in Ω, with:

$$\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{w}_0 \text{ on } \Gamma \text{ and replace } \boldsymbol{u}_0 \text{ by } \boldsymbol{v}\text{, such that: } \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{v} \text{ in } \Omega + \Gamma.$$

The analogy with our above result is disconcerting. In fact: for $\boldsymbol{u}_0 \equiv 0$, our ψ_N^0 is the Gresho λ, but our result is only true for the unsteady external aerodynamics problem, when for the acoustics problem (5.37), (5.40) and (5.43) we have the behaviour (5.45)!

For the unsteady internal aerodynamics problem, this behaviour (5.45) seems (!) true only when the wall of the body is set in movement progressively during an interval of time $O(1)$ - see, for instance, §8.4 in Chapter 8.

5.2.1. The problem of the "acoustic viscous Rayleigh layer"

A final comment concerning the singular nature of the limit process (5.32), which gives the linear equations of acoustics (5.33a, b, c, d) with the slip condition (5.35). Indeed, in the *vicinity of the wall and near the initial time,* it is necessary to consider a new limit process, in place of (5.32), and derive a new set of consistent (the so-called *'Rayleigh'*) equations, in place of (5.33a, b, c).

In this case the slip condition (5.35) appears as a matching condition (relative to coordinate normal to the wall of the body). This problem is investigated in the §7.4 of Chapter 7, in the framework of high-Reynolds number asymptotics, and makes it possible to initialize the classical Prandtl boundary-layer equations (through an adjustment problem, the initial condition for the tangential component of the boundary-layer velocity vector must be asymptotically derived). Unfortunately, the matching (relative to time) between the Rayleigh (initial-layer, in the vicinity of the wall) and Prandtl (boundary-layer, far from initial time) equations is actually not completely elucidated!

5.3. THE ASSOCIATED FOURIER EQUATION

Considering (5.8), taking (5.16) and (5.21) into account, we find that $\theta_N(t, \boldsymbol{x})$, $\boldsymbol{u}_N(t, \boldsymbol{x})$, and $\boldsymbol{u}'(t, x)$, all three of them, occur in the limit form of that equation, namely:

$$\frac{\partial \theta_N}{\partial t} + \boldsymbol{u}_N \cdot \nabla \theta_N + (\gamma - 1)\nabla \cdot \boldsymbol{u}' = \frac{\gamma}{PrRe}\nabla^2 \theta_N + (\gamma - 1)\frac{2\gamma}{Re}[\boldsymbol{D}(\boldsymbol{u}_N) : \boldsymbol{D}(\boldsymbol{u}_N)]. \tag{5.49}$$

But, from (5.6), retaining terms proportional to M^2, we get according to (5.21):

$$\nabla \cdot \boldsymbol{u}' = -\left[\frac{\partial \omega_N}{\partial t} + \boldsymbol{u}_N \cdot \nabla \omega_N\right]. \tag{5.50}$$

Substituting (5.50) into (5.49), and using the obvious consequence of (5.9), namely: $\omega_N = \pi_N - \theta_N$, we derive the equation needed for $\theta_N\ (t, \boldsymbol{x})$, namely:

$$\frac{\partial \theta_N}{\partial t} + \boldsymbol{u}_N \cdot \nabla \theta_N - \frac{1}{PrRe}\nabla^2 \theta_N = \frac{\gamma - 1}{\gamma}\left[\frac{\partial \pi_N}{\partial t} + \boldsymbol{u}_N \cdot \nabla \pi_N\right] + \frac{\gamma - 1}{\gamma}\frac{2}{Re}[\boldsymbol{D}(\boldsymbol{u}_N) : \boldsymbol{D}(\boldsymbol{u}_N)] \tag{5.51}$$

to which, for obvious reasons, the name of Fourier is associated.

We emphasize that the right hand side of the above Fourier equation (5.51) for the temperature perturbation θ_N is known, beforehand, once the Navier IBVP, formulated in the §5.1 and §5.2, has been solved. Inasmuch as the latter one does not contain θ_N, our assertion is true. Of course, the full determination of θ_N, necessitates that we provide initial and boundary conditions. Warned by the discussion concerning the initial value of $\boldsymbol{u}_N$, we must be cautions concerning the initial value for θ_N, and we simply write

$$\theta_N(0,\boldsymbol{x})=\theta_N{}^{\circ}(\boldsymbol{x}), \tag{5.52}$$

leaving aside, for the moment, the problem of the determination of $\theta_N{}^{\circ}(\boldsymbol{x})$. Concerning the boundary condition, we look at (5.13) and we immediately see an inconsistency, unless we assume that κ° and Bi are large numbers such that:

$$\kappa^{\circ}=\frac{\kappa^{\circ *}}{M^2} \text{ and } Bi=\frac{Bi^*}{M^2}, \tag{5.53a}$$

with

$$\kappa^{\circ *} = O(1),\ Bi^* = O(1).$$

When (5.53a) holds, we get the full boundary condition for θ_N, namely:

$$\kappa^{\circ *}\frac{\partial\theta_N}{\partial n}+Bi^*\theta_N=\theta_W \text{, all along } \Gamma. \tag{5.53b}$$

But, if on contrary:

$$\kappa^{\circ *} = 0 \text{ or } Bi^* = 0, \tag{5.54a}$$

we have, rather, a degenerate (Dirichlet/Neumann) form:

$$\theta_N=\frac{1}{Bi^*}\theta_W \text{, or } \frac{\partial\theta_N}{\partial n}=\frac{1}{\kappa^{\circ *}}\theta_W \text{, all along } \Gamma. \tag{5.54b}$$

When κ° and Bi are both $O(1)$, we do not escape to the conclusion that θ_N is no longer of order M^2, but we should not be disturbed by that conclusion. Indeed, in this case, it is necessary to assume that in the 'exact' condition (5.13) the known function θ_W is proportional (as is the perturbation of temperature θ) to M^2.

As we have already mentioned, there are two causes of heating (or cooling) the gas. One comes from viscous dissipation and it causes θ to grow like M^2.

The second cause is through heat transfer at the wall Γ, and it is such a heat transfer process which gives rise to (5.13). If the heat transfer process at the wall is not consistent with either (5.53a) or (5.54a), and if $\theta_w \neq M^2\ \theta_W^*$, with $\theta_W^* = O(1)$, then we need reconsider the expansions (5.21). See, for instance, Zeytounian (1977) and Zank and Matthaeus (1990), for some information concerning the case when θ_N is no longer of order M^2. Indeed, if κ^o, Bi and θ_N are $O(1)$ we might expect that θ is also $O(1)$. But, with π being $O(M^2)$ we get that: $\omega + \theta + \theta\,\omega = 0$, and we run into difficulties, due to such a relation being inconsistent with two equations, derived from (5.6) and (5.8), for ω and θ.

Considering the Fourier equation (5.51), with (5.52) [assuming that we have been able to compute $\theta_N{}^o(\boldsymbol{x})$] and either (5.53b) or (5.54b), we get the well-posed Fourier IBVP. We may state that, under the main limit (5.28), with (5.16) and (5.21), either (5.53a) or (5.54a), both limit problems: the Navier IBVP and the Fourier IBVP are *decoupled* from each other.

This conclusion is well known for liquids: except when strong heating or cooling occurs at the wall, the motion may be computed first using the Navier incompressible equations, then the temperature is obtained from the Fourier equation, and appropriate conditions. Here we find a corresponding situation for gases. Of course we know that, for some fluids, when, the viscosity is very sensitive to temperature, it may happen that heating by viscous dissipation has an influence on the motion through variation of the viscosity. This happens, for example, in lubrication theory - but in this case the Reynolds number is also a small parameter, as this is assumed in the Chapter 9.

5.3.1. Some remarks concerning the determination of $\theta_N{}^o(x)$

Going back to equations (5.3la, b, c, d), we remind the reader of our warning about $\theta_N{}^o(\boldsymbol{x})$. Let us try to examine this issue, assuming that $\boldsymbol{u}_a$, ω_a, π_a, θ_a are expanded in the following form:

$$\boldsymbol{u}_a = \boldsymbol{u}_{a,0} + M\boldsymbol{u}_{a,1} + \dots, \tag{5.55a}$$

$$(\omega_a, \pi_a, \theta_a) = (\omega_{a,0}, \pi_{a,0}, \theta_{a,0}) + M(\omega_{a,1}, \pi_{a,1}, \theta_{a,1}) + \dots, \tag{5.55b}$$

In this case, for the functions: $\boldsymbol{u}_{a,1}$, $\omega_{a,1}$, $\pi_{a,1}$, $\theta_{a,1}$ we derive the following second order inhomogeneous acoustic equations, namely:

$$\frac{\partial \omega_{a,1}}{\partial \tau} + \nabla \cdot \boldsymbol{u}_{a,1} = \frac{\partial Q_{a,0}}{\partial \tau}, \tag{5.56a}$$

$$\frac{\partial \boldsymbol{u}_{a,1}}{\partial \tau} + \nabla\left(\frac{\pi_{a,1}}{\gamma}\right) = \nabla P_{a,0}, \tag{5.56b}$$

$$\frac{\partial \theta_{a,1}}{\partial \tau} + (\gamma - 1)\nabla \cdot \boldsymbol{u}_{a,1} = \frac{\partial R_{a,0}}{\partial \tau}, \tag{5.56c}$$

$$\pi_{a,1} - (\omega_{a,1} + \theta_{a,1}) = (\gamma - 1)\left(\frac{\partial \phi_{a,0}}{\partial \tau}\right)^2, \tag{5.56d}$$

where

$$Q_{a,0} = \nabla \psi_N{}^{\circ} \cdot \nabla \phi_{a,0} + \frac{1}{2}(\nabla \phi_{a,0})^2 + \frac{1}{2}\left(\frac{\partial \phi_{a,0}}{\partial \tau}\right)^2, \tag{5.57a}$$

$$P_{a,0} = \frac{1}{2}\left(\frac{\partial \phi_{a,0}}{\partial \tau}\right)^2 - \frac{1}{2}\left|\nabla \psi_N{}^{\circ} + \nabla \phi_{a,0}\right|^2 + \frac{1}{2Re}\left(\frac{4}{3} + \sigma^{\circ}\right)\nabla^2 \phi_{a,0}; \tag{5.57b}$$

$$R_{a,0} = (\gamma - 1)\left[\nabla \psi_N{}^{\circ} \cdot \nabla \phi_{a,0} + \frac{1}{2}(\nabla \phi_{a,0})^2 + \frac{\gamma}{2}\left(\frac{\partial \phi_{a,0}}{\partial \tau}\right)^2\right] - (\gamma - 1)\frac{\gamma}{PrRe}\nabla^2 \phi_{a,0} \tag{5.57c}$$

It is easily checked that the following formulae solve (5.56a, b, c, d):

$$\boldsymbol{u}_{a,1} = \nabla \frac{\partial \phi_{a,1}}{\partial \tau}, \; \omega_{a,1} = -\nabla^2 \phi_{a,1} + Q_{a,0}, \tag{5.58a}$$

$$\theta_{a,1} = -(\gamma - 1)\nabla^2 \phi_{a,1} + R_{a,0}, \; \pi_{a,1} = -\gamma \frac{\partial^2 \phi_{a,1}}{\partial \tau^2} + \gamma P_{a,0}, \tag{5.58b}$$

provided $\phi_{a,1}$ be a solution to the inhomogeneous d'Alembert's equation:

$$\frac{\partial^2 \phi_{a,1}}{\partial \tau^2} - \nabla^2 \phi_{a,1} = P_{a,0} - \frac{1}{\gamma}\left[Q_{a,0} + R_{a,0}\right] - \frac{\gamma - 1}{\gamma}\left(\frac{\partial \phi_{a,0}}{\partial \tau}\right)^2 \tag{5.59}$$

to which we must add:

$$\tau = 0: \ \phi_{a,1} = 0\,, \ \frac{\partial \phi_{a,1}}{\partial \tau} = 0\,, \text{ everywhere outside the body,} \tag{5.60}$$

$$\frac{\partial \phi_{a,1}}{\partial n} = 0\,, \text{ all along } \Gamma. \tag{5.61}$$

From the value of $P_{a,0}$, $Q_{a,0}$ and $R_{a,0}$, according to (5.57a, b, c), we conclude that the right hand side of the inhomogeneous d'Alembert's equation (5.59) tends to zero when $|\boldsymbol{x}| \to \infty$ at τ fixed. Relying on the same physical argument as for (5.44), we may add:

$$\phi_{a,1} \to 0\,, \text{ when } |\mathbf{x}| \to \infty\,, \tag{5.62}$$

getting, with (5.60), (5.61) and (5.62), a well-posed initial boundary value problem for the equation (5.59) relative to the function $\phi_{a,1}$.

Now, the matching of the Fourier IBVP solution with the solution of the above inhomogeneous acoustics problem (5.59)-(5.62), gives the following relation for the initial condition for the Fourier IBVP:

$$\theta_N{}^{\circ}(\boldsymbol{x}) = -(\gamma - 1)\nabla^2\left[\lim_{\tau \to \infty} \phi_{a,1}\right]. \tag{5.63}$$

A rough argument suggests (!), that: the solution of the equation (5.59), for $\phi_{a,1}(\tau, \boldsymbol{x})$, tends to zero as $\tau \to \infty$, and as a consequence:

$$\theta_N{}^{\circ}(\boldsymbol{x}) = 0\,. \tag{5.64}$$

Indeed, such a statement necessitates a rigorous proof, and for this the result (5.64) is, in fact, only a conjecture for the moment.

But, from known results concerning the d'Alembert's equation (5.37), for $\phi_{a,0}$, we may be more precise about the behaviour (5.45). Namely [see, again Wilcox (1975)]: at any fixed point, $\phi_{a,0}$ (and also $\partial\phi_{a,0}/\partial\tau$) vanish identically within some vicinity (depending on the point considered) of $\tau = \infty$, and the same holds true for $P_{a,0}$, $Q_{a,0}$ and $R_{a,0}$. Considering the above problem, (5.59) to (5.62), for $\phi_{a,1}$, we may construct its solution in two steps. The *first step* is a retarted potential solution of the inhomogeneous equation (5.59) ignoring initial and boundary conditions (5.60) and (5.61), corresponding to an extension of the right hand side of (5.59) by zero for the time values $\tau < 0$ and/or within the body. The *second step* is to solve a homogeneous problem [with zero as a right hand side in (5.59)] with initial and boundary data according to (5.60), (5.61) - obviously the solution of this second step homogeneous problem is identically zero. Seeing that the right hand side of the inhomogeneous d'Alembert's equation (5.59) tends to zero as τ tends to infinity, we think that the solution of the first step inhomogeneous problem tends also to zero when the time $\tau \to \infty$.

In reality this conclusion is not sufficiently rigorous - behaviour of this particular solution as τ tends to infinity is strongly related to the nature (rapidity ?) of the behaviour (to zero) of the right hand side of the equation (5.59) when $\tau \to \infty$, but here we should not be disturbed by this mathematical problem. The above conjoncture (5.64) relating to the behaviour of the solution of the above problem (5.59) to (5.62) for $\phi_{a,1}(\tau, x)$ when $\tau \to \infty$, puts an end to our discussion concerning the Fourier IBVP.

5.4. THE INFLUENCE OF A WEAK COMPRESSIBILITY

As a useful by-product of the solution of the Fourier IBVP, we may get an equation for the following terms u' and π' in the expansions of velocity and pressure, occurring in the asymptotic expansions (5.16). To do so, we take into account the asymptotic expansions (5.21), and derive from the dominant equation of motion (5.7) the following 'second-order' linear equation:

$$\frac{\partial u'}{\partial t} + u' \cdot \nabla u_N + u_N \cdot \nabla u' + \nabla\left(\frac{\pi'}{\gamma}\right) - \frac{1}{Re}\nabla^2 u'$$

$$= -\omega_N\left[\frac{\partial u_N}{\partial t} + u_N \cdot \nabla u_N\right]$$

$$+\frac{1}{Re}\left\{2\left(\frac{d\mu}{d\theta}\right)_{\theta=0} \nabla \cdot [D(u_N)\theta_N] - \left[\frac{1}{3} + \sigma^\circ\right]\nabla\left[\frac{\partial \omega_N}{\partial t} + u_N \cdot \nabla \omega_N\right]\right\} \quad (5.65a)$$

where ω_N is computed according to:

$$\omega_N = \pi_N - \theta_N, \quad (5.65b)$$

and with [see (5.50)]:

$$\nabla \cdot \boldsymbol{u} = -\left[\frac{\partial \omega_N}{\partial t} + \boldsymbol{u}_N \cdot \nabla \omega_N\right]. \tag{5.65c}$$

Thus, we obtain a second-order closed system (5.65a, b, c) for $\boldsymbol{u}'$ and π'.

Of course we need initial and boundary conditions. It seems obvious that on the body wall we have:

$$\boldsymbol{u}' = 0, \text{ all along } \Gamma, \tag{5.66}$$

but what should be enforced at *infinity* and also at *initial time* is less obvious and we leave this as an open problem for future research. Indeed, concerning the initial condition, it is necessary to investigate the acoustic region near $t = 0$ and the adjustment problem up to the term $O(M^2)$ for the 'acoustic' velocity!

An interesting application of the above results is to compute, from a known solution $(\boldsymbol{u}_N, \pi_N)$, of the Navier model equations (5.23a, b), the companion solution θ_N, of the Fourier problem, (5.51), with (5.53b)/(5.54b) and (5.64). Then the above second-order equations (5.65a) and (5.65c), for $\boldsymbol{u}'$ and π', with (5.65b), conditions (5.66) and conditions at infinity and for $t = 0$ (probably $\boldsymbol{u}' = 0$!), makes it possible to take into account the effects of a weak compressibility.

In fact, with all known incompressible and viscous (Navier) solutions we can associate a solution of the Fourier equation for the temperature perturbation and to take into account the effects of a weak compressibility to obtain a weakly compressible solution of the physical problem under consideration.

CHAPTER 6

THE INVISCID/NONVISCOUS EULER MODEL AND SOME HYDRO-AERODYNAMICS PROBLEMS

Below, in §6.1, first, we consider the high (large) Reynolds numbers NS-F fluid flow and the corresponding Euler model, and then, in §6.2, the initial, boundary conditions and unicity problem for this Euler model. The §6.3 is devoted to linearization and associated wave phenomena. In §6.4 we derive some nonlinear 2D wave equations (in particular, for isochoric and Boussinesq fluid flows). The nonlinear long-surface-waves on water (potential theory) is considered in §6.5 and in §6.6, the incompressible rotational turbomachinery flows - when the blades within a row are very closed spased - is discussed from an asymptotic point of view. The model problems for transonic (Mach number near 1) and hypersonic (high Mach number) gasdynamics flows are derived in §6.7 and §6.8. Finally, an asymptotic theory for the rolled-up vortex sheets - when the turns of the sheet are very closely spaced (tightly rolled-up vortex sheet) - is given in §6.9. Below, in the figure 6.1, we have schematically sketched the various sub-models which are derived from the Euler inviscid model in the framework of this Chapter 6.

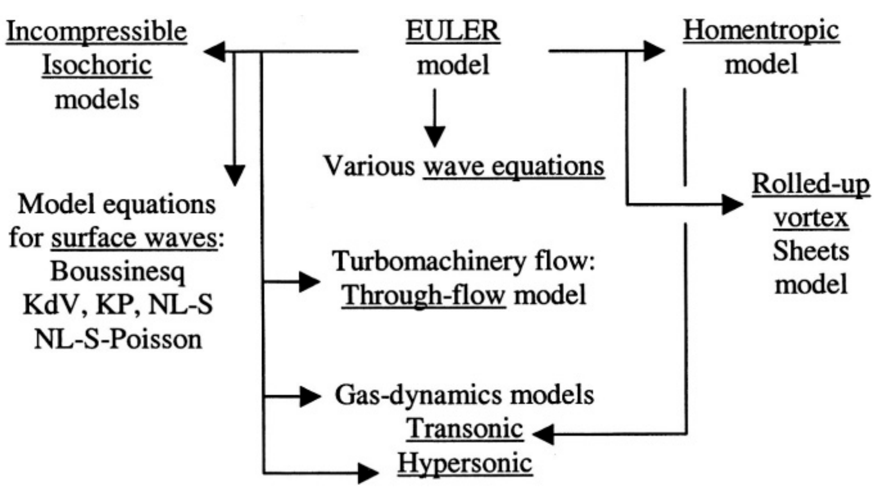


Fig. 6.1 From Euler nonviscous model to some hydro-aerodynamics models

6.1. HIGH REYNOLDS NUMBER FLUID FLOWS AND THE EULER LIMIT

First, instead of the NS-F equations (2.56a, b, c), with (2.56d), we shall use a simplified set of equations - the so-called inviscid and non-heat conducting Euler equations for a perfect gas. These equations are most simply obtained by rubbing out of (2.56b) and (2.56c) all the terms which have $1/Re$ in front of them [see, for instance, equations (4.7), of the §4.2, in Chapter 4]. More precisely, the Euler equations are derived from the NS-F equations (2.56a)-(2.56c), with (2.56d), when we consider the high Reynolds numbers ($Re >> 1$), asymptotics under the following Euler limit:

$$lim^E = [Re \uparrow \infty \text{ with } \boldsymbol{x} \text{ and } t \text{ fixed, and for } S, M, \gamma, Bo \text{ and } Pr, \text{ all } O(1)], \tag{6.1}$$

and in this case for the limit functions

$$(\boldsymbol{u}_E, p_E, \rho_E, T_E) = lim^E (\boldsymbol{u}, p, \rho, T), \tag{6.2}$$

we derive (again - see (4.7)) the Euler compressible, inviscid and adiabatic equations:

$$S\frac{D\rho_E}{Dt} + \rho_E \nabla \cdot \boldsymbol{u}_E = 0; \tag{6.3a}$$

$$\rho_E S\frac{D\boldsymbol{u}_E}{Dt} + \frac{1}{\gamma M^2}\nabla p_E + \frac{Bo}{\gamma M^2}\rho_E \boldsymbol{k} = 0, \tag{6.3b}$$

$$\rho_E S\frac{DT_E}{Dt} + (\gamma - 1)p_E \nabla \cdot \boldsymbol{u}_E = 0, \tag{6.3c}$$

with

$$p_E = \rho_E T_E. \tag{6.3d}$$

In place of the third equation we can write an equation for the specific entropy $S_E(\boldsymbol{x}, t)$:

$$S\frac{DS_E(\boldsymbol{x},t)}{Dt} = 0. \tag{6.4}$$

In this case, in place of: $p_E = \rho_E T_E$, we can write, for a perfect (ideal) gas the following dimensionless equation of state,

$$p_E = \rho_E^{\gamma} \exp(S_E). \tag{6.5}$$

As a consequence for: $\boldsymbol{u}_E$, p_E and ρ_E, we derive a closed Eulerian system:

$$S\frac{D\rho_E}{Dt} + \rho_E \nabla \cdot \boldsymbol{u}_E = 0;$$

$$\rho_E S\frac{D\boldsymbol{u}_E}{Dt} + \frac{1}{\gamma M^2}\nabla p_E + \frac{Bo}{\gamma M^2}\rho_E \boldsymbol{k} = 0, \tag{6.6}$$

$$S\frac{D}{Dt}\left(\frac{p_E}{\rho_E^{\gamma}}\right) = 0.$$

According to equation (6.4), for S_E, with the equation of state (6.5):

In the continuous motion of an inviscid perfect (ideal) gas, specific entropy is convected with the fluid in its motion.

In other words, specific entropy remains constant for a given particle - such flows are appropriately called *isentropic flows.* As a consequence:

In a body of inviscid ideal gas in a continuous motion has uniform specific entropy at time t = 0, then its specific entropy remains uniform at all times and

$$p_E = \rho_E^{\gamma}. \tag{6.7}$$

6.1.1. Homentropic motion

Such gas motions, which obey (6.7), are called *homentropic* and are of much practical interest since many motions develop from a uniform thermodynamic state. Therefore the *homentropic motion* of inviscid ideal gas (when a flow field has uniform specific entropy in space and time) is governed by the following two equations, for $\boldsymbol{u}_h$ and ρ_h:

$$\left(S\frac{\partial}{\partial t}+\boldsymbol{u}_h\cdot\nabla\right)\rho_h+\rho_h\nabla\cdot\boldsymbol{u}_h=0; \tag{6.8a}$$

$$\left(S\frac{\partial}{\partial t}+\boldsymbol{u}_h\cdot\nabla\right)\boldsymbol{u}_h+\frac{1}{M^2}\rho_h^{\gamma-2}\nabla\rho_h+\frac{Bo}{\gamma M^2}\boldsymbol{k}=0. \tag{6.8b}$$

6.1.2. Incompressible and isochoric motions

The term 'incompressible motion' denotes a fluid motion such that: $\rho\equiv 1$, with nondimensional quantities. In fluid dynamics, incompressibility thus refers not to a property of the fluid but to a property of the representation by which its real motion is approximated. A less restrictive definition, making incompressibility equivalent to the statement that the density $\rho=\rho(\boldsymbol{x}(\boldsymbol{a},t),t)$, is a function (noted ρ_i) only of the Lagrangian label $\boldsymbol{a}$ is used for the study of so-called " stratified fluids" in atmospheric fluid dynamics and ocean dynamics [see, for instance, the books by Yih (1980) and Zeytounian (1991)].

With the above definition

$$\left(S\frac{\partial}{\partial t}+\boldsymbol{u}_i\cdot\nabla\right)\rho_i=0\Rightarrow\nabla\cdot\boldsymbol{u}_i=0, \tag{6.9}$$

and in this case we have an " isopycnic" or "isochoric" motion.

The corresponding system of equations for an isochoric motion, for the functions $\boldsymbol{u}_i$, p_i and ρ_i are:

$$S\frac{\partial\rho_i}{\partial t}+\boldsymbol{u}_i\cdot\nabla\rho_i=0;$$

$$\nabla\cdot\boldsymbol{u}_i=0\ ; \tag{6.10}$$

$$\rho_i S\frac{\partial\boldsymbol{u}_i}{\partial t}+(\boldsymbol{u}_i\cdot\nabla)\boldsymbol{u}_i+\frac{1}{M^*}(\nabla p_i+Bo\rho_i\boldsymbol{k})=0,$$

and in this case we have the following similarity relation:

$$\gamma M^2=M^*=O(1),\ \text{when}\ \gamma>>1\ \text{and}\ M<<1. \tag{6.11a}$$

Indeed, for: $\gamma \to \infty$, the equation (6.4) is equivalent to the first of equations (6.9) for ρ_i, if we assume that:

$$\gamma \to \infty \leftrightarrow C_p = O(1), \text{ but } C_V \to 0. \qquad (6.11b)$$

From (6.10), we derive the classical (so-called, 'incompressible') Euler nonviscous equations for divergence-free motion, for the functions:

$$(v, \pi, 1) = \lim_{\rho_i \to 1} (ui, p_i, \rho_i),$$

namely:

$$S\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla \pi = 0, \ \nabla \cdot v = 0. \qquad (6.12)$$

when $Bo = 0$ and $M^* \equiv 1$.

On the other hand, if the motion is irrotational:

$$v = \nabla \phi \text{ and } \omega = \nabla \wedge v = 0, \qquad (6.13a)$$

then we derive, from the incompressibility condition, $\nabla \cdot v = 0$, the Laplace equation for the velocity-potential function $\phi(t, x)$; namely:

$$\nabla^2 \phi = 0, \qquad (6.13b)$$

and also the Bernoulli (incompressible) integral (see (6.95)).

6.1.3. The Euler model with the slip condition

From an asymptotic point of view, the Eulerian fluid flow is an "outer" flow, when the viscosity tends to zero ($Re \uparrow \infty$, with x and t fixed), which matches with the "inner" boundary-layer fluid flow valid near the solid wall. In this case, from the MMAE we derive in a coherent way the slip boundary condition on the solid boundary for the outer Eulerian fluid flow (see in Chapter 7, §7.2 and §7.3). A simple form of this slip condition, for a steady motion is:

$$u_E \,.\, n = 0, \text{ on the motionless solid wall of the body.} \qquad (6.13c)$$

Indeed, it is important to note that the steady Euler equations, involving only first-order partial differential coefficients with respect to *x, y,* and *z,* need only the one condition (6.13c), at a motionless solid boundary in order to determine fully a flow which, in general, involves substantial tangential motion at the solid surface.

The relationship of the Euler model, which neglects all effects of viscosity, to real fluid flows with very thin boundary layers (layers whose thickness tends to zero with a vanishing viscosity) is, then, that a flow calculated using the Euler model is a close representation of the flow outside the boundary layer.

For the fluids of small viscosity, including air and water, many flows do involve only very thin boundary layers, so that all of their (large-dominant) scale features can be predicted well by the Euler model, as is very well presented and discussed in the book by Lighthill (1986).

This is particularly important because the flow that is well predicted by the Euler model possess features advantageous in many applications (for example, incompressible waves on the free-surface of the water or compressible, baroclinic, lee waves in the atmosphere downstream of a mountain).

Unfortunately, however, there are also very many flows for which it is impossible for the associated boundary layers to remain thin, however small the fluid's viscosity may be.

6.2. INITIAL AND BOUNDARY CONDITIONS AND EXISTENCE AND UNIQUENESS PROBLEMS FOR THE EULER MODEL

In the above Euler compressible equations (6.3a, b, c) we have time derivatives, respectively, for the velocity vector $\boldsymbol{u}_E$, density ρ_E and temperature T_E. These Euler equations are *hyperbolic, evolution equations.* As a consequence a significant problem for these hyperbolic Euler equations is the classical *Cauchy* (initial data) problem (in L^2-norm, for example) with the following initial conditions:

$$t = 0: u_E = u^\circ(x), \rho_E = \rho^\circ(x), T_E = T^\circ(x), \tag{6.14}$$

where, $\rho^\circ(x) > 0$ and $T^\circ(x) > 0$. Moreover, when considering a free-boundary problem or an unsteady flow in a bounded cavity, $\Omega(t)$, with a boundary depending on time, an initial condition for the (moving) boundary $\partial\Omega(t)$, has to be specified. Again, we stress that it is necessary to elucidate the problems of unsteady adjustment to isentropic, isochoric and incompressible equations, since the limiting processes which lead to these approximate equations filter out some time derivatives (mainly those corresponding to acoustic waves).

6.2.1. Boundary conditions

Several boundary conditions could be considered with respect to different physical situations. If we consider, as simple example, the motion of an Eulerian fluid in a *rigid* container Ω (with $\partial\Omega$ independent of time *t*), a bounded connected open subset of $\mathbf{R}^d$ (where $d \geq 1$ is the physical dimension), then the mathematical structure of the Euler equations makes it necessary leads to impose only the *slip* boundary condition:

$$w_E = \mathbf{u}_E \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \tag{6.15}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$, in the following, denotes the unit outward normal vector to $\partial\Omega$. This (steady) slip boundary condition (6.15) is valid for both compressible and incompressible steady Eulerian fluid flows. No boundary condition has to be imposed on temperature T and on pressure p, if (6.15) is satisfied, since in this case the temperature and the pressure are not subjected to transport phenomena through the boundary as a consequence of the adiabaticity [according to equation (6.3c)]. The above stationary slip boundary condition (6.15) is, in fact, a further expression of continuity: it states that there is a zero rate of disappearance of fluid, or creation of new fluid, at a solid boundary at rest.

We are also interested in moving solid boundaries and in this case the absence of flow through the boundary depends on the relative velocity between fluid and solid body having a zero normal component. If $\mathbf{U}_W$ is the velocity of the point of the solid body in motion, this relative velocity is $\mathbf{v}_{rE} = \mathbf{u}_E - \mathbf{U}_W$, and the boundary condition is therefore:

$$\text{on the moving solid boundary: } (\mathbf{u}_E - \mathbf{U}_W) \cdot \mathbf{n} \equiv \mathbf{v}_{rW} \cdot \mathbf{n} = 0. \tag{6.16}$$

A certain alternative form of the condition (6.16) which is very often useful involves an interesting application of the material derivative operator D/Dt. It can be applied if the geometrical equation of the solid surface in Cartesian coordinates (x, y, z) is known at time t, in (say) the form: $F(t, x, y, z) = 0$, and the separate dependence (derivative) on t in this above equation would disappear only if the solid boundary were at rest. In this case we obtain:

$$\text{on } F = 0\text{: } S\frac{DF}{Dt} \equiv S\frac{\partial F}{\partial t} + \boldsymbol{u}_E \cdot \nabla F = 0\,. \tag{6.17}$$

Although the conditions in (6.16) and (6.17) look very different they are, in fact, mathematically equivalent.

Being concerned with the motion of the fluid in a domain extending to a neighbourhood of infinity, we need some conditions relative to this limit. It seems obvious that for the external aerodynamics, we require again:

$$U \equiv \left(w,\theta,\pi,|\boldsymbol{u}|\right) \rightarrow 0,\ when\ |\boldsymbol{x}| \rightarrow \infty, \tag{6.18}$$

for any finite time, where (ω, θ, π) are the thermodynamic perturbations relative their constant values (in fact, unity, with dimensionless quantities) at infinity, and we stress, again, that (6.18) raises some difficulties which come from the wake trailing behind the body.

For the external aerodynamics, there are two new issues when the fluid domain extends to infinity. First, in addition to the usual initial and boundary conditions there needs to be some prescription of fluxes or pressure drops when the fluid flow domain has several "exits to infinity". Secondly, the solutions of interest often have infinite energy integrals and recently a technique of integral estimates to deal with this problem has been developed. These estimates are called "Saint Venant's type" because the method was first used in the study of Saint Venant's principle in elasticity.

6.2.2. The Kutta-Joukovski and Villat condition

For bodies with a sharp trailing edge, experimental observation of subsonic flows show that, in general, the action of viscosity causes the flow to leave the trailing edge smoothly, and that a thin wake is formed downstream from the trailing edge by the retarded layers of fluid from the body surface. As the viscosity tends to zero, this situation is idealized by the assumptions that the wake is infinitely thin as it leaves the body, and that the velocity at the sharp trailing edge is finite.

This last condition is due to Kutta and Joukovski, but also to Villat [the K-J-V condition; see, in §12.1, of Chapter 12, a discussion related to the triple deck concept].

In fact, according to the triple deck model (considered in Chapter 12), contrary to the accepted K-J-V scheme, the flow does not leave the profile right at the trailing edge, but, rather, there is a slight warping of the flow, with an inviscid stagnation point on the leeward side of the profile, at a distance $O(Re^{-3/8})$ ahead of the trailing edge (with $Re >> 1$). Indeed, this K-J-V condition applies at sharp trailing edges in subsonic flows. In supersonic flows, the condition is found to be satisfied automatically at supersonic trailing edges [Ward (1955)]. The velocity varies rapidly through the thin wake in real flows, so the idealized wake can be taken as a vortex sheet from the time it leaves the trailing edge. At a distance from the body this vortex sheet is supposed to roll up under the action of its own induced velocity, until it ultimately assumes the form of distinct vortex cores of finite diameter at large distances downstream. The greater the strength of the vortex sheet the more rapidly it roll up, and experimental evidence generally supports this supposition, but the action of viscosity ultimately dissipates the vortex cores, so they do not extend downstream to "infinity" in practice.

In linearized theory, the K-J-V condition is that:

The component of the perturbation velocity normal to the mean body surface must be finite in a neighbourhood of any trailing edge,

since the usual condition of finite velocity is too restrictive in linearized theory.

This causes the appearance in the mathematical solution of surfaces stretching downstream from the trailing edges on which a tangential component of velocity is discontinuous. For a very pertinent discussion of the K-J-V condition, in relation to the so-called d'Alembert's paradox, see the paper by Stewartson (1981). In the book by Lighthill (1986, Chapters 10 and 11) the reader can find a very pertinent quantitative analysis, through the classical theory of the conformal mapping and the Joukovski transformation [see, for instance, the book by Glauert (1947)], concerning aerofoils at incidence and the derivation of the classical formula for the value of the circulation at which the flow will stabilize itself, leading to smooth flow at the trailing edge. For a finite flat plate, in this case, the leading-edge separation can be avoided for sufficiently small angles of incidence and at the trailing edge, on the other hand, the self-regulation of the circulation by vortex shedding may, if the trailing edge remains sharp, lead to a predictable value for the circulation which can eliminate separation in that region as well. Indeed, an aerofoil shape with a rounded leading-edge and sharp trailing edge avoids separation not only in symmetrical flow around the aerofoil; it may also, for sufficiently small angles of incidence, avoid separation both at the leading and trailing edges after the circulation has adjusted to a value for which the velocity at the trailing edge is finite (K-J-V condition!). In the book by Chorin and Marsden (1993) the reader can find a deeper study of the relationship between viscous and non-viscous flows, with a more detailed study of inviscid irrotational flows, i.e. potential flows.

6.2.2a. The case of a 2D steady potential flow past an aerofoil

For a circulatory 2D steady potential flow past an aerofoil we can prove easily the following theorem [see, for instance, the book by Shinbrot (1973; Chapter 6)]:

Let V be an exterior domain with a connected boundary and at least two boundary points. Let $F(z)$, with $z = x + iy = r\exp(i\theta)$, be an analytic function in V, having a simple pole at infinity, mapping V conformally onto the exterior of the unit disc and having the properties, $F(\infty) = \infty$, $F'(\infty) > 0$. Then, the function

$$U_\infty\left[F(z)+\frac{1}{F(z)}\right]-\frac{C}{2\pi i}\log[F(z)], \tag{6.19a}$$

is a weak complex velocity potential of a flow in V with circulation $(-C)$. The velocity at infinity of this flow is horizontal and has magnitude $U_\infty F'(\infty)$.

Naturally, there is nothing special about the horizontal direction - let the desired velocity at infinity be the vector (U_∞, V_∞). Write $V_\infty = U_\infty + iV_\infty$ and let V^*_∞ be the complex conjugate of V_∞. A review of the argument leading to the above theorem then shows that a weak complex velocity potential for the domain V of the above theorem is:

$$\frac{V^*_\infty}{F'(\infty)}\left[F(z)+\frac{1}{F(z)}\right]-\frac{C}{2\pi i}\log[F(z)]. \tag{6.19b}$$

Incidentally we remark that this last formula shows that there is a single-valued velocity associated with each point of V, since its derivative is single-valued.

Our experience so far leads us to believe that there exist flows past bodies with continuous derivatives in the closure of the domain of the fluid, except possibly for the point $z°$ on the trailing edge where ∂V is not smooth, which resemble those of the most important application, the wing of an aeroplane [as, for example, for the Joukowski aerofoil, shaped like the usual aerodynamic profile - for the details of the corresponding Joukowski transformation as well as extensions of it, see the book by Glauert (1947)].

At $z°$, the angle (measured through the fluid) between the upper and lower surfaces of the wing exceeds π and this fact leads us to believe that the velocity of the flow is generally infinite at $z°$! Now, according to the above formula (6.19b), the absolute value of the derivative is:

$$F'(z)\left\{\frac{V_\infty^*}{F'(\infty)}\left[1-\frac{1}{F^2(z)}\right]-\frac{C}{2\pi i F(z)}\right\}; \quad (6.20)$$

which gives the speed.

Let $z°$ be the point on the trailing edge of the wing. Since $F(z)$ maps the domain of the fluid onto the domain $|w| > 1$, $|F(z°)| = 1$, where $w = u - iv$ and (u, v) is the components of the velocity vector attached to the point z of the flow. Therefore, $F(z°) = exp\,(i\beta)$, where $\beta = arg\ F(z°)$ is real. Finally, set:

$$C = -\frac{4\pi V^*_\infty}{F'(\infty)} sin\,\beta = C^*; \quad (6.21)$$

then the quantity multiplying the derivative $F'(z)$ in (6.20) vanisches when $z = z°$.

Thus, the choice (6.21) for C (which gives the value C^* for the circulation around the aerofoil) will produce the desired effect of bounding (6.20) if the singularity of the derivative $F'(z)$ at $z°$ is not too severe. The above K-J-V hypothesis suggests and 'ad hoc' value C^* for the circulation valid only for wings having a single sharp trailing edge. Unfortunately, there is no theoretical justification for it, and there is no generalization to be applied if the trailing edge is ever so slightly rounded.

6.2.3. Existence and uniqueness

From a rigorous mathematical point of view, existence of a (unique) classical solution to the Euler equation requires that the initial vorticity be Hölder continuous and when this condition is not satisfied, weak solutions, i.e., solutions in the sense of distributions, can possibly be defined, mainly because the Euler equation can be written in a conservative form. This is the case if the initial condition u^0 is such that the energy and the L^p-norm of initial vorticity are finite with $p > 2$. Furthermore this solution is unique if the supremum of vorticity is bounded. The solution is obtained (by a compactness procedure) as the limit when the viscosity ν_0 tends to zero of the solution $\boldsymbol{u}_N$ of the Navier equation where the usual rigid boundary condition, $\boldsymbol{u}_N = 0$ has been replaced by the (steady) slip conditions: $\boldsymbol{u}_E \cdot \boldsymbol{n} = 0$ and $\boldsymbol{Curlu}_E = 0$, to prevent the formation of boundary layers. The solution of the initial value problem associated with the (incompressible) Euler equations, in 2D for arbitrary times and in 3D for short times, has long been known. There is a large literature on this subject and in Marchioro and Pulvirenti's (1994) book, the reader can find various references. For instance, we know that when the vorticity is initially bounded (in Hölder norm), there exists for all time a unique solution with remains as smooth as the initial data. But, this condition is not satisfied in the context of the classical Kelvin-Helmholtz instability where the initial velocity is discontinuous across a vortex sheet! In the latter case, existence for a short time of the vortex sheet is assured only if the sheet and the vorticity density are initially analytic. In this case, if the total energy of the flow is finite and the linear vorticity density absolutely integrable on the surface, then there exists a weak solution to the Euler equations according to *C.* Sulem *et al.* (1981). We note again that, in 3D, smooth as the initial conditions may be, the existence of a unique (classical) incompressible solution to the Euler equations has been proven only for a finite time $[0, t°[$ of the order of the inverse Hölder norm of the initial vorticity. During, this period, the solution remains as differentiable as the initial conditions; if the initial data are analytic, the solution is also analytic [see, for a review, Bardos (1978)]. It is still unknown whether the problem remains well-posed even in a weak sense for $t > t°$. In fact, existence and uniqueness for all time of classical solutions is assured, provided the initial vorticity be Holder continuous, and a geometrical proof has been given by Ebin and Marsden (1970); it uses the fact that the motion of an ideal incompressible fluid is a geodesic flow on an infinite dimensional manifold of volume preserving diffeomorphisms. Concerning the vorticity generation and trailing-edge condition, see the interesting paper by Morino (1986). In Stewartson (1981; section 5) the reader can find a deep discussion related with to trailing-edge flows. In inviscid fluid dynamics a typical example is the potential flow past a finite flat plate. If the flow is not normal to the plate, then the K-J-V condition is invoked to define both the vortex sheet and the circulation. But if the flow is normal to this finite flat plate, then we remove the plate to obtain the type of vortex sheet considered by Saffman and Baker (1979). The effect of the vortex sheet is always simply to introduce vorticity into the fluid at the no-slip surface. Whether this vorticity flux causes the tangential pressure gradient, or vice versa, is a moot point and probably subservient to the fact that the no-slip boundary condition generally causes both, does velocity cause vorticity or does vorticity "induce" velocity? If the viscosity is zero, any vortex sheet deemed to be present initially is bound (for t small) to remain unchanged, perhaps this is where the term bound vorticity came from. Concerning mathematical results for incompressible Euler fluid flows, in the following two recent books by: Chemin (1995) and Marchioro and Pulvirenti (1994), the reader can find various classical and new (solvability) results relative to existence, uniqueness and regularity of solutions. A possible criticism of the contents of these books is that 2D fluid flows are treated in much more mathematical detail than 3D ones, which are, physically speaking, much more interesting for the aerodynamicists who are confronted with 3D fluid flow simulations around and downstream of various bodies. Unfortunately, for a mathematical treatise, it cannot be otherwise because the mathematical theory of genuine 3D fluid flows is, at present, still poor compared with the rather rich analyses of the 2D case wich has attracted much attention and efforts from theorticians. For example, if we consider the 3D Eulerian fluid flows, then the conservation laws (energy and circulation or helicity) are not capable of preventing the development of singularities (in the sense of a blow-up of the $\boldsymbol{L}_\infty$ norm of the vorticity) during the motion. Another feature of 3D Eulerian fluid flow is the possibility of an extremely complicated topology; namely, the vorticity (vortex) tubes tend to become bent many times on themselves in complicated geometries, and the vorticity fields may assume a very large values, in small regions of space, without any violation of the conservation laws. Numerical simulations, actually very difficult, delicate and far from being conclusive, seem to show this tendency (but unfortunately do not provide a conclusive answer to the problem of singularity formation). In fact, the behaviour of a vortex tube and its ability to stretch and, eventually, in create singularities, is strongly related to the essential problem of turbulence and various routes (scenarios) to chaos [actually, concerning these routes, there exists a large number of review papers, which are often "initiated" by the Eckmann (1981) pioneer paper]. The proof of rigorous mathematical results for a compressible fluid flow, is obviously harder. As regards the Cauchy problem for the compressible Euler equations (in the whole space, for simplicity), we observe that it is more difficult than the incompressible case when the density is constant. Thus we cannot hope to have existence and uniqueness of solutions for all times. Furthermore, the lack of conservation of vorticity makes difficult even the solvability in the main (Eulerian) part of the Cauchy problem in the compressible 2D case, and, actually, this problem is still unsolved (as far as we know!). Naturally, the case when we choose a small, smooth enough, initial density, and a smooth initial velocity which forces particles to spread out, is easier and actually there are a few results of global existence in multidimensional gas dynamics [see, for instance, Serre (1997a, b)]. In our recent review paper, Zeytounian (1999), the reader can find a discussion concerning the well-posedness of problems in fluid dynamics (in fact, 'a fluid-dynamical point of view'), with various classical and more recent references (up to 1997).

6.3. LINEARIZATION AND SMALL AMPLITUDE WAVE PHENOMENA

6.3.1. Linearization and the linear problem

Indeed, when we consider high Reynolds number, it is necessary to write an asymptotic expansion:

$$U = U_E + \delta_1(Re)\, U_E + \ldots\ldots, \tag{6.22}$$

where $U = (\boldsymbol{u}, p, \rho, T)^T$, and $\delta_1(Re)$ tends to zero when Re tends to infinity.

Obviously for U_E we derive again, from the NS-F equations, the Euler equations:

$$S\frac{D\rho_E}{Dt} + \rho_E \nabla\cdot\boldsymbol{u}_E = 0;$$

$$\rho_E S\frac{D\boldsymbol{u}_E}{Dt} + \frac{1}{\gamma M^2}\nabla p_E + \frac{Bo}{\gamma M^2}\rho_E \boldsymbol{k} = 0; \tag{6.23}$$

$$\rho_E S\frac{DT_E}{Dt} + (\gamma - 1)p_E \nabla\cdot\boldsymbol{u}_E = 0;$$

$$p_E = T_E\,\rho_E.$$

Now if

$$\delta_1 Re \neq O(1), \tag{6.24}$$

then U_1 is a solution of following linearized Euler equations:

$$S\frac{\partial\rho_1}{\partial t} + \boldsymbol{u}_E\cdot\nabla\rho_1 + \boldsymbol{u}_1\cdot\nabla\rho_E + \rho_E\nabla\cdot\boldsymbol{u}_1 + \rho_1\nabla\cdot\boldsymbol{u}_E = 0, \tag{6.25a}$$

$$S\left[\rho_E\frac{\partial \boldsymbol{u}_1}{\partial t}+\rho_1\frac{\partial \boldsymbol{u}_E}{\partial t}\right]+\rho_E(\boldsymbol{u}_E\cdot\nabla)\boldsymbol{u}_1+[(\rho_1\boldsymbol{u}_E+\rho_E\boldsymbol{u}_1)\cdot\nabla]\boldsymbol{u}_E$$
$$+\frac{1}{\gamma M^2}[\nabla p_1+Bo\rho_1\boldsymbol{k}]=0, \qquad (6.25b)$$

$$S\left[\rho_E\frac{\partial T_1}{\partial t}+\rho_1\frac{\partial T_E}{\partial t}\right]+\rho_E(\boldsymbol{u}_E\cdot\nabla)T_1+[(\rho_1\boldsymbol{u}_E+\rho_E\boldsymbol{u}_1)\cdot\nabla]T_E$$
$$+(\gamma-1)[p_E\nabla\cdot\boldsymbol{u}_1+p_1\nabla\cdot\boldsymbol{u}_E]=0, \qquad (6.25c)$$

with

$$p_1 = T_E\,\rho_1 + T_1\,\rho_E. \qquad (6.25d)$$

But, it is necessary to linearize not only the equations, but also the initial and boundary conditions. For example, if we consider an unsteady fluid flow around a solid body, with the equations

$$F(t, \boldsymbol{x};\ \eta) = 0 \Rightarrow z = \eta\, h(t, \boldsymbol{P}), \qquad (6.26)$$

where $\boldsymbol{P} = (x, y)$, in a Cartesian system of coordinates: $\boldsymbol{x} = \boldsymbol{P} + z\,\boldsymbol{n}$, then the slip condition for the nonlinear Euler equations, according to (6.23), is:

$$\text{on } z = \eta\, h(t, \boldsymbol{P}):\ \left[S\frac{\partial}{\partial t}+\boldsymbol{u}_E\cdot\left(\boldsymbol{D}+\frac{\partial}{\partial z}\boldsymbol{n}\right)\right][z-\eta h(t,\boldsymbol{P})]=0. \qquad (6.27)$$

Now, if: $\boldsymbol{u}_E = \boldsymbol{v}_{TE} + w_E\,\boldsymbol{n}$, $\boldsymbol{v}_{TE}\cdot\boldsymbol{n} = 0$, and $\boldsymbol{v}_{TE} = (u_{TE}, v_{TE})$, then, in place of (6.27), we obtain the following slip condition:

$$\text{on } z = \eta\, h(t, \boldsymbol{P}):\ \left[S\frac{\partial}{\partial t}+\boldsymbol{v}_{TE}\cdot\boldsymbol{D}+w_E\frac{\partial}{\partial z}\right][z-\eta h(t,\boldsymbol{P})]=0. \qquad (6.28)$$

In (6.26)-(6.28), the parameter $\eta << 1$ is, in fact, a small parameter related to the geometry of the solid body and in (6.27) it is assumed that: $\nabla \equiv \boldsymbol{D} + \partial/\partial z\,\boldsymbol{n}$, with $\boldsymbol{D}\cdot\boldsymbol{n} = 0$ where $\boldsymbol{D} = (\partial/\partial x, \partial/\partial y)$.

In this case, with the condition: $\eta\, Re \neq O(1)$, the above linear equations (6.25), are derived directly from the nonlinear Euler equations (6.23) for

$U_E\ (x, y, z;\ \eta)$, when we assume the existence of an asymptotic expansion such that:

$$U_E\,(x, y, z;\ \eta\,) = U_0 + \eta\, U_1 + \ldots, \tag{6.29}$$

and the equations for U_0 are again the nonlinear Euler equations (6.23), while for U_1 we obtain, again the linearized system of Euler equations (6.25), relative to U_E. Now, from (6.28), since $\eta << 1$, we derive the following two slip linearized conditions:

$$\boldsymbol{u}_0 \cdot \boldsymbol{n}_0 = w_0 = 0, \text{ on } z = 0; \tag{6.30a}$$

and

$$\boldsymbol{u}_1 \cdot \boldsymbol{n} = w_1 = S\frac{\partial h}{\partial t} + \boldsymbol{v}_{T0} \cdot \boldsymbol{D}h + h\frac{\partial w_0}{\partial z}, \text{ on } z = 0. \tag{6.30b}$$

6.3.2. D'Alembert's equation of acoustics

Usually, when $Bo = 0$, we have simply:

$$U_0 \equiv [u_{T0} = 1,\ v_{T0} = 0,\ w_0 = 0,\ p_0 = \rho_0 = T_0 = 1]$$

and in this case, from (6.25) we derive for U_1 the following linear system of equations:

$$S\frac{\partial \rho_1}{\partial t} + \frac{\partial \rho_1}{\partial x} + \nabla \cdot \boldsymbol{u}_1 = 0,$$

$$S\frac{\partial \boldsymbol{u}_1}{\partial t} + \frac{\partial \boldsymbol{u}_1}{\partial x} + \frac{1}{\gamma M^2}\nabla p_1 = 0, \tag{6.31a}$$

$$S\frac{\partial T_1}{\partial t} + \frac{\partial T_1}{\partial x} + (\gamma - 1)\nabla \cdot \boldsymbol{u}_1 = 0,$$

$$p_1 = \rho_1 + T_1, \tag{6.31b}$$

with the slip condition:

$$\boldsymbol{u}_1 \cdot \boldsymbol{n} = S\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}, \text{ on } z = 0. \tag{6.31c}$$

From (6.31a, b) it is easy to derive a single equation. Firstly, for ρ_1 and $\boldsymbol{u}_1$ we obtain:

$$\left(S\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\rho_1+\nabla\cdot\boldsymbol{u}_1=0,\ \text{and}\ M^2\left(S\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)\boldsymbol{u}_1+\nabla\rho_1=0,$$

and then for ρ_1 the following wave equation:

$$M^2\left[S\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right]\left(S\frac{\partial\rho_1}{\partial t}+\frac{\partial\rho_1}{\partial x}\right)-\nabla^2\rho_1=0, \tag{6.32}$$

and we note that the velocity vector $\boldsymbol{u}_1$ is, in fact, derived from a velocity potential function: $\boldsymbol{u}_1 = \nabla\phi_1$, such that,

$$\frac{p_1}{\gamma}=\rho_1=-M^2\left(S\frac{\partial\phi_1}{\partial t}+\frac{\partial\phi_1}{\partial x}\right). \tag{6.33}$$

as a consequence, for the function $\phi_1(\tau,\ \xi = x - \tau,\ y,\ z)$, with $\tau = t/S$, the following equation of linear homogeneous acoustics (the so-called, "d'Alembert equation") is derived:

$$M^2S^2\frac{\partial^2\phi_1}{\partial^2\tau}-\Delta\phi_1=0, \tag{6.34}$$

where Δ is the Laplacian 3D operator: $\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

6.3.3. Internal waves and filtering

Below, we consider small (of order η) wave motions relative to a reference Eulerian flow (with subscript '$_0$') where the velocity vector $\boldsymbol{u}_0 \equiv 0$ and

$$p_0 = p_0(z),\ \rho_0 = \rho_0(z)\ \text{and}\ T_0 = 1, \tag{6.35a}$$

a so-called "iso-temperature" motion with:

$$p_0(z) = \rho_0(z) = exp\ [-\ Bo\ z]. \tag{6.35b}$$

In this case for: $\boldsymbol{u}_1$, $\pi_1 = p_1/p_0(z)$, $\omega_1 = \rho_1/\rho_0(z)$ and $\theta_1 = T_1$, we derive from (6.25) the following linear system (with $Bo \neq 0$):

$$S\frac{\partial \omega_1}{\partial t} + \nabla \cdot \boldsymbol{u}_1 - Bo(\boldsymbol{u}_1 \cdot \boldsymbol{n}) = 0,$$

$$S\frac{\partial \boldsymbol{u}_1}{\partial t} + \frac{1}{\gamma M^2}[\nabla \pi_1 + Bo\,\omega_1 \boldsymbol{k}] = 0, \tag{6.36}$$

$$S\frac{\partial}{\partial t}\left[\theta_1 - (\gamma - 1)\frac{\pi_1}{\gamma}\right] + \nabla \cdot \boldsymbol{u}_1 + (\gamma - 1)\frac{Bo}{\gamma}(\boldsymbol{u}_1 \cdot \boldsymbol{n}) = 0,$$

$$\omega_1 = \pi_1 - \theta_1.$$

Now, it is easy to eliminate all the unknown functions from (6.36) - all but one, of course, and if the latter is, for instance, π_1, then the following wave equation results:

$$S^2\frac{\partial^2}{\partial t^2}\left[S^3 M^2\frac{\partial^3 \pi_1}{\partial t^3} - S\nabla^2\frac{\partial \pi_1}{\partial t} + SBo\frac{\partial^2 \pi_1}{\partial z \partial t}\right]$$
$$-(\gamma - 1)\left[\frac{Bo}{\gamma M}\right]^2 SD^2\frac{\partial \pi_1}{\partial t} = 0. \tag{6.37}$$

The above wave equation (6.37), for the case of a heavy, stratified fluid motion, (when $Bo \neq 0$) generalizes the classical aerodynamic acoustic wave equation (6.34). We note that in (6.37): $\nabla^2 \equiv \Delta = D^2 + \partial^2/\partial z^2$, with $D^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

If now, we note that, between w_1 $(= \boldsymbol{u}_1 \cdot \boldsymbol{n})$ and π_1 we have the following relation:

$$\gamma M^2 S^2\frac{\partial^2 w_1}{\partial t^2} + (\gamma - 1)\frac{Bo^2}{\gamma}w_1 + \left[\frac{\partial}{\partial z} - (\gamma - 1)\frac{Bo}{\gamma}\right]S\frac{\partial \pi_1}{\partial t} = 0,$$

we can explain why, to equation (6.37), we should add [since, according to (6.30b): $w_1 = S\partial h/\partial t$, on $z = 0$] the slip boundary condition:

$$\left[\frac{\partial}{\partial z}-(\gamma-1)\frac{Bo}{\gamma}\right]S\frac{\partial \pi_1}{\partial t}=-\gamma M^2 S^3\frac{\partial^3 h}{\partial t^3}+(\gamma-1)S\frac{Bo^2}{\gamma}\frac{\partial h}{\partial t},$$

$$\text{on } z=0. \tag{6.38}$$

Obviously, the case

$$M \downarrow 0 \text{ with } S \text{ and } Bo \text{ fixed} \tag{6.39a}$$

is very singular, and gives a strongly degenerate two dimensional Laplace equation for $S\partial\pi_1/\partial t$, in place of (6.37); namely (when $\gamma \neq 1$):

$$SD^2\frac{\partial \pi_1}{\partial t}=0, \tag{6.39b}$$

the vertical structure in z being filtered! On the other hand the Boussinesq case (see §4.7, in Chapter 4), when:

$$M \downarrow 0 \text{ and } Bo \downarrow 0, \text{ such that: } Bo/M = B^*, \text{ and } S \text{ fixed,} \tag{6.40a}$$

filters the acoustic waves and gives, in place of (6.37), the following approximate linear equation, 'à la Boussinesq':

$$\left\{S^2\frac{\partial^2}{\partial t^2}(\nabla^2)+\frac{\gamma-1}{\gamma^2}B^{*2}D^2\right\}S\frac{\partial \pi_1}{\partial t}=0. \tag{6.40b}$$

6.3.4. The three-dimensional linear Boussinesq wave equation

When we consider a three-dimensional obstacle,

$$z = \nu h(x, y), \tag{6.41}$$

then it is necessary to study a 3D flow around and downstream of this obstacle. A relatively simple case is related with the hypothesis that:

$$\nu << 1, \tag{6.42}$$

and in such a case we can linearise the nonlinear Boussinesq equations (4.68) derived in §4.7 of Chapter 4. With the dimensionless variables, if

$$u_0 = 1,\ w_0 = 0 \text{ and } \theta_1 = 0,\ \pi_2 = 0, \text{ at } x \to -\infty,$$

then the steady linear solution of the Boussinesq equations (4.68) has the form:

$$u_0 = 1 + \nu u' + ...,\ v_0 = \nu v' + ...,\ w_0 = \nu w' + ..., \tag{6.44a}$$

$$\theta_1 = \nu\theta' + ...,\ \pi_2 = \nu\pi' + ... \tag{6.44b}$$

In this case, we obtain, in place of the Boussinesq equations (4.68), for the steady case (when $S\partial/\partial t = 0$), taking account (6.44a, b) and negecting the higher order terms (proportional to ν^2), the following linear system:

$$\frac{\partial u'}{\partial x} + \frac{1}{\gamma}\frac{\partial \pi'}{\partial x} = 0;$$

$$\frac{\partial v'}{\partial x} + \frac{1}{\gamma}\frac{\partial \pi'}{\partial y} = 0;$$

$$\frac{\partial w'}{\partial x} + \frac{1}{\gamma}\frac{\partial \pi'}{\partial z} = \frac{B^*}{\gamma}\theta'; \tag{6.45}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0;$$

$$\frac{\partial \theta'}{\partial x} + \Lambda^\circ w' = 0$$

where

$$\Lambda^\circ \equiv B^*\left\{\frac{\gamma - 1}{\gamma} - \Gamma^*(0)\right\} = const.$$

From the first two and the fourth equations of (6.45), we derive a relation between w' and π', namely:

$$\frac{\partial}{\partial x}\left(\frac{\partial w'}{\partial z}\right) = \frac{1}{\gamma}\left[\frac{\partial^2 \pi'}{\partial x^2} + \frac{\partial^2 \pi'}{\partial y^2}\right]. \tag{6.46a}$$

A second relation, between w' and π', is derived from the third and the last equation of the above system (6.45):

$$\frac{\partial^2 w'}{\partial x^2}+\frac{B^*}{\gamma}\Lambda^\circ w'=-\frac{1}{\gamma}\frac{\partial}{\partial x}\left(\frac{\partial \pi'}{\partial z}\right). \tag{6.46b}$$

Now, by eliminating π' from (6.46a, b) we derive a single equation for w':

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\frac{\partial^2 w'}{\partial x^2}+K_0^{\,2}\left[\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial y^2}\right]=0, \tag{6.47a}$$

where

$$K_0^{\,2}=\frac{B^*}{\gamma}\Lambda^\circ. \tag{6.47b}$$

For the above equation (6.47a) we must write, relative to z, the following two linearized conditions:

$$z=0:\ w'=\frac{\partial h}{\partial x}\ \text{ and } z=1:\ w'=0, \tag{6.48}$$

when we consider the motion in a curvilinear channel with a bottom given by the equation (6.41), and limited by an upper plane $z = H_c$ (with dimension).

The problem (6.47), (6.48) is considered in detail in our book, [Zeytounian (1991, §15)], and for various references concerning wave phenomena in the atmosphere, see the "background reading" in the end of Chapter III, in Zeytounian (1991).

The lee-waves problem (the influence of a mountain in a baroclinic, stratified, atmosphere) is strongly influenced by the relief slip condition and also by the upstream flow conditions. In an unbounded atmosphere the radiation [see, Guiraud (1979) for the 3D steady case] condition at infinity (in altitude) plays also an essential role.

6.4. TWO-DIMENSIONAL NONLINEAR WAVE EQUATIONS

6.4.1. Two-dimensional nonlinear Boussinesq steady waves: Long's problem

In the 2D case (in the plane *(x, z)),* according to the continuity equation (4.68d), in the Boussinesq system (4.68), where: $v_0 = 0$ and $\partial/\partial y = 0$, we can introduce a steady stream-function $\psi_B(x, z)$, such that:

$$u_0 = \frac{\partial \psi_B}{\partial z} \text{ and } w_0 = -\frac{\partial \psi_B}{\partial x}. \tag{6.49}$$

In this case from the equation (4.68e), for θ_1, when $S\partial/\partial t = 0$ (steady case), and $v_0 = 0$, $\partial/\partial y = 0$, we derive the following (first) integral:

$$\theta_1 + \Lambda^\circ z = H(\psi_B). \tag{6.50}$$

On the other hand, from the Boussinesq equations of motion for u_0 and w_0 (the equations (4.68a) and (4.68c)) we can eliminate the terms with π_2, and in such case we derive the following second-vorticity-(first) integral:

$$\frac{\partial^2 \psi_B}{\partial x^2} + \frac{\partial^2 \psi_B}{\partial z^2} - \frac{B^*}{\gamma} z \frac{dH(\psi_B)}{d\psi_B} = F(\psi_B), \tag{6.51}$$

where $H(\psi_B)$, in (6.50), and $F(\psi_B)$, in (6.51), are two arbitrary functions of ψ_B only. If, now, we assume that at infinity upstream, when $x \to -\infty$,

$$u_0 = 1,\ w_0 = 0 \text{ and } \theta_1 = 0, \tag{6.52a}$$

with dimensionless variables, then we obtain:

$$\psi_B = z \equiv \psi_{B\infty}, \text{ at } x \to -\infty,$$

$$H(\psi_{B\infty}) = \Lambda^\circ \psi_{B\infty},\ F(\psi_{B\infty}) = -\frac{B^*}{\gamma} \Lambda^\circ \psi_{B\infty}. \tag{6.52b}$$

Finally, we derive, for $\psi_B(x, z)$, the following linear Helmholtz equation:

$$\frac{\partial^2 \psi_B}{\partial x^2} + \frac{\partial^2 \psi_B}{\partial z^2} + K_0^{\,2}[\psi_B - z] = 0 . \qquad (6.53)$$

The dominant feature, from the mathematical point of view, is that the linearity of the equation (6.53) is not related to any one hypothesis of small perturbations and linearization!

But an important difficulty remains: it is that the slip boundary condition on the wall of the obstacle, $z = \nu h(x)$:

$$\psi_B(x, \nu h(x)) = 0, \qquad (6.54)$$

is *nonlinear and cannot be linearized without invoking the hypothesis of small disturbances.* Namely, if:

$$\psi_B - z = -\delta(x, z), \qquad (6.55)$$

then for the (linear!) Helmholtz equation:

$$\frac{\partial^2 \delta}{\partial x^2} + \frac{\partial^2 \delta}{\partial z^2} + K_0^{\,2}\delta = 0 , \qquad (6.56)$$

we can write the following boundary conditions, in the framework of the lee-wave problem [in a bounded, at $z = 1$, duct with a curvilinear bottom $z = \nu h(x)$]:

$$z = \nu h(x):\ \delta = \nu h(x),$$

$$z = 1:\ \delta = 0,$$

$$x \to -\infty:\ \delta \to 0, \qquad (6.57)$$

$$x \to +\infty:\ |\delta| < \infty$$

and the first slip condition above is strongly nonlinear. Namely:

$$\delta(x, z = \nu h(x)) = \nu h(x)!$$

The reader can find in the book by Zeytounian (1991, §26) a detailed resolution of the above (so-called "Long's") problem; see also, for instance, Long's (1953) pioneer paper.

6.4.2. The Boussinesq model as an inner approximation. Mile's problem and the Guiraud-Zeytounian double-scale approach

In an unbounded atmosphere, it is necessary to impose for δ, the solution of (6.56), a radiation condition (à la Sommerfeld) when $r = [x^2 + z^2]^{1/2} \to \infty$: namely:

$$\delta \approx \left[\frac{2K_0}{\pi r}\right]^{\frac{1}{2}} \sin\theta \, Real\left\{G(\cos\theta) exp\left[i\left(K_0 r - \frac{\pi}{4}\right)\right]\right\}, \quad (6.58a)$$

where the function $G(\cos\theta)$ is arbitrary and depends on the form of the relief via the function $h(x)$.

So as to satisfy the upstream behaviour at infinity (when $x \to -\infty$), the following condition must also be imposed:

$$G(\cos\theta) = 0, \text{ for } \cos\theta < 0. \quad (6.58b)$$

It is pointed out that the polar coordinates, r, θ, in the upper half-plane $z > 0$ are defined such that: $x = r\cos\theta$ and $z = r\sin\theta$. It is also important to note that the "infinity in altitude" relative to z, for the Boussinesq model problem, must be understood as a boundary condition for the 'inner' vertical Boussinesq (dimensionless) coordinate z, which is matched with an outer coordinate; namely: $\zeta = M z$. This outer vertical coordinate ζ, taking into account the upper condition at the top of the troposphere, $H^* = RT^*(0)/g >> H_B$, given by (4.69), is lost in the framework of the Boussinesq 'inner" problem. We note that the dimensionless equation of the top of the troposphere is, in fact, $z = H^*/H_c = 1/Bo$, and as a consequence the outer region is bounded by: $\zeta = 1/B^*$, with $B^* = Bo/M = O(1)$.

The reader can find, in Guiraud and Zeytounian (1979) paper, an asymptotic analysis of this very interesting problem relative to the lee-waves in the whole troposphere and the role of the upper boundary condition. The inner problem is, in fact, the problem considered by Miles (1968) and also by Kozhevnikov (1963, 1968), with the conditions (6.58a, b), which express that "no waves are radiated inwards". In Guiraud and Zeytounian (1979) the associated outer problem is considered and it is shown that the upper and lower boundaries of the troposphere alternately reflect internal short gravity waves excited by the lee-waves arising from the inner (Boussinesq) problem, with a wavelength of the order of the Mach number, the characteristic scale in the outer region.

As a consequence, there is a *double scale* built into the solution and we must take care of it. The important point is that:

These short gravity waves propagate downstream and no feedback occurs on the inner flow close to the mountain (to lower order at least!).

Thus, we should understand that the imposed upper boundary at the top of the troposphere is an *artificial one,* having asymptotically no effect on the inner Boussinesq flow which is the only really interesting one. More precisely, in order to obtain an outer approximation of the exact lee-wave 2D steady Eulerian compressible adiabatic problem for the vertical displacement of the streamline

$$\delta = \frac{1}{\nu}\left(Bo\, z - z^*\right), \tag{6.59}$$

with $Bo = B^* M$, such that the upper (exact) condition, on the top of the troposphere, $\delta(x, 1/Bo) = 0$, may be applied, it is necessary to introduce the outer variables:

$$\xi = M x,\ \zeta = M z,\ \Delta(\xi,\zeta,M) = \frac{1}{M^{1/2}}\delta, \tag{6.60}$$

and the choice of scaling for δ is dictated by matching with (6.58a). In this case we have: $\Delta(\xi,\ 1/B^*) = 0$. As a consequence in the outer region a dominant equation is derived. For the particular case of $\Gamma^*_0 \equiv -(dT^*/dz^*) = const$, we obtain the following equation:

$$\frac{\partial^2\Delta}{\partial\xi^2} + \frac{\partial^2\Delta}{\partial\zeta^2} + \frac{\Phi(\zeta)}{Ma^2}\Delta = \frac{B^*}{\gamma}\left[\frac{d\log\Phi(\zeta)}{d\zeta}\Delta + \left[1 + \left(\frac{\gamma K_0}{B^*}\right)^2\right]\frac{\partial\Delta}{\partial\zeta}\right], \tag{6.61a}$$

where (since $\Gamma^*_0 \equiv const$)

$$0 < \Phi(\zeta) = \frac{K_0^{\,2}}{1 - \frac{R}{g}\Gamma_0^* B^* \zeta}. \tag{6.61b}$$

Indeed, it is possible to consider (in the asymptotic sense) the above equation (6.61a), for $\Delta(\partial\xi, \zeta)$, as an outer one and the Helmholtz equation (6.56), for $\delta(x, z)$, as an inner equation. Therefore, the upper condition: $w = 0$ on $z = 1/Bo$, belongs to an outer limiting process:

$$Bo = B^* M; \; M \to 0, \text{ with } \xi, \zeta \text{ and } B^* \text{ fixed.} \tag{6.62}$$

Such an outer asymptotic approximation is worked out by Guiraud and Zeytounian (1979).

6.4.3. The isochoric case

If we consider as simple example, again, the case of a steady 2D isochoric flow then, in place of equations (4.48) - see §4.5 in Chapter 4 - we obtain the following system of equations (written without the '$_i$') for the velocity components $u(x, z)$ and $w(x, z)$, the pressure $p(x, z)$ and the density $\rho(x, z)$, where $x = x_1$ and $z = x_3$, $u = u_1$ and $w = u_3$:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \tag{6.63a}$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0, \tag{6.63b}$$

$$u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0, \tag{6.63c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{6.63d}$$

This above nonlinear system of equations (6.63a, b, c, d) is written with the dimensional quantities. According to (6.63d), we can define a stream-function:

$$\psi = \psi(x, z) \Rightarrow u = -\frac{\partial \psi}{\partial z}, \; w = \frac{\partial \psi}{\partial x} \tag{6.64}$$

and ρ is conservative along the streamlines:

$$\rho = R(\psi). \tag{6.65}$$

But, from the above steady 2D isochoric dynamical equations (6.63a,b) we can also derive the following conservation equation along the streamlines (if we eliminate the pressure p):

$$\left[\frac{\partial\psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\right]\left\{R(\psi)\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right]\right\}$$
$$+\left[\frac{\partial\psi}{\partial z}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\right]\left\{\frac{dR(\psi)}{d\psi}\left[\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2\right] + gz\right]\right\} = 0 \tag{6.66}$$

and as a consequence we obtain for the function $\psi(x, z)$ the following PD equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{d\log R(\psi)}{d\psi}\left[\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] + gz\right] = F(\psi). \tag{6.67}$$

In (6.67), the two functions $R(\psi)$ and $F(\psi)$, which depend only on ψ, can be determined from the boundary conditions. In particular, when we consider (again!) the isochoric motion in a duct having a curvilinear bottom, confined to the vicinity of $x = 0$ (for example, between the two abscissae, namely: $x = -L°/2$ and $x = +L°/2$), then these functions can be determined from the conditions far away upstream at infinity: for $x \to -\infty$, where:

$$z = z_\infty,\ \psi = \psi_\infty(z_\infty) \text{ and } R(\psi) = R(\psi_\infty). \tag{6.68}$$

More precisely, below, we assume that at infinity upstream, for $x \to -\infty$, we have:

$$u = U_\infty(z_\infty),\ w = 0,\ \rho = \rho_\infty(z_\infty), \tag{6.69}$$

and in this case, since $u = -\partial\psi/\partial z$, we can write:

$$\psi = -\int_0^{z_\infty} U_\infty(z_\infty)dz = \psi_\infty(z_\infty), \text{ at infinity upstream}, \tag{6.70}$$

and as a consequence

$$z_\infty = (\psi_\infty)^{-1}(\psi) \equiv z_\infty(\psi). \tag{6.71}$$

From (6.68)-(6.71) we can now determine the function $F(\psi)$ - for this it is necessary, first, to write our above PD equation (6.67) at infinity upstream; namely:

$$-\frac{dU_\infty(z_\infty)}{dz_\infty} + \frac{1}{2}\left(\frac{1}{R(\psi_\infty)}\frac{dR(\psi_\infty)}{d\psi_\infty}\right)\left[(U_\infty(z_\infty))^2 + 2gz_\infty(\psi_\infty)\right] = F(\psi_\infty).$$

But, since $F(\psi_\infty)$ and $R(\psi_\infty)$ are functions *only* of ψ, then obviously:

$$F(\psi) = U_\infty\frac{dU_\infty}{d\psi} + \frac{1}{\rho_\infty}\frac{d\rho_\infty}{d\psi}\left[\frac{1}{2}U_\infty{}^2 + gz_\infty(\psi)\right]. \tag{6.72}$$

where the known functions U_∞ and ρ_∞, according to (6.69) and (6.71), are dependent on ψ through $z_\infty(\psi)$. On the other hand:

$$R(\psi_\infty) = \rho_\infty(z_\infty(\psi)) \equiv \rho_\infty(\psi). \tag{6.73}$$

Finally, for the 2D steady stream function $\psi(x, z)$ we derive the following single nonlinear elliptic second-order partial differential equation:

$$\begin{aligned}\Delta\psi + \frac{1}{2\rho_\infty}\frac{d\rho_\infty}{d\psi}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 + 2gz\right] \\ = U_\infty\frac{dU_\infty}{d\psi} + \frac{1}{\rho_\infty}\frac{d\rho_\infty}{d\psi}\left[\frac{1}{2}U^2 + gz_\infty(\psi)\right]\end{aligned} \tag{6.74}$$

When, for simplicity, $U_\infty = U^\circ = constant$, then

$$z_\infty(\psi) = -\frac{1}{U^\circ}\psi \text{ and } \frac{d\rho_\infty}{d\psi} = -\frac{1}{U^\circ}\frac{d\rho_\infty}{dz_\infty}. \tag{6.75}$$

If now we introduce the variation of the vertical position of a streamline relative to its unperturbed position at upstream infinity:

$$\delta = z + \frac{1}{U^\circ}\psi \equiv z - z_\infty(\psi), \tag{6.76}$$

then, for the function $\delta(x, z)$, we derive the following (elliptic) nonlinear (quasilinear!) equation:

$$\frac{\partial^2 \delta}{\partial x^2} + \frac{\partial^2 \delta}{\partial z^2} - \frac{g}{\rho_\infty (U^\circ)^2}\frac{d\rho_\infty}{dz_\infty}\delta$$
$$= \frac{1}{2}\frac{dLog\rho_\infty}{dz_\infty}\left[\left(\frac{\partial \delta}{\partial x}\right)^2 + \left(\frac{\partial \delta}{\partial z}\right)^2 - 2\frac{\partial \delta}{\partial z}\right] \tag{6.77}$$

in place of (6.74).

Again, a very simple case is:

$$-\frac{dLog\rho_\infty}{dz_\infty} = \frac{1}{g}(N^\circ)^2 = constant, \tag{6.78}$$

where N° is the internal frequency of Brünt-Väisälä, which corresponds to:

$$\rho_\infty(z_\infty) = \rho_\infty(0)exp\left[-\frac{1}{g}(N^\circ)^2 z_\infty\right]. \tag{6.79}$$

In such a case, in place of (6.77), we derive an elliptic equation with constant coefficients:

$$\frac{\partial^2 \delta}{\partial x^2} + \frac{\partial^2 \delta}{\partial z^2} + \left[\frac{U^\circ}{N^\circ}\right]^2 \delta + \frac{1}{2g}(N^{\circ 2})\left[\left(\frac{\partial \delta}{\partial x}\right)^2 + \left(\frac{\partial \delta}{\partial z}\right)^2 - 2\frac{\partial \delta}{\partial z}\right] = 0. \tag{6.80}$$

For an isochoric motion in a duct having a curvilinear bottom, confined to the vicinity of $x = 0$, between the abscissae $x = -L^\circ/2$ and $x = +L^\circ/2$, and an upper flat roof, at $z = H^\circ$, we can write the following boundary conditions for the equation (6.80):

$$\delta\left(x, h^{\circ}h\left(\frac{x}{L^{\circ}}\right)\right) = h^{\circ}h\left(\frac{x}{L^{\circ}}\right),\ x \in \left[-\frac{L^{\circ}}{2}; +\frac{L^{\circ}}{2}\right]; \tag{6.81a}$$

$$\delta(-\infty, z_{\infty}) = 0; \tag{6.81b}$$

$$\delta(x, H^{\circ}) = 0; \tag{6.81c}$$

$\delta(x, z)$ is bounded at the downstream infinity. (6.81d)

The last condition (6.81d) is the only (physically) admissible condition, when x tends to infinity downstream, because of the *lee-wave phenomenon far downstream* at the curvilinear bottom, with the equation:

$$z = h^{\circ}h\left(\frac{x}{L^{\circ}}\right) \text{ when } x \in \left[-\frac{L^{\circ}}{2}; +\frac{L^{\circ}}{2}\right], \tag{6.82a}$$

$$z = 0 \text{ when } x > +\frac{L^{\circ}}{2} \text{ and } x < -\frac{L^{\circ}}{2}. \tag{6.82b}$$

It is interesting to note that the generalized Bernoulli equation considered for the 2D steady isochoric waves, which are governed by equation (6.77), is obtained by setting the expression:

$$-\frac{1}{2\rho_{\infty}}\frac{d\rho_{\infty}}{dz_{\infty}}\left[\left(\frac{\partial\delta}{\partial x}\right)^2 + \left(\frac{\partial\delta}{\partial z}\right)^2 - 2\frac{\partial\delta}{\partial z} + \frac{2g}{(U^{\circ})^2}\delta\right],$$

in (6.77), equal to zero. Consequently, the dynamical boundary condition at the free upper surface, $z = H^{\circ} + \delta$, is:

$$\left(\frac{\partial\delta}{\partial x}\right)^2 + \left(\frac{\partial\delta}{\partial z}\right)^2 - 2\frac{\partial\delta}{\partial z} + \frac{2g}{(U^{\circ})^2}\delta = 0. \tag{6.83}$$

Since the free surface is a streamline, the kinematic condition there is satisfied identically. In the paper by Weidman (1978) the reader can find a study of internal solitary wave solutions of the equation (6.80) with the two boundary conditions in z; namely (6.83) on $z = H^{\circ} + \delta$, and $\delta(x, z = 0) = 0$.

The results given in Weidman (1978) indicate the dramatic changes which take place due to free-surface effect.

6.4.4. Weakly nonlinear long internal waves in stratified flows

The propagation of long nonlinear internal waves, including solitary waves, is possible whenever either the basic flow profile or an adjacent boundary acts as a horizontal waveguide. Such waves are probably generated fairly freqently both in the oceanic thermocline [or pycnocline - a region of thickness *O(h)* of sharp density variation exists within the stratified isochoric fluid flow] and the atmospheric tropopause [a thin transition region between the troposphere (the lower part of the atmosphere) and the stratosphere (which is the region above the tropopause)]. An important advance in the theory occurred with the simultaneous appearance of papers by Benjamin (1967) and Davis and Acrivos (1967) dealing with solitary internal waves in unbounded fluids. In that case it was found that the amplitude evolution of the wave was governed by the so-called BDAO equation [Ono (1975) subsequently derived conservation laws whose existence indicates that solitary wave solutions, of the BDAO equation (see below the equation (6.84)), possess the soliton property- i.e., individual solitons can emerge unaffected from collisions apart from a spatial phase shift vis-a-vis their pre-collision trajectories)]:

$$\frac{\partial A}{\partial \tau}+\gamma A\frac{\partial A}{\partial \xi}-\delta\frac{\partial^2}{\partial \xi^2}\left\{\frac{1}{\pi}P\int_{-\infty}^{\infty}\left[\frac{A(\xi',\tau)}{\xi-\xi'}\right]d\xi'\right\}=0, \qquad (6.84)$$

where P denotes the Cauchy principal value. The amplitude A in (6.84) is related to the perturbation stream function by:

$$\psi^* = A(\xi, \tau)\,\phi(z^*),$$

where

$\tau = \varepsilon^p t$ is a slow time scale and $\xi = \varepsilon^q \theta$, with $\theta = x - ct$,

describe slow spatial modulation in a coordinate system moving at the wave speed, namely: $\partial/\partial t \to c\varepsilon^q \partial/\partial\xi + \varepsilon^p \partial/\partial\tau$, where p and q are to be specified subsequently [see, for instance, Maslowe and Redekopp (1979)] and the contraction of the horizontal coordinate being required in order to deal with long waves. In order to formulate a long-wave, finite-amplitude theory it is necessary to assume that: $\varepsilon = a/h << 1$ and $\mu = h/\lambda << 1$, where a and λ are the amplitude and wavelength, respectively. For the function $\phi(z^*)$ we have an eigenvalue problem for the so-called, Taylor-Goldstein (T-G) dimensionless equation:

$$\frac{d}{dz^*}\left[\rho^*\frac{d\phi}{dz^*}\right]+\left\{\frac{r^*(z^*)}{(U^*-c)^2}-\frac{1}{U^*-c}\frac{d}{dz^*}\left[\rho^*\frac{dU^*}{dz^*}\right]\right\}\phi=0, \tag{6.85}$$

where $U^*(z^*)$ and $\rho^*(z^*)$ are the background dimensionless flow velocity (in the horizontal direction) and density. The parameter ($\Delta\rho$ is a reference value for the perturbation of the density ρ'):

$$r^*(z^*)=-\frac{d\rho^*}{dz^*}\frac{h}{\sigma},\ \sigma=\frac{\Delta\rho}{\rho(0)}, \tag{6.86}$$

in the above T-G equation (6.85), is related to the Brunt-Vaisälä frequency. The Boussinesq case corresponds to:

$$\sigma = O(\varepsilon) << 1 \text{ and } \rho^*(z^*) = 1 + \sigma\rho^\circ(z^*), \tag{6.87}$$

and in this case the infuence of $\rho^*(z^*)$ is negligible in the T-G equation (6.85). In the paper by Tung, Ko and Chang (1981) an evolution equation in a finite-depth fluid for weakly nonlinear long internal waves is derived in a stratified and sheared medium. The governing equations are the 2D unsteady equations for an isochoric fluid flow, and the perturbation of the density ρ', relative to the background density, is scaled relative to $\Delta\rho$. The evolution equation derived in Tung, Ko and Chang (1981) reduces to the KdV equation when the depth is small compared to the wavelength, and to the BDAO equation when the depth is large compared to the wavelength. But in the equation derived in Tung, Ko and Chang (1981) the presence of a critical level is do not take into account - when a critical level is present, the effects of shear on the wave depend critically on the relative magnitudes of nonlinearity and viscosity within the critical layer. For the shearless case, the evolution equation for a weakly nonlinear internal wave on a thin pynocline in a finite-depth fluid has been derived by Kubota, Ko and Dobbs (1978). In Maslowe and Redekopp (1980), the theory is generalized to allow for a radiation condition when the region outside the stratified shear layer is unbounded and weakly stratified. In this case, the evolution equation contains a damping term describing energy loss by radiation which can be used to estimate the persistence of solitary waves or nonlinear wave packets in realistic environments. In the book by Miropolsky (1981, pp.202-207), for the weakly nonlinear waves in a narrow pycnocline, when:

$$r^*(z^*) = r_0^*(z^*)\,,\ \text{for } -1 < z^* < +1,$$

$$r^*(z^*) = 0,\ \text{for } +1 < z^* < +\infty, \tag{6.88}$$

$$r^*(z^*) = 0,\ \text{for } -\infty < z^* < -1,$$

the following equation (à la Benjamin) is derived:

$$\frac{1}{\pi}\left[\rho^*(+1)W(+1) - \rho^*(-1)W(-1)\right]\frac{d}{d\xi}\left\{P\int_{-\infty}^{\infty}\frac{U(\xi')}{\xi-\xi'}d\xi'\right\}$$
$$= \alpha U(\xi) + \beta[U(\xi)]^2 \tag{6.89}$$

where the function $W(z^*)$ is the solution of the following Sturm-Liouville problem (for a non-Boussinesq flow, when $\delta \neq 0$):

$$W'' - \delta r_0^*(z^*)\,W' + \gamma r_0^*(z^*)\,W = 0;\ W(\pm 1) = 0, \tag{6.90}$$

and in the narrow pycnocline: $\psi^* \sim U(\xi)\,W(z^*)$. In the book (unfortunately in Russian!) of Miropolsky (1981) the author considers modern theoretical methods for investigating wave motions, results of their application to internal wave problems, as well as oceanic observational data. Nonlinear effects in connection with the presence of thin structures in the vertical density distribution are analysed. Oceanic observational data interpretation is given as well as methods for separating internal waves and turbulence. This book contains many Russian references up to 1980. Finally, in the recent paper by Staquet and Sommeria (1996) the reader can find a very pertinent discussion concerning various aspects of internal (gravity) waves in stably stratified flows.

6.4.5. From the isochoric equations to the Boussinesq equations

For simplicity we consider only the steady 2D case and in this case for isochoric motion we have the elliptic equation (6.80). Now, first, we introduce the following dimensionless function and variables:

$$\Delta = \frac{\delta}{h^\circ},\ \xi = \frac{x}{L^\circ} \text{ and } \zeta = \frac{z}{H^\circ}, \tag{6.91}$$

where h°, L° and H° are three length scales which are present in the boundary conditions (6.81a) and (6.81c). For the dimensionless perturbation of the streamline, $\Delta(\xi, \zeta)$ we obtain the following dimensionless isochoric equation:

$$\varepsilon^2 \frac{\partial^2 \Delta}{\partial \xi^2} + \frac{\partial^2 \Delta}{\partial \zeta^2} + \sigma^2 \Delta + \frac{1}{2} \nu Bo \left[\varepsilon^2 \left(\frac{\partial \Delta}{\partial \xi} \right)^2 + \left(\frac{\partial \Delta}{\partial \zeta} \right)^2 \right] - Bo \frac{\partial \Delta}{\partial \zeta} = 0, \tag{6.92}$$

In the above dimensionless equation (6.92) the following dimensionless parameters appear:

$$\varepsilon = \frac{H^\circ}{L^\circ},\ \nu = \frac{h^\circ}{H^\circ},\ \sigma^2 = \frac{Bo}{(Fr_{H^\circ})^2};\ Bo = \frac{H^\circ (N^\circ)^2}{g}. \tag{6.93a}$$

where

$$Fr_{H^\circ} = \frac{U^\circ}{\sqrt{g H^\circ}}, \tag{6.93b}$$

is the Froude number formed with H°. When Bo tends to zero we obtain again the Helmholtz equation, as in the case of the Boussinesq 2D steady case [see the equation (6.56)]. But for this it is necessary that

$$\sigma^2 = O(1), \text{ or } \frac{H^\circ N^\circ}{U^\circ} = O(1). \tag{6.94}$$

We observe that the nonlinear terms (proportional to Bo!) in the isochoric equation (6.92) disappear when we pass to the Boussinesq case. As consequence, we can consider the Boussinesq case as a particular degenerate case of the isochoric case.

6. 5. NONLINEAR LONG SURFACE WAVES ON WATER (POTENTIAL THEORY)

6.5.1. Potential Flows

From 1840 to 1915, the study of potential flows [governed by Laplace's equation] was the most active branch of analytical (Lagrange's) fluid mechanics. In 1851, Riemann gave an intuitive "proof" of the fact that the potential flow around any connected two-dimensional obstacle can be obtained from the flow around a circular cylinder by a conformal map. At about the same time, Lagrange's "proof"of the permanence of potential flow in an in viscid fluid was greatly clarified and generalized by Kelvin's theorem (1869) on the in variance of circulation (defined as the kinematic analogue of work): *"circulation was invariant in time for any closed path C moving with the fluid"*. His result implies the persistence of potential flow in an inviscid fluid of constant density. For unsteady potential flows of such a fluid, moreover, Kelvin derived the generalized Bernoulli equation, namely (we assume that the external force - in fact the gravity - is derived from a potential U):

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\nabla\phi\cdot\nabla\phi+\frac{p}{\rho_0}+U=constant. \tag{6.95}$$

A fuller proof of the general surface slip condition:

$$\frac{D(z-\zeta)}{Dt}=0\text{, on } z=\zeta(t,x,y), \tag{6.96}$$

where: $D/Dt = \partial/\partial t + \phi_x\partial/\partial x + \phi_y\partial/\partial y + \phi_z\partial/\partial z$, and $z = \zeta(t, x, y)$ is the free (unknown) surface, is due to Kelvin (1848) and for the first investigation of progressive waves in a channel, see Green (1839) and also Airy's Treatise (1845).

The theory of potential flows with a free surface at constant pressure (waves on the free surface of the water) also developed brilliantly and we indicate below some main points of this development [following to Zeytounian review paper (1995), dedicated by the Editorial Board of the journal: "Uspekhi Fizicheskikh Nauk" (Russian Academy of Sciences) to the 30th anniversary of the publication of the paper by Zabusky and Kruskal (1965), in which the term "soliton" was mentioned for the first time in the scientific literature].

6.5.2. Formulation of the water-waves problem

Obviously, for the Laplace equation:

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = \Delta \phi = 0\,, \tag{6.97}$$

which governs an incompressible, irrotational Eulerian unsteady (potential) fluid flow, we cannot impose any initial conditions! But this Laplace equation is very appropriate for the investigation of waves on (incompressible) water and in this case it is necessary to consider a free-boundary problem, i.e. a problem for which the fluid (water) is not contained in a given domain but can move freely. Usually, for Laplace's equation (6.97), one boundary condition is given (on the contour line containing the fluid), but only when the boundary is known. Otherwise two unsteady (dynamic and kinematic) conditions are needed (and also two initial conditions) at the free surface (or interface): $z = \zeta(t, x, y)$, because the surface position $\zeta(t, x, y)$ has to be determined as well as the potential function $\phi(t, x, y, z)$. For the free-surface problem [for the two unknown functions $\phi(t, x, y, z)$ and $\zeta(t, x, y)$] governing nonlinear waves on water, we can consider two physical problems. First the so-called "signalling" (two - dimensional) problem and in this case we have as initial conditions (we assume that the flow is initially at rest in a semi-infinite channel $x > 0$):

$$\phi(0, x, y, z) = 0, \text{ and } \zeta(0, x) = 0, \text{ when } x > 0, \tag{6.98a}$$

and at initial time $t = 0$ an idealized wave-maker at $x = 0$ will generate a horizontal velocity disturbance

$$\frac{\partial \phi}{\partial x} = W^\circ B\left(\frac{t}{t^\circ}\right), \text{ for } x = 0 \text{ and } t > 0, \tag{6.98b}$$

where W° and t° are the characteristic velocity and time scales associated with the wave-maker idealized by the function $B(t/t^\circ)$. A second category of problem for water waves in the infinite channel is obtained by specifying an initial surface shape but zero velocity:

$$\text{for } t = 0\text{: } \zeta = a^\circ \zeta^\circ\left(\frac{x}{l^\circ}, \frac{y}{m^\circ}\right), \text{ and } \phi(0, x, y, z) = 0, \tag{6.98c}$$

where $l°$ and $m°$ are the characteristic wavelengths (in the x and y directions) for the three-dimensional water wave motion and $a°$ is a characteristic amplitude for the initial elevation of the free surface characterized by the given function $\zeta°(x/l°, y/m°)$. For the above free surface problem, i.e. a problem for which the fluid is not contained in a given domain but can move freely, two conditions (kinematic and dynamic) are necessary. Namely, for an inviscid incompressible fluid (water) when we consider the wave on the water (in this case the problem for an irrotational flow is governed by the Laplace equation) an obvious physical simple condition is (if we assume that the surface tension is negligible):

$$p = p°, \text{ at the interface between water and air above,} \tag{6.98d}$$

where $p°$ denotes here the (constant) air pressure on the interface. Now, since the equation of the interface is $\zeta(t, x, y)$, in a Cartesian system of coordinates $(0; x, y, z)$, then from the incompressible Bernoulli integral (6.95), for the potential fluid flow, we obtain the following dynamic condition on the interface [with a convenient choice of the constant $\equiv p°/\rho_0$ and $U = g\, z$, in (6.95)]:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla \phi)^2 + g\zeta = 0, \text{ on } z = \zeta(t, x, y), \tag{6.98e}$$

and since the interface is a material wave surface we also have a kinematic condition (consequence of (6.96)):

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y}, \text{ on } z = \zeta(t, x, y). \tag{6.98f}$$

Finally, if we assume that the water rests on a horizontal and impermeable bottom of infinite extend at $z = - h_0$, where $h_0 = constant$, is supposed finite, then we have the following simple (flat) bottom boundary condition for the Laplace equation:

$$\frac{\partial \phi}{\partial z} = 0, \text{ on } z = - h_0. \tag{6.98g}$$

Now, if we introduce the following dimensionless quantities using the depth of the water h_0 and the characteristic velocity $c_0 = (gh_0)^{1/2}$:

$$\phi^* = \frac{\phi}{h_0 c_0}; \; (x^*, z^*) = \frac{(x, z)}{h_0}; \; t^* = \frac{c_0}{h_0} t; \; \zeta^* = \frac{\zeta}{h_0}, \qquad \textbf{(6.98h)}$$

then we obtain the dimensionless classical 2D water waves problem (dropping the asterisks *):

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \; 0 < z < 1 + \zeta(t, x);$$

$$\frac{\partial \phi}{\partial z} = 0, \text{ on } z = 0; \qquad \textbf{(6.98i)}$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x}, \text{ on } z = 1 + \zeta(t, x);$$

$$\frac{\partial \phi}{\partial t} + \zeta + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] = 0, \text{ on } z = 1 + \zeta(t, x).$$

6.5.3. From Cauchy and Poisson to Airy and Stokes

The theory of the (infinitesimally small) waves produced in deep water by local disturbances of a free surface was investigated in two classical memoirs by Cauchy (1815 - but published in 1827) and Poisson (1816), and concerning this problem see, also, the papers by Rayleigh (1876) and Popoff (1858). The determination of the waveforms which satisfy the conditions of uniform propagation without change of type, when the restriction to "infinitesimally small" amplitude waves is abandoned, forms the subject of the classical research by Stokes (1847) and of many subsequent investigations (Stokes expansion). For this problem, see also Rayleigh's (1876) results. In fact, the validity of the Stokes expansion requires that: (a) the amplitude must be smaller than the wavelength; (b) the amplitude of water waves must be less than the depth, or the wave properties must vary little over a distance of the same order as the depth. The convergence proofs of the Stokes expansion were given by Levi Civita (1925) and Struik (1926) - but *convergence does not imply stability and the Stokes waves in deep water are, in fact, unstable*! A system of exact equations, expressing a possible form of wave motion when the depth of the fluid is infinite, was given so as long ago as 1802 by Gerstner (1809) and at a later period independently by Rankine (1863). Indeed, the Gerstner "trochoidal waves" are *exact rotational* (vorticity is not zero!) solutions of the Euler equations (for an inviscid and incompressible fluid). The "shallow water" theory is governed by a system of equations favoured by Airy (1845), who first formulated the limit model equations for the analysis of very long waves of finite amplitude in shallow water. In the one-dimensional case, these *Airy equations* are the *Saint-Venant* (1871) *hydraulic equations.*

6.5.4. The Boussinesq and KdV equations

However, the effects of dispersion do not appear in the Airy equations. These dispersion effects are present in the Boussinesq (1871, 1872 and 1877) equations. Russell (1844), in his interesting experimental investigations paid great attention to a particular type of wave which called the solitary wave and concerning some historical facts related to these waves, see the paper by Miles (1980). This is a wave consisting of a single elevation, of height not necessarily small compared with the depth of the fluid, which (if properly started) may travel for a considerable distance along a uniform canal with little or no change. But this description of the wave as a solitary elevation of finite amplitude and constant profile contradicts Airy's shallow water theory prediction that a wave of finite amplitude cannot propagate without change of profile!

The conflict between Russell's observations and Airy's shallow water theory (and also Stokes' expansion, for oscillatory waves of constant profile) was resolved independently by Boussinesq (in the papers published during the years 1871-1877) and Rayleigh (1876), by appropriately allowing for the vertical acceleration - which is ultimately responsible for dispersion, but is neglected in the Airy's shallow water theory.

6.5.4a. TheBoussinesq equation

As well as for finite amplitude, the Boussinesq equation, leads to the solution:

$$\zeta = a^\circ sech^2\left[\frac{x-ct}{l^\circ}\right], \tag{6.99a}$$

where

$$\frac{a^\circ}{h^\circ} = \varepsilon << 1\,,\ \delta^2 = \left(\frac{h^\circ}{l^\circ}\right)^2 = O(\varepsilon), \tag{6.99b}$$

for a flat bottom simulated by the equation: $z = - h^\circ = const.$ In (6.99a, b), a° is a characteristic amplitude [for the initial elevation of a free surface characterized by the function $\zeta^\circ(x/l^\circ)$] and l° is the characteristic wavelength, in the horizontal x-direction. The wave velocity is:

$$c = [g(h^\circ + a^\circ)]^{1/2}, \tag{6.99c}$$

and in fact this characteristic wavelength, l°, is determined by the Ursell (1953) criterion:

$$Ur \equiv \frac{3\varepsilon}{4\delta^2} = 1, \tag{6.100}$$

and the essential quality of the solitary wave is then the balance between nonlinearity and dispersion.

The Boussinesq equation (with dimensional quantities) for $\zeta(t, x)$ is:

$$\zeta_{tt} = gh^\circ\left[\zeta_{xx} + \frac{3}{2h^\circ}\left(\zeta^2\right)_{xx} + \frac{1}{3}h^{\circ 2}\zeta_{xxxx}\right], \tag{6.101}$$

and for a consistent asymptotic derivation of the Boussinesq equation and its generalization, see our review paper [Zeytounian (1994, §3, pp. 268-272)].

Here, we give only an asymptotic derivation of the Boussinesq equation from the dimensional problem (6.98i). For this we introduce in (6.98i) the following new quantities:

$$\xi = \varepsilon^{1/2} x,\ \tau = \varepsilon^{1/2} t,\ \zeta = \varepsilon\eta(\tau, \xi),\ \phi = \varepsilon^{1/2} F(\tau, \xi, z\ ; \varepsilon), \tag{6.101a}$$

where the new variable τ and ξ are the pertinent variables for the free surface function $\zeta = \varepsilon\eta(\tau, \xi)$, and, in this case, in place of (6.98i) we derive the following dimensionless problem for the two functions F and η:

$$\varepsilon\frac{\partial^2 F}{\partial \xi^2} + \frac{\partial^2 F}{\partial z^2} = 0,\ 0 < z < 1 + \varepsilon\eta(\tau, \xi; \varepsilon); \tag{6.101b$_1$}$$

$$\frac{\partial F}{\partial z} = 0, \text{ on } z = 0; \tag{6.101b$_2$}$$

$$\frac{\partial F}{\partial z} = \varepsilon \frac{\partial \eta}{\partial \tau} + \varepsilon^2 \frac{\partial F}{\partial \xi}\frac{\partial \eta}{\partial \xi}, \text{ on } z = 1 + \varepsilon\eta(\tau, \xi; \varepsilon); \qquad (6.101b_3)$$

$$\frac{\partial F}{\partial \tau} + \eta + \frac{1}{2}\left[\varepsilon\left(\frac{\partial F}{\partial \xi}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right] = 0, \text{ on } z = 1 + \varepsilon\eta(\tau, \xi; \varepsilon). \quad (6.101b_4)$$

The Laplace equation is the only equation which contains z in its solution and this variation may be made explicit by formally expanding its solution in powers of ε and writing

$$F = F^0(\tau, \xi, z) + \varepsilon F_1(\tau, \xi, z) + \varepsilon^2 F_2(\tau, \xi, z) + \ldots, \qquad (6.101c)$$

and for $F_0(\tau, \xi, z)$ we obtain as solution:

$$F_0 = B(\tau, \xi).$$

For simplicity, we assume here that the arbitrary function $B(\tau, \xi)$ is in fact the unknown value of the velocity potential F at the bottom, $z = 0$! In this case:

$$F_1(\tau, \xi, 0) = 0, \ F_2(\tau, \xi, 0) = 0, \ldots,$$

and we find, from the above Laplace equation, with (6.101c), as solution for $F_1(\tau, \xi, z)$, and $F_2(\tau, \xi, z)$:

$$F_1(\tau, \xi, z) = -\frac{1}{2} z^2 \frac{\partial^2 B}{\partial \xi^2}, \qquad (6.101d)$$

$$F_2(\tau, \xi, z) = \frac{1}{24} z^4 \frac{\partial^4 B}{\partial \xi^4}, \qquad (6.101e)$$

since $\partial F_1/\partial z = 0$ and $\partial F_2/\partial z = 0$, at bottom, $z = 0$. Now, from (6.101c, d) with $F_0 = B(\tau, \xi)$ and (6.101d) and (6.101e), we obtain, in place of two free surface boundary conditions ((6.101b$_3$) and (6.101b$_4$)) the following two approximate equations:

$$\frac{\partial \eta}{\partial \tau}+\frac{\partial}{\partial \xi}\left[(1+\varepsilon\eta)\frac{\partial B}{\partial \xi}\right]-\frac{1}{6}\varepsilon\frac{\partial^4 B}{\partial \xi^4}=O(\varepsilon^2); \qquad (6.101f)$$

$$\frac{\partial B}{\partial \tau}-\frac{1}{2}\varepsilon\frac{\partial}{\partial \tau}\left(\frac{\partial^2 B}{\partial \xi^2}\right)+\eta+\frac{1}{2}\varepsilon\left(\frac{\partial B}{\partial \xi}\right)^2=O(\varepsilon^2). \qquad (6.101g)$$

From (6.101f, g) we can write two approximate relations:

$$\eta+\frac{\partial B}{\partial \tau}=O(\varepsilon),\ \frac{\partial^2 B}{\partial \xi^2}-\frac{\partial^2 B}{\partial \tau^2}=O(\varepsilon)$$

and also

$$\frac{\partial}{\partial \xi}\left(\eta\frac{\partial B}{\partial \xi}\right)=\frac{\partial}{\partial \xi}\left[\left(\frac{\partial B}{\partial \tau}\right)\left(\frac{\partial B}{\partial \xi}\right)\right]+O(\varepsilon)$$

$$=\frac{1}{2}\frac{\partial}{\partial \tau}\left[\left(\frac{\partial B}{\partial \xi}\right)^2+\varepsilon\left(\frac{\partial B}{\partial \tau}\right)^2\right]+O(\varepsilon)$$

As a consequence, with an error of $O(\varepsilon^2)$ we derive the following one-dimensional Boussinesq equation, for the function $B(\tau, \xi)$ - the value of velocity potential at the bottom, $z = 0$,

$$\frac{\partial^2 B}{\partial \xi^2}-\frac{\partial^2 B}{\partial \tau^2}+\varepsilon\frac{\partial}{\partial \tau}\left[\left(\frac{\partial B}{\partial \xi}\right)^2+\frac{1}{2}\left(\frac{\partial B}{\partial \tau}\right)^2\right]-\frac{1}{3}\varepsilon\frac{\partial^4 B}{\partial \xi^2\partial \tau^2}=0. \qquad (6.101h)$$

6.5.4b. The KdV equation

From the Boussinesq equation (6.101), invoking the prior assumption of unidirectional propagation, and integrating with respect to x, we obtain the famous Korteweg-de Vries [KdV (1895)] equation:

$$\zeta_t+(gh^\circ)^{1/2}\left[\frac{3}{2h^\circ}\zeta\zeta_x+\frac{1}{6}h^{\circ 2}\zeta_{xxx}\right]=0. \qquad (6.102)$$

But this KdV equation (6.102) admits only wave solution moving to the right.

In fact, the KdV equation emerges "very naturally" from a consistent asymptotic expansion, with respect to the small parameter ε (see (6.99b)), when we start from the dimensionless free-surface water wave problem (6.98i), with judicious asymptotic expansions for the dimensionless potential velocity function and surface position - indeed, the KdV equation is a "compatibility relation" for the consistency of an asymptotic derivation connected with associated asymptotic expansions [see, for instance, Zeytounian (1995, Section 4.1, pp. 1345-1346) for the details and discussion]. Below, we give such an asymptotic derivation, writing:

$$\phi = \varepsilon^{1/2} [\ \phi_1 + \varepsilon\, \phi_2 + \varepsilon^2\, \phi_3 + \dots], \tag{6.102a}$$

$$\zeta = \varepsilon\, \zeta_1 + \varepsilon^2\, \zeta_2 + \dots, \tag{6.102b}$$

with the news variables

$$\xi = \varepsilon^{1/2} (x - t) \text{ and } \tau = \varepsilon^{3/2} t. \tag{6.102c}$$

By substitution, from (6.98i) with the above relations (6.102a, b, c) we derive at different orders of ε, respectively, the following solutions:

$$\phi_1 = K(\tau, \xi) \text{ and } \zeta_1 = \frac{\partial K}{\partial \xi};$$

$$\phi_2 = -\frac{1}{2} z^2 \frac{\partial^2 K}{\partial \xi^2} + G(\tau, \xi), \tag{6.102d}$$

$$\phi_3 = \frac{1}{4!} z^4 \frac{\partial^4 K}{\partial \xi^4} - \frac{1}{2!} z^2 \frac{\partial^2 G}{\partial \xi^2} + H(\tau, \xi),$$

with

$$\frac{\partial G}{\partial \xi} = \zeta_2 + \left(\frac{\partial K}{\partial \tau}\right)_{z=1} + \frac{1}{2} \frac{\partial^3 K}{\partial \xi^3} + \frac{1}{2} \zeta_1{}^2 .$$

But, at the order ε^2, from the free surface dynamic condition (the fourth equation in (6.98i)) we derive also the following relation between the functions: ϕ_1, ϕ_2, ϕ_3, ζ_1 and ζ_2:

$$\left(\frac{\partial \phi_3}{\partial z}\right)_{z=1} + \zeta_1 \left(\frac{\partial^2 \phi_2}{\partial z^2}\right)_{z=1} - \left(\frac{\partial \zeta_1}{\partial \xi}\right)\left(\frac{\partial \phi_1}{\partial \xi}\right)_{z=1} = \left(\frac{\partial \zeta_1}{\partial \tau}\right) - \left(\frac{\partial \zeta_2}{\partial \xi}\right).$$

As a direct consequence of this last relation, when we utilize the above solutions (6.102d) and also the relation (derived at order ε)

$$\zeta_2 = \left(\frac{\partial \phi_2}{\partial \xi}\right)_{z=1} - \frac{1}{2}\zeta_1^2 - \left(\frac{\partial \phi_1}{\partial \tau}\right)_{z=1},$$

we derive the following KdV equation for the function $\zeta_1(\tau, \xi)$, which is, in fact, a compatibilty condition for the consistency of the asymptotic derivation connected with the expansions (6.102a, b), with (6.102c):

$$\frac{\partial \zeta_1}{\partial \tau} + \frac{3}{2}\zeta_1 \frac{\partial \zeta_1}{\partial \xi} + \frac{1}{6}\frac{\partial^3 \zeta_1}{\partial \xi^3} = 0. \tag{6.102e}$$

Hence, we confirm that the KdV equation (6.102e) emerges very naturally from a consistent asymptotic expansion, with respect to the small parameter ε, when we start from the exact dimensionless free-surface problem (6.98i). On the contrary, the above derived Boussinesq equation (6.101h) is an inconsistent model equation, since the small parameter ε is present in this equation. But, this Boussinesq model (inconsistent) equation is also interesting, since from this equation we can again derive an asymptotically consistent KdV equation. To do so it is sufficient to introduce the following new variables: $\sigma = \xi - \tau$ and $T = \varepsilon\,\tau$.

In this case for the function , $B(T/\varepsilon,\ \sigma + T/\varepsilon) = f(T, \sigma)$ we derive the following equation from (6.101h):

$$\frac{\partial^2 f}{\partial T \partial \sigma} + \frac{3}{4}\frac{\partial}{\partial \sigma}\left(\frac{\partial f}{\partial \sigma}\right)^2 + \frac{1}{6}\frac{\partial^4 f}{\partial \sigma^2} = O(\varepsilon).$$

But, at the leading order, according to

$$\eta + \frac{\partial B}{\partial \tau} = O(\varepsilon), \text{ and } \frac{\partial B}{\partial \tau} = \varepsilon\frac{\partial f}{\partial T} - \frac{\partial f}{\partial \sigma},$$

we obtain:

$$\frac{\partial f}{\partial \sigma} = \zeta(T,\sigma) + O(\varepsilon), \text{ with } \zeta(T,\sigma) = \eta\left(\frac{T}{\varepsilon}, \sigma + \frac{T}{\varepsilon}\right).$$

Finally, for the function $\zeta(T, \sigma)$ we derive again the above KdV equation (6.102e), namely:

$$\frac{\partial \zeta}{\partial T} + \frac{3}{2}\zeta\frac{\partial \zeta}{\partial \sigma} + \frac{1}{6}\frac{\partial^3 \zeta}{\partial \sigma^3} = 0.$$

It is interesting to note [private communication of M. G. Velarde (Madrid)] that, indeed, in Boussinesq (1877, see his equation (283bis)) the above KdV equation, with dimensions, is derived by Boussinesq himself, from the "Boussinesq equation " (281), page 354 of Boussinesq (1877).

This KdV equation, (281), "à la Boussinesq", is written in the following form for *h'(t, s)*:

$$\frac{dh'}{dt} + \omega_0 \frac{d}{ds}\left[h' + \frac{k''}{2}\left[\frac{2+k}{2}\frac{h'^2}{H} + \frac{k' H^2}{3}\frac{d^2 h'}{ds^2}\right]\right] = 0.$$

Interest waned after the resolution of the Airy-Stokes paradox by Boussinesq and Rayleigh and was sporadic prior to Zabusky and Kruskal's (1965) discovery that the solitary waves (called "solitons" by Z-K) typically dominate the asymptotic solution of the KdV equation. Current interest stems from that discovery and is intense. The theory of solitons is attractive and exciting:

It brings together many branches of mathematics, some of which touch upon profound ideas and several of its aspects are amazing and beautiful [see, for instance, the book by Newell (1985)].

6.5.5. Soliton dynamics: KP, NLS and NLS-Poisson equations

As is noted in Newell's (1985) book [see Chapter 1: The History of the Soliton]: "...In this first stage of discovery, the primary thrust was to establish the existence and resilience of the wave. The discovery of its *universal nature* and its additional properties was to await a new day and an *unexpected result* from another experiment designed to answer a totally different question [the so- called Fermi-Pasta-Ulam (FPU) experiment; see, in Newell's (1985) book, §1b]...". We note that, Kruskal (1974) and Zabusky (1981) [see, for instance, the book by Drazin and Johnson (1990)] approached the FPU problem (why do solids have finite heat conductivity? - the solid being modelled by a one-dimensional lattice, a set of masses coupled by springs!) from the continuum viewpoint and demonstrate that for this it is sufficient to consider the following reduced KdV equation:

$$u_t + u\, u_X + \kappa^2 u_{XXX} = 0. \tag{6.103a}$$

They solved the above KdV equation (6.103a) with the initial condition:

$$u(x,0) = \cos\pi x,\ 0 \leq x \leqq 2, \tag{6.103b}$$

and

$$u,\ u_X,\ u_{XX},\ \text{periodic on } [0,\ 2] \text{ for all } t; \tag{6.103c}$$

for instance, they chose $\kappa = 0.022$. A set of their results is reproduced in the book by Drazin and Johnson (1990; page 14). After a short time the wave steepens and almost produces a shock, but the dispersive term $(\kappa^2 u_{XXX})$ then becomes significant and some sort of local balance between nonlinearity and dispersion ensues. At later times the solution develops " a train of eight well-defined waves", each like $sech^2$ functions, with the faster (taller) waves for ever catching-up and overtaking the slower (shorter) waves. At the heart of these observations is the discovery that these nonlinear waves can interact strongly and then continue thereafter almost as if there had been no interaction at all. This persistence of the wave led Z-K to coin the name "Soliton" to emphasise the particle - like character of these waves which seem to retain their identities in a collision. The discovery has led, in turn, to an intense study over the last twenty five years and, for instance, I can mention the following important topics: the conservation laws and the Miura transformation, the inverse scattering transform (IST), the Lax equation, the Bäcklund transformation, Hirota's method,.... Concerning "Soliton Mathematics", I can recommend three books: Newell (Solitons in Mathematics and Physics - 1985), Drazin and Johnson (Solitons: an Introduction - 1990) and Dodd, Eilbeck, Gibbon and Morris (Solitons and Nonlinear Equations -1982).

6.5.5a. The KP equation

Naturally, for the cases when the nonlinear surface waves in weakly dispersing shallow water are not strictly one-dimensional, the KdV equation no longer applies! In fact, it is necessary to derive a new approximate model equation for this case. This KP equation is of the following form:

$$\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial t}+\frac{3c_0}{2h_0}u\frac{\partial u}{\partial x}+\frac{c_0}{6}(h_0)^2\frac{\partial^3 u}{\partial x^3}\right]+\frac{c_0}{2}\frac{\partial^2 u}{\partial y^2}=0, \qquad (6.104)$$

and for a formal consistent derivation of the KP equation see the paper by Freeman and Davey (1975). Indeed, if we consider, in place of the two-dimensional problem (6.98i), a three-dimensional problem when the unknown functions ϕ and ζ are also dependent on y, then by analogy with the asymptotic derivation performed in above Section 6.5.4b, we derive the following dimensionless consistent KP equation:

$$\frac{\partial}{\partial \xi}\left[\frac{\partial G}{\partial \theta}+6G\frac{\partial G}{\partial \xi}+\frac{\partial^3 G}{\partial \xi^3}\right]+3\frac{\partial^2 G}{\partial \eta^2}=0, \qquad (6.104a)$$

where $\eta=\varepsilon y$, and $\zeta = (3/2)\ \varepsilon\ G + O(\varepsilon^2)$. In expansions (6.102a) and (6.102b) we have:

$$\phi_1(\tau, \xi, \eta, z) = K(\tau, \xi, \eta),\ \zeta_1=\frac{\partial K}{\partial \xi}.$$

and in the KP equation (6.104a) for the function G we have the following relation

$$G(\theta,\xi,\eta)=\frac{2}{3}\frac{\partial K}{\partial \xi},\ \text{with}\ \theta=\frac{\tau}{6}. \qquad (6.104b)$$

According to Kadomtsev and Petviashvili [(1970), originally in Russian], if the y dependence is weak, the KdV equation can be easily corrected by adding a small term to the KdV equation. In their (1970) paper, Kadomtsev and Petviashvili deduced the form of this additional linear(!) term from a consideration of the two-dimensional long wave dispersion relation - but they did not verify that there were no additional nonlinear terms!

6.5.5b. The NLS equation

Now it is necessary to note that the IST and the structure of the KdV equation would have remained a mathematical curiosity, if further important model equations (for water waves) had not been found that were solvable in this way. However, in 1972, in a paper of fundamental importance, Zakharov and Shabat (1972) showed that the nonlinear Schrödinger (NLS) equation:

$$-iA_t + \alpha A_{XX} + \beta |A|^2 A = 0, \tag{6.105}$$

could also be solved by the IST for initial data which decayed sufficiently fast at infinity, $|x| \to \infty$. For the water wave problem the NLS equation (6.105) was derived first for finite depth (the classical problem) by Hasimoto and Ono (1972). A similar NLS equation was deduced earlier, but for infinite depth, by Zakharov (1968). But, in fact, a NLS equation also emerges from the KdV equation. Indeed, for this it is judicious to consider the above reduced KdV equation (6.103a), with $\kappa^2 = 1$, and expanding u as

$$u = \alpha u_1 + \alpha^2 u_2 + \alpha^3 u_3 + \ldots, \tag{6.105a}$$

where α is a small parameter.

Below we want to know the development of a slowly varying envelope modulating a fast carrier wave, and in this case the small parameter α is the ratio of the typical wavelength of the carrier wave to the typical wavelength of the envelope. Since, in equation (6.103a), x and t are normal time and space variables for the two-dimensional carrier wave, we can define a set of "slow" space and time variables: $X_n = \alpha^n x$ and $T_n = \alpha^n t$, these slow space and time variables are the variables for the envelope motion and from now on will be considered as independent variables according to the multiple scale method. Accordingly, in (6.105a) we assume that the functions

$$u_n = u_n(t, x; X_1, T_1, T_2, \ldots), n = 1, 2, 3, \ldots$$

and in this case

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial T_1} + \alpha^2 \frac{\partial}{\partial T_2} + \ldots\ldots,$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \alpha \frac{\partial}{\partial X_1} + \alpha^2 \frac{\partial}{\partial X_2} + \ldots\ldots$$

At order $O(\alpha)$ we find:

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial^3}{\partial x^3}\right] u_1 = 0$$

and

$$u_1 = A(X_1, T_1, T_2, ...)\, E + C.C., \tag{6.105b}$$

with $E = exp(i\theta)$, and $\theta = kx - \omega t$. At $O(\alpha^2)$ we find:

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial^3}{\partial x^3}\right]u_2 = -\left[\frac{\partial A}{\partial T_1}+\left(1-3k^2\right)\frac{\partial A}{\partial X_1}\right]E + C.C.$$
$$-2ikA^2E^2 + 2ikA^{*2}E^{-2}$$

where A^* is the complex conjugate of the amplitude function A. Obviously the term proportional to E is secular in the above equation for u_2, and we take

$$Z = X_1 - \left(1-3k^2\right)T_1 \rightarrow A = A\left(Z, X_2, T_2, \ldots\right). \tag{6.105c}$$

Then integrating the equation for u_2 with (6.105c), we obtain the following solution:

$$u_2 = \frac{1}{3k^2}\left[A^2E^2 + A^{*2}E^{-2}\right] + B, \tag{6.105d}$$

where the function B is in fact an integration constant with respect to the fast scales x and t, but can be made a function of the slow scales. At $O(\alpha^3)$ we find:

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial^3}{\partial x^3}\right]u_3 = -\left[3\frac{\partial^3}{\partial x \partial X_1^2}+\frac{\partial}{\partial T_2}\right]\left(AE + A^*E^{-1}\right)$$
$$-\left[3\frac{\partial^3}{\partial x^2\partial X_1}+\frac{\partial}{\partial T_1}+\frac{\partial}{\partial X_1}\right]\left[B+\frac{1}{3k^2}\left(A^2E^2 + A^{*2}E^{-2}\right)\right]$$
$$-\frac{\partial}{\partial X_1}\left[A^2E^2 + A^{*2}E^{-2} + 2|A|^2\right]$$
$$-2\frac{\partial}{\partial x}\left[ABE + \frac{1}{3k^2}\left[A|A|^2E + A^3E^3\right] + C.C.\right]. \tag{6.105e}$$

There are two types of secular termes in (6.105e). The first are ones which are functions of the slow scales only and which give rise to terms explicitly in x and t in u_3. Removal of these gives:

$$\left[\frac{\partial}{\partial T_1}+\frac{\partial}{\partial X_1}\right]B+2\frac{\partial|A|^2}{\partial X_1}=0. \tag{6.105f}$$

When the NLS equation for A is derived (see the equation (6.105g) below), then equation (6.105f) determine the function *B*. Next, removal of the secular term proportional to *E*, in (6.105e), we obtain for *A* the following equation:

$$3ik\frac{\partial^2 A}{\partial Z^2}+\frac{\partial A}{\partial T_2}-\left(\frac{4}{3k^2}\right)ikA|A|^2+\frac{2}{3k^2}ikA|A|^2=0.$$

Finally, for the amplitude $A(\tau,\ \xi)$, with $\tau = kT_2$ and $\xi = Z/(3)^{1/2}$, we obtain the following NLS equation:

$$-i\frac{\partial A}{\partial \tau}+\frac{\partial^2 A}{\partial \xi^2}+\frac{2}{3k^2}A|A|^2=0. \tag{6.105g}$$

Therefore, the time scale on which the envelope NLS equation operates is quite long, since one unit of time on the τ scale is $1/\alpha^2$ units of real time (a "far-field" equation!). The amplitude $A(\tau,\ \xi)$ is a complex function and therefore contains information about the phase of the wave. Equation (6.105g) may be expressed in real functions letting

$$A(\tau,\ \xi) = f\ exp\ [i\int W\, d\xi], \tag{6.105h}$$

where $f = f(\tau,\ \xi)$, and $W = W(\tau,\ \xi)$. Separating the real and imaginary parts, we obtain:

$$\frac{\partial f^2}{\partial \tau}-2\frac{\partial(f^2 W)}{\partial \xi}=0; \tag{6.105i}$$

$$\frac{\partial W}{\partial \tau}-\frac{\partial}{\partial \xi}\left[\frac{1}{f}\frac{\partial^2 f}{\partial \xi^2}-W^2+\frac{2}{3k^2}f^2\right]=0. \tag{6.105j}$$

Both of the above equations (6.105i, j) are in the form of conservation laws and they were derived by Chu and Mei (1970) and also by Whitham (1967), but without the term, $(1/f)\partial^2 f/\partial\xi^2$ in (6.105j)! The connection between equations (6.105i, j) and the NLS equation (6.105g) was pointed out by Davey (1972).

Finally, we note that the NLS equation has the following solution representing a nonlinear plane wave, namely:

$$A = A^\circ \exp[i(\alpha\tau - \kappa\xi)], \text{ with } A^\circ = const \text{ and } \alpha = \kappa^2 - \frac{2}{3k^2}|A^\circ|^2,$$

and in the so-called "shallow-water limit", the nonlinear plane wave corresponds to a weak cnoidal wave. Indeed, the NLS equation yields a rich variety of nonlinear wave structure, namely: solitons, rarefaction solitons, several kinds of periodic nonlinear waves, and a pair of shocks. But, as a result of this overabundance, scientists are not sure that all these solutions correspond to physical waves!

6.5.5c. The NLS-Poisson equations

For two-dimensional surface water waves, in place of the NLS equation (6.105g), Benney and Roskes (1969) and Davey and Stewartson (1974), derive a system of two coupled equations, the so-called NLS-Poisson system:

$$iA_t + \lambda A_{xx} + \mu A_{yy} = \chi|A|^2 A + \chi_1 A B_x, \tag{6.106a}$$

$$aB_{xx} + B_{yy} = -b\left(|A|^2\right)_x. \tag{6.106b}$$

For the capillary-gravity water waves (when we take into account the surface tension in the classical problem), expressions for the various constant coefficients in (6.106a, b) are given by Djordjevic and Redekopp (1977) and Ablowitz and Segur (1979) - see also the book by Craik (1985; Chapter 6).

It is interesting to note also that for the long waves (in shallow-water) Freeman and Davey (1975) derive a generalization of the KP equation, which is valid as

$$\delta \to 0 \text{ with } \frac{\delta^2}{\varepsilon} = \kappa_0 \text{ finite (fixed).} \tag{6.107}$$

where δ and ε are defined in (6.99b). If, now, $(1/\kappa_0) \to 0$, then the long wave limit (for $\delta \to 0$) of equations (6.106a, b) is recovered at $O(1/\kappa_0)$ after a further slight rescaling (matching between KP and NLS-Poisson system, in long wave limit). In fact, the double limit $\varepsilon \to 0$ and $\delta \to 0$ is non-uniform! The result depends on the order in which these limits are taken. But Freeman and Davey (1975) showed that the introduction of the similarity parameter $\Delta = 1/\kappa_0$, in place of ε, leads to an uniform double limit, namely: $\Delta \to 0$, and $\delta \to 0$.

Concerning the derivation of these water waves evolution equations (KdV, KP, NLS and NLS-Poisson) see the books by Newell (1985), Craik (1985), Mei (1983), Infeld and Rowlands (1992) and our two review papers, Zeytounian (1994, 1995).

6.5.6. Complementary remarks

It is also possible to derive the Boussinesq, KdV and KP equations if the bottom of the channel is uneven. Concerning the Boussinesq equations for a variable depth, see the paper by Peregrine (1967). For the modified (by a variable depth) KdV equation see the papers by Ono (1972) and Johnson (1973) and in this case, in place of the classical KdV equation (6.103a) the following 'perturbed' KdV equation is derived:

$$u_t + u\,u_X + \kappa^2 u_{XXX} = \nu(h)\,u, \tag{6.108}$$

where the function $\nu(h)$ represents the effect of variable depth. It was found numerically and confirmed experimentally that a (KdV) soliton travelling from one constant depth to another constant but smaller depth, disintegrates into several solitons of varying sizes, trailed by an oscillatory tail. This fission is clearly related to the result of the IST- see Gardner *et al.* (1974) - and the 'perturbed' KdV equation (6.108) predicts the soliton fission that occurs as a solitary wave moves into a shelving region [Masden and Mei (1969)]. In particular, the phenomenon of the shelf that appears behind the solitary waves is now well-understood [Knickerbocker and Newell (1980)]. Concerning the soliton interactions in two dimensions I mention the review paper by Freeman (1980).

The case of free surface water waves in a channel with a rough bottom:

$$z = -h(x^*)\,,\ \text{with}\ x^* = \frac{x}{\varepsilon^{1/2}}\,,\ \varepsilon = \frac{a_0}{h_0} << 1\,, \tag{6.109}$$

is very interesting in relation to the application of the multiple scale asymptotic method - see, for instance, the paper by Rosales and Papanicolaou (1983). The application of this MS method gives an unsurprising result - a KdV equation governing again the evolution of free surface one-dimensional disturbances, as in the usual flat bottom case, but now, however, the coefficients in this KdV equation are not given explicitly! For the determination of these coefficients it is necessary to solve four auxiliary problems. In the recent paper by Benilov (1992), three types of bottom topography are distinguished, allowing a simplification of the basic (two-dimensional) shallow-water-wave equations and for two of them, asymptotic equations of KdV type are derived. In the paper by Xue-Nong Chen (1989) a unified KP equation is derived asymptotically, in which viscous (where the effects of viscosity can be considered only in the boundary layer near the bottom), topographic, and transverse modulational effects are incorporated. In Zeytounian (1994), quasi-one-dimensional generalizations of different forms of the Boussinesq equations are asymptotically derived, the influence of bottom topography on the KP equation is elucidated and a significant second-order approximation for quasi-one-dimensional long nonlinear waves in shallow-water is derived [in this case it is possible to introduce the notion of "dressed KP solitons" and concerning the notion of a "dressed KdV soliton", that is the KdV soliton with higher-order corrections, see the paper by Sugimoto and Kakutani (1977) and the references cited in Jeffrey and Kawahara's book (1982; Section 7.2). Finally, we note that the above mentioned problems are discussed in detail in our Preprint, Zeytounian (1993).

6.6. TURBOMACHINERY FLOWS

Fluid flows in turbines and compressors have defied analysis at the level of sophistication given by the partial differential equations because of the complex configuration of the fluid flows. One may even wonder, whether there exist good mathematical models for describing such a complex kind of fluid flow.

We ignore these difficulties here and assume that the model of Eulerian inviscid and incompressible fluid flow is adequate and accept, consequently, any contradiction which could result from the inadequacy of this simplified model. Starting from this statement one cannot escape the fact that: *blades are usually very closely spaced* and be tempted to take advantage of that. Such a close spacing has two main consequences: one is that the flow is *forced* to follow a more or less restricted geometry which is dictated by the *blades,* and the second is that such a constraint is actually accomplished at the expense of *forces* that are either *applied* to the fluid or that the fluid applies to the blades. Therefore, one may speculate that were the blades *infinitely close to each other*, the flow would be *guided along* some *manifold* and be unchanged by any rotation around the axis of the turbo-machine. But this state could not be maintained unless a body force is applied to the fluid. Whenever friction or losses by mixing is ignored the forces have to be perpendicular to the trajectories (referred to the fixed frame of the turbo-machine) of fluid particles. Such a conclusion was found in Guiraud and Zeytounian (1971a) as a consequence of asymptotic modelling relying on two-scale analysis of the problem of flow *within a row* either fixed or rotating.

Below, we discuss, first, the various facets of an asymptotic theory of turbo-machinery fluid flow and then the so-called "through-flow" model and the flow at the inlet/outlet (leading/trailing edges) of a row. Finally, we give some complementary remarks.

6.6.1. Various facets of an asymptotic theory

The main basic small parameter κ (given by (2.61)), is the *reciprocal of the number of blades* encoutered when marching along the periphery of the row. A small scale is introduced, in the azimuthal direction, chosen in such a way that, according to this small scale coordinate, the blades are spaced by $O(1)$. One may view the approach as one relying on homogenization. Averaging is most simply realized through applying periodicity from blade to blade, a condition imposed on the solution, which might be inadequate for some situations. The localization here is very simple, amounting, indeed, to linear variations of various quantities in the space from blade to blade. The body force which was alluded to previously is a result of the small spatial variation of pressure, which, through periodicity, is recovered as a *discontinuity* on crossing the blade. The *thinness* of the blades and the fact that the only stresses are pressures suffice to justify the normality of the pressure forces alluded to previously. The model is applicable, with some simplifying features in the space between the rows, under the assumption, found to be self-consistent, that the structure imposed by the *closeness* of the blades persists outside the row. Guiraud and Zeytounian (1971b) have investigated the *thin transition* between the free and the blade-dominated regions; they gave the proper *transmission conditions* that have to be used in any code designed to compute a flow involving both types of regions, as in Veuillot (1971). All these works were aimed at applying asymptotic modelling for building a mathematical model, derived from three-dimensional flow, that approximates the more or less classical model of *through flow*. This goal was rather nicely achieved in 1974 by Guiraud and Zeytounian, but the modelling was, in some sense, unrealistic! As a matter of fact, it applies only to *rows* which are very thick in comparaison with the blade to blade spacing. A more realistic approach would be to look for an asymptotic model in which the thickness of each row is not too much larger than the blade to blade spacing. This is the geometrical assumption which is made in justifying the mathematical model of *cascade flow*. It was made the basis of an asymptotic model of the flow in a turbo-machine involving many rows by Guiraud and Zeytounian (1974). The result is that one does not find the cascade model but one involving an *infinity of cascades alternately fixed and moving*. Averaging over many blades one finds again the through-flow model. The basic difficulty with such an analysis is that the flow over the series of pairs of cascades involves an infinity of vortex sheets (concerning the vortex sheets, see §6.9, at the end of this Chapter) which accumulate and which are not taken into account in the analysis. To be sure, the vorticity involved on each sheet is vanishingly small in the limit which generates the asymptotic model, but one ignores the possibility that the total vorticity involved is not infinitesimal in the limit! On the other hand, even if this difficulty is ignored, the localization problem of flow over an infinite number of pairs of cascade is an open one. More work should be done on the asymptotic modelling of flows of this type but one may wonder whether such sophistication is actually justified without taking into account large scale turbulent structures which are generated by the numerous instability mechanisms which operate in the flow. Going back to the idealized situation of the flow over one row, Guiraud and Zeytounian (1978) have investigated the behaviour of the vortex sheets which persists in the limit when the blades are vibrating. The free flow downstream of the row has a *fine-scale structure* which involves not only the coordinate transverse to the sheets but also one along the trajectories of the averaged through-flow. If one goes back to the previous model involving an infinity of pairs of cascades one finds that any sheet generates unsteadiness downstream and from the next cascade encountered zeroth order vortex sheets are shed into the flow downstream. All this work, which was given impetus by a research programme on the computation of rotating machine flows at ONERA (Office National d'Etudes et de Recherches Aérospatiales) in the Aerodynamics Department during the early seventies, is related in Chapter IX of Zeytounian Notes (1974) – except the results of the 1978 paper.

The asymptotic theory relies on inviscid incompressible Eulerian flow equations and slip boundary conditions on the blades and is derived from the assumption that the number of blades per row, N, is much greater than one.

The ratio: $h = l°/L$, is the *thickness* of a row divided by the length L of the compressor and is assumed to be small, as is the ratio: $e/D = \pi/N$, the *mean* *blade -to-blade distance* over the mean diameter D of the row. The basic assumption is that the ratio: $e/l°$, which is related to the pitch of the cascade configuration is near one or smaller ($l°$ is the thickness of the row). This means that one small parameter

$$\kappa = \frac{1}{N} << 1, \tag{6.110}$$

controls the flow. Obviously, we may encounter a number of situations, which lead to various theories. The *actuator-disc* theory is obtained when $e/l°$ is near unity while $\delta/l°$ is large, where δ is the distance between two rows. When $e/l°$ is small

$$\frac{e}{l°} << 1, \tag{6.111}$$

we approach the through-flow theory taking account of the forces exerted on the fluid by the blades - this is the situation that we shall consider below. On the other hand, when $e/l°$ is near one while $\delta/l°$ is also near one we encounter the situation which leads to a *coupling between cascade flow and through-flow.*

6.6.2. The through-flow model

We use throughout cylindrical coordinates r, θ, z with relative velocity components u, v, w and pressure p. The inviscid incompressible flow equations are written on matrix dimensionless form as:

$$\frac{\partial T}{\partial t} + \frac{\partial R}{\partial r} + \frac{\partial Z}{\partial z} + \frac{1}{r}\frac{\partial S}{\partial \theta} + \frac{H}{r} = 0. \tag{6.112}$$

The situation considered corresponds to "long blades" as compared with the breadth of the channel from blade-to-blade and we want to attempt a formalisation (through the formal technique of asymptotic expansions) of the obvious idea that the flow is close to *axisymmetric* when the number of blades N is *very large.*

First we make the change of variables from (t, r, θ, z) to (t, r, χ, z) as shown in (6.113a, b):

$$\theta = \Theta(t, r, z) + 2\pi\kappa(k + \chi), \tag{6.113a}$$

$$f(t, r, \theta, z; \kappa) = f_k(t, r, \chi, z; \kappa), \tag{6.113b}$$

with the idea in mind that the flow in the through-flow limit will be independent of χ (whereas r will appear as a parameter for cascade flow). In (6.113a), the surfaces: $\theta = \Theta(t, r, z)$ are, in fact, the skeleton of the blades in the row, when the small parameter $\kappa \to 0$, and outside of the row are material surfaces which are the extension of the skeleton of the blades. Without loss of generality the flow has to be periodic in χ and we enforce this by (6.114a,b):

$$f_k(t, r, \chi + 1, z; \kappa) = f_{k+1}(t, r, \chi, z; \kappa); \qquad (6.114a)$$

$$f_{k+N}(t, r, \chi, z; \kappa) = f_k(t, r, \chi, z; \kappa). \qquad (6.114b)$$

We use for convenience the index k which runs from 1 to N. Accordingly, χ is between zero and one. Now, we expand, formally, in powers of κ, but we need two such expansions. For the *through-flow* (*outer expansion*) we have:

$$f_k(t, r, \chi, z; \kappa) = f_{k,0}(t, r, \chi, z) + \kappa f_{k,1} + \ldots, \qquad (6.115)$$

and (6.115) will fail near both ends of the row where *inner expansions* are needed, namely:

$$f_k(t, r, \chi, h(r) + \kappa z^*; \kappa) \equiv f^*_k(t, r, \chi, z^*; \kappa) = f^*_{k,0}(t, r, \chi, z^*) + \kappa f^*_{k,1} + \ldots, \qquad (6.116)$$

where $z = h(r)$ is the locus of *leading/trailing* edges of a row, and:

$$z^* = \frac{z - h(r)}{\kappa}. \qquad (6.117)$$

Next, when the change of variables from (t, r, θ, z) to (t, r, χ, z), according to (6.113a, b), is made in the matrix equation (6.112), this equation is converted to:

$$\frac{\partial \Gamma_k}{\partial \chi} + 2\pi\kappa r L_k = 0, \qquad (6.118)$$

with an operator

$$L_k = \frac{\partial T_k}{\partial t} + \frac{\partial R_k}{\partial r} + \frac{\partial Z_k}{\partial z} + \frac{H_k}{r}, \qquad (6.119a)$$

and the following matrix

$$\Gamma_k = \begin{bmatrix} \gamma_k u_k + r\dfrac{\partial \Theta}{\partial r} p_k \\ \gamma_k v_k - \dfrac{1}{\lambda} p_k \\ \gamma_k w_k - r\dfrac{\partial \Theta}{\partial z} p_k \\ \gamma_k + \mu r \dfrac{\partial \Theta}{\partial t} \end{bmatrix} \qquad (6.119b)$$

with

$$\gamma_k = \lambda v_k - r u_k \frac{\partial \Theta}{\partial r} - r w_k \frac{\partial \Theta}{\partial z} - \mu r \frac{\partial \Theta}{\partial t}, \qquad (6.119c)$$

and

$$\lambda = \frac{\omega^\circ D}{w_\infty} \text{ and } \mu = \frac{D}{T^\circ w_\infty}, \qquad (6.120)$$

where T° is the reference time, D is the diameter of the row, ω° is the reference value of the angular velocity ω of the row, and w_∞ is the upstream, uniform axial velocity. We note that (in dimensionless form): $p = (p' - p'_\infty)/\rho^\circ w_\infty^2$, with ρ° the constant density of the incompressible fluid and p' the dimensional pressure. Two facts deserve to be stressed at the outset: *First,* if we assume axially symmetric flow, $\partial/\partial\chi = 0$, we get:

$$L_k = 0, \qquad (6.121a)$$

and it may be checked that: $L_k = 0$ is the matrix form of axially symmetric flow; *second,* if we set, by "brute force", $\kappa = 0$ in (6.118) we do not get the equations of axially symmetric flows but, rather, the highly degenerate equation, namely:

$$\frac{\partial \Gamma_k}{\partial \chi} = 0. \qquad (6.121b)$$

This is somewhat strange but is not unexpected! The above definitions imply that, when κ is small, variations in the χ direction are magnified by $1/\kappa$ in comparaison with variation with *t, r*, or *z*. Indeed, substituting the outer expansion (6.115) in (6.118), we get the following hierarchy of approximate equations:

$$\frac{\partial \Gamma_{k,0}}{\partial \chi} = 0\,, \qquad (6.122a)$$

$$\frac{\partial \Gamma_{k,1}}{\partial \chi} + 2\pi r L_{k,0} = 0\,, \qquad (6.122b)$$

....

which must be solved in turn. We choose, as appropriate to the problem, the solution of the first equation of the hierarchy (6.122), $\partial \Gamma_{k,0}/\partial \chi = 0$, for which $u_{k,0}$, $v_{k,0}$, $w_{k,0}$, $p_{k,0}$, are all independent of χ. At this step we don't known the way in which these functions depend on *t, r* and *z*. Then, if we use the second equation of (6.122), in order to compute $u_{k,1}$ and so on, we encounter compatibility conditions arising from periodicity, which enforces $L_{k,0} = 0$, and we have thus obtained through-flow, axially symmetric theory. In fact, the interesting point is that we may go a step further and get through-flow theory to order κ inclusively. For this, with the assumption that the channel between two consecutive blades is

$$0 \le \chi_e \le \chi \le \chi_i \le 1, \text{ with } \Delta(r, z) = \chi_i - \chi_e,$$

we define:

an average of f, namely: $< f >$, and also the bracket of *f*: $[f]$,
which is the jump of *f* from blade to blade,

as shown in (6.123):

$$< f > = \frac{1}{\Delta}\int_{\chi_i}^{\chi_e} f d\chi\,,\ [f] = f_i - f_e\,. \qquad (6.123)$$

If f denotes, for example, the pressure, p, then the bracket of the p may be viewed as the pressure difference between the two sides of one and the same blade.

Finally, the basic result of the Guiraud-Zeytounian (1971a) approximate through-flow theory is the following:

Up to first order in κ, the average of velocity $< \boldsymbol{v} >$ and pressure $<p>$:

$$<\boldsymbol{v}> = \boldsymbol{v}_{k,0} + \kappa <\boldsymbol{v}_{k,1}>, \; <p> = p_{k,0} + \kappa <p_{k,1}> \qquad (6.124a)$$

satisfy, with an error of $O(\kappa^2)$, axially symmetric through-flow equations:

$$\frac{\partial \Delta}{\partial t} + div(\Delta < \boldsymbol{v} >) = O(\kappa^2), \qquad (6.124b_1)$$

$$\frac{\partial < \boldsymbol{v} >}{\partial t} + \{Curl < \boldsymbol{v} > + 2\Omega \boldsymbol{e}_z\} \wedge < \boldsymbol{v} > + grad(J) - \frac{\Pi}{\Delta} grad(\Sigma) = O(\kappa^2), \qquad (6.124b_2)$$

$$J = < p > + \frac{1}{2} |< \boldsymbol{v} >|^2 - \frac{1}{2} \Omega^2 r^2 ; \; \Omega = \lambda \omega. \qquad (6.124b_3)$$

Again, two points deserve to be stressed: *first*, the breadth of the channel from blade-to-blade, Δ, enters in the continuity equation in an obvious way, *second* there is a source term in the momentum equation, namely:

$$\boldsymbol{F} = \frac{\Pi}{\Delta} \nabla \Sigma, \qquad (6.124c)$$

which is proportional to the jump in pressure,

$$\Pi = \frac{1}{2\pi} \{ [p_{k,1}] + \kappa [p_{k,2}] \}, \qquad (6.124d)$$

and is orthogonal to:

$$\Sigma = S + 2\pi\kappa \left[\frac{1}{2} (\chi_i + \chi_e) \right] = const, \qquad (6.124e)$$

with $S = \Theta - \theta$, a surface which is just in the middle of the channel between blades, which is a material surface:

$$\frac{\partial \Sigma}{\partial t} + < v > \cdot grad(\Sigma) = O(\kappa^2), \tag{6.124f}$$

The force $\boldsymbol{F}$ is such that:

$$\boldsymbol{F} \cdot Curl\boldsymbol{F} = 0, \tag{6.124g}$$

and as has long been known in classical through-flow theory: *it occurs from redistribution (homogenization) of forces applied to the flow by the blades.*

The above derivation is illuminating with regard to the error involved in the approximation. Finally we note that, to order one, there is a "fine" dependency on χ which may be computed once the through-flow is known [see, for instance, Guiraud and Zeytounian (1971a, §5)]. As a conclusion:

The derived set of equations and relations, (6.124a)-(6.124g), give a closed system for functions governing the through-flow to $O(\kappa^2)$ in the case of a direct problem, when the geometric definition of blades in the row is given.

6.6.3. Flow analysis at the leading/trailing edges of a row

The classical through flow in an axial turbo-machine is invalid near the locus of leading/trailing edges of a row. According to Guiraud and Zeytounian (1971b), a local asymptotic study reveals the nature of the flow in their neighbourhood and leads to a system of transmission conditions. As a consequence these transmission conditions must be added to the above system of partial differential equations of the trough-flow, (6.124b) for the functions (6.124a), with (6.124c)-(6.124g), in order to get a well-posed problem for the whole of the turbo-machine. Therefore, near the leading/trailing edges of a blade we make a local analysis by setting (see (6.117)), $z = h(r) + \kappa z^*$, and rewrite the matrix equation (6.112) in the form:

$$\frac{\partial \Gamma^*_k}{\partial \chi} + 2\pi r \frac{\partial G^*_k}{\partial z^*} + 2\pi \kappa r M^*_k = 0, \tag{6.125a}$$

where

$$G^*_k = Z^*_k - \frac{dh}{dr} R^*_k, \tag{6.125b}$$

and

$$M^*_k = \frac{\partial T^*_k}{\partial t} + \frac{\partial R^*_k}{\partial r} + \frac{H^*_k}{r}. \tag{6.125c}$$

Next, it is necessary to expand in κ, according to (6.116). To zeroth order we get the equations of cascade flow,

$$\frac{\partial \Gamma^*_{k,0}}{\partial \chi} + 2\pi r \frac{\partial G^*_{k,0}}{\partial z^*} = 0, \tag{6.126}$$

but the configuration is that of semi-infinite cascade flow. For a detailed expression of (6.126) adapted to a local frame related to the curve

$$\Gamma: \{z = h(r),\ \theta = \Theta(t, r, h(r))\}, \tag{6.127}$$

see the paper by Guiraud and Zeytounian (1971b, §III,4). As a matter of fact, we get an inner expansion (see (6.116)) which has to be matched with the outer one considered previously (see (6.115)). More precisely, the semi-infinite local cascade flow fills the gap between the external, force-free, axially symmetric through-flow (outside of the row), and the internal (in the row) through - flow with the source term $\boldsymbol{F}$ (which is a fictive force). Matching provides *transmissions* conditions between these two disconnected through- flows. The necessity of such conditions appears readily as soon as any numerical treatment of the whole through-flow in a two-row stage is attempted. To zeroth order these *transmissions* conditions are rather simple and indeed obvious on physical grounds, namely:

They mean that mass flow is conserved as is the component of momentum parallel to the leading/trailing edge:

$$[[\boldsymbol{v}^*_{k,0} \cdot \boldsymbol{\tau}]] = 0,$$

$$[[2\pi r \cos\alpha\, \Delta^*_0 \boldsymbol{v}^*_{k,0} \cdot \boldsymbol{\mu}]] = 0, \tag{6.128}$$

where $\boldsymbol{\tau}$ is the unit vector tangential to the curve Γ defined by (6.127) and

$$\boldsymbol{\mu} = \boldsymbol{\tau} \wedge \boldsymbol{\nu}, \text{ with } \boldsymbol{\nu} \cdot \boldsymbol{\tau} = 0. \tag{6.129a}$$

In (6.128), $[[f]] = f_{+0} - f_{-0}$, is the jump of f at the leading/trailing edges of a row.

We note also that:

$$\Delta^*_0 \equiv 1 \text{ for } z^* < 0 \text{ and } \Delta^*_0 = \chi^*_{i,0} - \chi^*_{e,0} \text{ for } z^* > 0.$$

Finally, for the angle α we have the following relation:

$$tan\,\alpha = -r\left[\frac{\partial\Theta}{\partial r} + \frac{dh}{dr}\frac{\partial\Theta}{\partial z}\right]_{z=h(r)}. \tag{6.129b}$$

The analysis has also been carried to first order but without a simple interpretation for the result, in Guiraud and Zeytounian (1971b, §IV).

6.6.4. The incompressible and irrotational steady case

In the case of an incompressible and irrotational steady fluid flow, when far upstream the flow is uniform and constant, the governing equations are:

$$\nabla\cdot\boldsymbol{u} = 0 \text{ and } \nabla\wedge\boldsymbol{u} = 0, \tag{6.130}$$

where $\nabla = \partial/\partial r\, \boldsymbol{e}_r + (1/r)\partial/\partial\theta\, \boldsymbol{e}_\theta + \partial/\partial z\, \boldsymbol{e}_z$ and $R° \le r \le r°$, where $R°$ and $r°$ are constant values of the variable r. Below, the main small parameter is (in place of $\kappa = 1/N$)

$$\varepsilon = \frac{2\pi}{N}, \tag{6.131}$$

and thanks to the divergenceless velocity condition, $\nabla\cdot\boldsymbol{u} = 0$, we have for the velocity vector the following representation (in the row):

$$\boldsymbol{u} = \nabla(\varepsilon\Phi)\wedge\nabla\Psi, \text{ with } \boldsymbol{u}\cdot\nabla\Phi = \boldsymbol{u}\cdot\nabla\Psi = 0. \tag{6.132}$$

The choice of the stream-function $\Phi = const.$ is such that, in a row-channel between two consecutive blades, with an error of $O(\varepsilon^2)$ we have:

$$\theta = \Theta(r,z) + \varepsilon\Delta(r,z)\Phi \Rightarrow \varepsilon\Phi = \frac{[\theta - \Theta(r,z)]}{\Delta(r,z)}, \qquad (6.133)$$

where $\theta = \Theta(r, z)$ is the skeleton of blades and $\Delta(r, z)$ is the contraction factor of this channel (the breadth of the inter-blade channel), and this inter-blade channel is limited by the two surfaces, $\Phi = \pm 1/2$. As a consequence:

$$\boldsymbol{u} = \nabla\left[\frac{\theta - \Theta(r,z)}{\Delta(r,z)}\right] \wedge \nabla\Psi(r,\theta,z). \qquad (6.134)$$

We note that the divergenceless condition for $\boldsymbol{u}$ and the slip condition on the blades of the inter-blade row-channel, are both satisfied thanks to (6.134). If we take into account the relation (6.133), then it is judicious to make the following change of variables:

$$(r, \theta, z) \equiv (r, \Theta + \varepsilon\Phi, z), \qquad (6.135a)$$

$$\boldsymbol{u} \Rightarrow \boldsymbol{U}(r, \Phi, z; \varepsilon), \quad \Psi = F(r, \Phi, z; \varepsilon). \qquad (6.135b)$$

In this case we can write the following formulae:

$$\frac{\partial\Psi}{\partial s} = \frac{\partial F}{\partial s} - \frac{1}{\Delta}\frac{\partial F}{\partial\Phi}\left[\frac{1}{\varepsilon}\frac{\partial\Theta}{\partial s} + \Phi\frac{\partial\Delta}{\partial s}\right], \; s = r, z; \qquad (6.135c)$$

$$\frac{\partial\Psi}{\partial\theta} = \frac{1}{\varepsilon\Delta}\frac{\partial F}{\partial\Phi}, \qquad (6.135d)$$

and in place of (6.134) we obtain the following three scalar relations:

$$U = \frac{1}{r\Delta}\frac{\partial F}{\partial z}, \; W = -\frac{1}{r\Delta}\frac{\partial F}{\partial r}, \qquad (6.135e)$$

$$V = rU\left(\frac{\partial\Theta}{\partial r} + \varepsilon\Phi\frac{\partial\Delta}{\partial r}\right) + rW\left(\frac{\partial\Theta}{\partial z} + \varepsilon\Phi\frac{\partial\Delta}{\partial r}\right), \qquad (6.135f)$$

with

$$\boldsymbol{U} = U\boldsymbol{e}_r + V\boldsymbol{e}_\theta + W\boldsymbol{e}_z. \qquad (6.135g)$$

As a consequence of the above relations, the equation: $\nabla \wedge \boldsymbol{u} = 0$, gives the following three scalar equations for U, W, and $\Gamma = rV$, namely:

$$\frac{1}{r\Delta}\left[\frac{\partial U}{\partial \Phi}+\frac{\partial \Theta}{\partial r}\frac{\partial \Gamma}{\partial \Phi}\right]=\frac{\varepsilon}{r}\left[\frac{\partial \Gamma}{\partial r}-\frac{\Phi}{\Delta}\frac{\partial \Delta}{\partial r}\frac{\partial \Gamma}{\partial \Phi}\right]; \tag{6.135h$_1$}$$

$$\frac{1}{r\Delta}\left[\frac{\partial W}{\partial \Phi}+\frac{\partial \Theta}{\partial z}\frac{\partial \Gamma}{\partial \Phi}\right]=\frac{\varepsilon}{r}\left[\frac{\partial \Gamma}{\partial z}-\frac{\Phi}{\Delta}\frac{\partial \Delta}{\partial z}\frac{\partial \Gamma}{\partial \Phi}\right]; \tag{6.135h$_2$}$$

$$\frac{1}{\Delta}\left[\frac{\partial \Theta}{\partial z}\frac{\partial U}{\partial \Phi}-\frac{\partial \Theta}{\partial r}\frac{\partial W}{\partial \Phi}\right]=\varepsilon\left(\frac{\partial U}{\partial z}-\frac{\partial W}{\partial r}\right)+\varepsilon\frac{\Phi}{\Delta}\left[\frac{\partial \Delta}{\partial r}\frac{\partial W}{\partial \Phi}-\frac{\partial \Delta}{\partial z}\frac{\partial U}{\partial \Phi}\right]. \tag{6.135h$_3$}$$

6.6.4a. Derivation of the through-flow equations

Now we expand U, W, Γ and F:

$$(U,\ W,\ \Gamma,\ F) = (U_0,\ W_0,\ \Gamma_0,\ F_0) + \varepsilon(U_1,\ W_1,\ \Gamma_1,\ F_1) + \ldots \tag{6.136}$$

and at zeroth-order we derive the following, leading-order approximate equations, from the above three equations (6.135h) with (6.135e, f, g) for the functions,

$$U_0 = \frac{1}{r\Delta}\frac{\partial F_0}{\partial z},\ W_0 = -\frac{1}{r\Delta}\frac{\partial F_0}{\partial r},\ \Gamma_0 = r^2\left(U_0\frac{\partial \Theta}{\partial r}+W_0\frac{\partial \Theta}{\partial z}\right);$$

namely:

$$\frac{\partial U_0}{\partial \Phi}+\frac{\partial \Theta}{\partial r}\frac{\partial \Gamma_0}{\partial \Phi}=0; \tag{6.137a}$$

$$\frac{\partial W_0}{\partial \Phi}+\frac{\partial \Theta}{\partial z}\frac{\partial \Gamma_0}{\partial \Phi}=0; \tag{6.137b}$$

$$\frac{\partial \Theta}{\partial z}\frac{\partial U_0}{\partial \Phi}-\frac{\partial \Theta}{\partial r}\frac{\partial W_0}{\partial \Phi}=0. \tag{6.137c}$$

The three above equations (6.137) are not independent since:

$$\begin{bmatrix} 0 & 1 & \dfrac{\partial\Theta}{\partial z} \\ 1 & 0 & \dfrac{\partial\Theta}{\partial r} \\ \dfrac{\partial\Theta}{\partial z} & \dfrac{\partial\Theta}{\partial r} & 1 \end{bmatrix} = 0 .$$

On the other hand, from the equations (6.137a, b, c) we derive easily the following two relations:

$$\frac{\partial\Gamma_0}{\partial\Phi} = r^2\left[\frac{\partial\Theta}{\partial r}\frac{\partial U_0}{\partial\Phi} + \frac{\partial\Theta}{\partial z}\frac{\partial W_0}{\partial\Phi}\right],$$

$$\frac{\partial\Gamma_0}{\partial\Phi}\left[1 + \left(r\frac{\partial\Theta}{\partial r}\right)^2 + \left(r\frac{\partial\Theta}{\partial z}\right)^2\right] = 0 .$$

As a consequence:

$$\frac{\partial\Gamma_0}{\partial\Phi} = 0 , \ \frac{\partial U_0}{\partial\Phi} = 0 , \ \frac{\partial W_0}{\partial\Phi} = 0 . \tag{6.138}$$

Therefore:

When ε tends to zero, and as a consequence of a uniform and constant steady flow far upstream of the row, in the leading-order the approximate limiting through-flow in the row is independent ofthe short -scale Φ.

So it is necessary to consider in (6.135h) the next order (terms proportional to ε).

In a such case, we obtain the following three equations:

$$\frac{1}{r}\frac{\partial\Gamma_0}{\partial z} = \frac{1}{r\Delta}\left[\frac{\partial W_1}{\partial\Phi} + \frac{\partial\Theta}{\partial z}\frac{\partial\Gamma_1}{\partial\Phi}\right]; \tag{6.139a}$$

$$\frac{\partial U_0}{\partial z}-\frac{\partial W_0}{\partial r}=\frac{1}{\Delta}\left[\frac{\partial \Theta}{\partial z}\frac{\partial U_1}{\partial \Phi}-\frac{\partial \Theta}{\partial r}\frac{\partial W_1}{\partial \Phi}\right]; \qquad (6.139b)$$

$$\frac{1}{r}\frac{\partial \Gamma_0}{\partial r}=\frac{1}{r\Delta}\left[\frac{\partial U_1}{\partial \Phi}+\frac{\partial \Theta}{\partial r}\frac{\partial \Gamma_1}{\partial \Phi}\right]. \qquad (6.139c)$$

From (6.139a, b, c), by elimination of the functions: U_1, W_1 and Γ_1, we derive the following compatibility relation,

$$\frac{\partial U_0}{\partial z}-\frac{\partial W_0}{\partial r}=\frac{\partial \Theta}{\partial z}\frac{\partial \Gamma_0}{\partial r}-\frac{\partial \Theta}{\partial r}\frac{\partial \Gamma_0}{\partial z},$$

and with,

$$U_0=\frac{1}{r\Delta}\frac{\partial F_0}{\partial z},\ W_0=-\frac{1}{r\Delta}\frac{\partial F_0}{\partial r}$$

we obtain an equation for the function, $F_0 = \Lambda(r, z)$, wich characterizes the limiting through-flow in the row, namely:

$$\frac{\partial}{\partial r}\left[\frac{1}{r\Delta}\frac{\partial \Lambda}{\partial r}\right]+\frac{\partial}{\partial z}\left[\frac{1}{r\Delta}\frac{\partial \Lambda}{\partial z}\right]=\frac{\partial \Theta}{\partial z}\frac{\partial \Gamma_0}{\partial r}-\frac{\partial \Theta}{\partial r}\frac{\partial \Gamma_0}{\partial z}. \qquad (6.140)$$

Now, because far upstream of the row, we have a uniform and constant steady incompressible flow: $\rho_0 = const$ and: $(1/2)\ |\boldsymbol{u}|^2 + p/\rho_0 = const$, then the jump of the pressure from blade to blade $[p] = p_{\Phi=1/2} - p_{\Phi=-1/2}$, gives:

$$[p] = -\,\varepsilon\rho_0\{[U_0[U_1] + W_0[W_1] + (\Gamma_0/r^2)[\Gamma_1]\} + O(\varepsilon^2).$$

As a consequence, $[p]$ is $O(\varepsilon)$, and we can write:

$$lim_{\varepsilon\to 0}\left(\frac{-[p]}{\varepsilon\rho_0}\right)=U_0\frac{\partial U_1}{\partial \Phi}+W_0\frac{\partial W_1}{\partial \Phi}+\frac{\Gamma_0}{r^2}\frac{\partial \Gamma_1}{\partial \Phi}=\Pi_0, \qquad (6.141)$$

since the functions U_1, W_1, and Γ_1 are linear functions of the fine variable Φ, according to (6.139). If now we eliminate the **Φ-derivatives** in (6.141), thanks to equations (6.139), we derive the following simple relation, *in the row:*

$$\Pi_0 = \Delta\left[U_0 \frac{\partial \Gamma_0}{\partial r} + W_0 \frac{\partial \Gamma_0}{\partial z}\right]. \tag{6.142}$$

Obviously, p outside of the row remains continuous, even in the presence of wakes, which in the Eulerian flow are only vortex sheets (contact discontinuities). As a consequence, taking into account the periodicity in $\boldsymbol{\Phi}$, we have *outside of the row*:

$$[p] = 0 \Rightarrow \Pi_0 = 0 \text{ and } U_0 \frac{\partial \Gamma_0}{\partial r} + W_0 \frac{\partial \Gamma_0}{\partial z} = 0 \Rightarrow \Gamma_0 = \Gamma_0(\Lambda). \tag{6.143}$$

But $\Gamma_0 = 0$ - far upstream, of the row, and as consequence by continuity we have:

$\Gamma_0 = 0$, *in the whole outside region upstream of the row.*

Naturally, in the *downstream region outside of the row* we can only write the relation: $\Gamma_0 = \Gamma_0(\Lambda)$.

In conclusion, the velocity vector of the (homogenized) through-flow is:

$$\boldsymbol{U}_0 = \frac{1}{\Delta}[\nabla(\theta - \Theta) \wedge \nabla \Lambda], \tag{6.144}$$

and, as a consequence:

The streamlines of the through-flow are obtained by crossing of median sufaces, $\theta = \Theta + const$, *in the inter-blade row-channel, with the cylindrical sufaces, resulting from the rotation around the axis* z *of the turbomachine of meridian streamlines* $\Lambda = const$.

For this through-flow we have the following equation for the function $\Lambda(r, z)$:

$$\frac{\partial}{\partial r}\left[\frac{1}{r\Delta}\frac{\partial \Lambda}{\partial r}\right] + \frac{\partial}{\partial z}\left[\frac{1}{r\Delta}\frac{\partial \Lambda}{\partial z}\right] = \frac{\partial \Theta}{\partial z}\frac{\partial \Gamma_0}{\partial r} - \frac{\partial \Theta}{\partial r}\frac{\partial \Gamma_0}{\partial z}, \tag{6.145a}$$

with

$$\Gamma_0 = 0, \text{ upstream of the row,} \tag{6.145b}$$

$$\Gamma_0 = \frac{r}{\Delta}\left[\frac{\partial\Theta}{\partial r}\frac{\partial\Lambda}{\partial z} - \frac{\partial\Theta}{\partial z}\frac{\partial\Lambda}{\partial r}\right], \text{ in the row,} \tag{6.145c}$$

$$\Gamma_0 = \Gamma_0(\Lambda), \text{ downstream of the row.} \tag{6.145d}$$

This axially-symmetric through-flow introduces a fictitious force (as a consequence of the homogenization):

$$\boldsymbol{F} = \frac{\Pi_0}{\Delta}\nabla(\theta - \Theta), \tag{6.145e}$$

which simulates the action of the blades in the row.

6.6.4b. Transmission conditions, local solution and matching

First, as in Section 6.6.3, it is necessary to perform a local analysis for the matching the upstream and downstream flows with the flow in the row!

For this we consider the region of entry (e) in the row and the region of exit (s) from the row. For our problem, first

$$\left[\left[\frac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho_0}\right]\right]_e = \left[\left[\frac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho_0}\right]\right]_s = 0, \tag{6.146a}$$

and then, obviously

$$[[\Delta W_0]]_e = [[\Delta W_0]]_s = 0. \tag{6.146b}$$

But unfortunately, these above "transmission conditions" are not sufficient for the existence and uniqueness of the solution of the whole through-flow problem, (6.145a, b, c, d), from upstream of the row to downstream of this row via the row!

If $z = z_e$ is the entry plane, then we introduce the following new local variables,

$$r = r,\ \Phi = \Phi,\ z = z_e + \varepsilon\,\zeta, \tag{6.146c}$$

$$\Theta(r,z)=\Theta(r,z_e+\varepsilon\zeta)=\Theta(r,z_e)+\varepsilon\zeta\left(\frac{\partial\Theta}{\partial z}\right)_{z=z_e}+O(\varepsilon^2). \quad (6.146d)$$

We consider again the system of three equations (6.135h), but written with the local variables (6.146c, d). In this case, at the order ε^0, the local equations are trivially satisfied (since the zeroth order functions in the row are independent of the fine scale Φ). At the order ε^1, we derive as local entry equations:

$$\left[\left(\frac{\partial\Theta}{\partial z}\right)_{z=z_e}\right]\frac{\partial\Gamma^*{}_1}{\partial\Phi}=\frac{\partial\Gamma^*{}_0}{\partial\zeta},$$

$$\left[\left(\frac{\partial\Theta}{\partial z}\right)_{z=z_e}\right]\frac{\partial U^*{}_1}{\partial\Phi}=\frac{\partial U^*{}_0}{\partial\zeta}, \quad (6.146e)$$

$$\frac{\partial U^*{}_1}{\partial\Phi}+\left[\left(\frac{\partial\Theta}{\partial r}\right)_{z=z_e}\right]\frac{\partial\Gamma^*{}_1}{\partial\Phi}=0,$$

where, the functions with '*' are dependent on the local variables $(r,\ \Phi,\ \zeta)$ and, for instance,

$$U^* = U^*{}_0(r,\ \zeta) + \varepsilon\, U^*{}_1(r,\ \Phi,\ \zeta) + \ldots,\ \Gamma^* = \Gamma^*{}_0(r,\ \zeta) + \varepsilon\, \Gamma^*{}_1(r,\ \Phi,\ \zeta) + \ldots,$$

$$F^* = \Lambda^*(r,\ \zeta) + \varepsilon\, F^*{}_1(r,\ \Phi,\ \zeta) + \ldots, \quad (6.146f)$$

where $\Lambda^*(r,\ \zeta)$, for the local problem, in (6.146f), corresponds to $\Lambda(r,\ z)$ in the through-flow problem. Next, by elimination of $\partial U^*{}_1/\partial\phi$ and $\partial\Gamma^*{}_1/\partial\phi$, from (6.146e) we derive, the following compatibility relation:

$$\frac{\partial}{\partial\zeta}\left[U^*{}_0+\left(\frac{\partial\Theta}{\partial r}\right)_{z=z_e}\Gamma^*{}_0\right]=0, \quad (6.146g)$$

and, as a consequence the quantity: $U^*{}_0 + [(\partial\Theta/\partial r)_{z=z_e}]\ \Gamma^*{}_0$, *is conservative* in *the entry region.* An analogous result is derived in the region near the exit plane:

$z = z_e + l^o = z_s$, where $l°$ is the tickness of the row.

Therefore, we can consider the singular regions near the entry and exit of the row as planes of discontinuity, if we impose the following new transmission condition:

$$\left[\left[\frac{1}{\Delta}\frac{\partial \Lambda}{\partial z}+r\frac{\partial \Theta}{\partial r}\Gamma_0\right]\right]_{e,s}=0. \qquad (6.146h)$$

Now, it is important to well understand that near the entry ($z = z_e$) and exit ($z = z_s$) planes of the row, it is necessary to work with two main expansions. wich are different on all sides of $z = z_e$ and $z = z_s$, and the limit functions Λ and Γ_0, solution of the through-flow problem: (6.144), (6.145), are also different on all sides of $z = z_{e,s}$. For this, below, we write: $\Lambda^{\pm}$ and $\Gamma_0^{\pm}$. Obviously, Λ^+ is *not* an analytical extension of Λ^- and likewise, Γ_0^+ is *not* an analytical extension of Γ_0^-, since one is valid in the row and the other is valid outside of the row!

As a consequence of the above discussion, the transmission conditions (6.146h), and the matching principle, we must write:

$$\left(\Lambda^{\pm}\right)_{z=z_e}=lim_{\zeta\to\pm\infty}\Lambda^* \text{ and } \left(\Lambda^{\pm}\right)_{z=z_s}=lim_{\zeta^*\to\pm\infty}\Lambda^*. \qquad (6.146i)$$

$$\left(\Gamma_0^{\pm}\right)_{z=z_e}=lim_{\zeta\to\pm\infty}\Gamma_0^* \text{ and } \left(\Gamma_0^{\pm}\right)_{z=z_s}=lim_{\zeta^*\to\pm\infty}\Gamma_0^*. \qquad (6.146j)$$

where $\zeta^* = (z - z_s)/\varepsilon$, is the local variable for the region near the exit of the row.

Indeed, the above transmission conditions (6.146h), must be interpreted as

$$\left[\left[\frac{1}{\Delta}\frac{\partial \Lambda}{\partial z}+r\frac{\partial \Theta}{\partial r}\Gamma_0\right]\right]_{e,s}=\left[\frac{1}{\Delta}\frac{\partial \Lambda}{\partial z}+r\frac{\partial \Theta}{\partial r}\Gamma_0\right]^+_{e,s}$$

$$-\left[\frac{1}{\Delta}\frac{\partial \Lambda}{\partial z}+r\frac{\partial \Theta}{\partial r}\Gamma_0\right]^-_{e,s}. \qquad (6.146k)$$

6.6.5. Complementary remarks

I turn now, briefly, to the theory which is intended to unify through-flow and cascade flow theories. In fact, we attempt to formalise a physically obvious idea that the flow results approximately (in an asymptotic point of view) from such a kind of cascade-like flow superposed on a basic through-flow, unless we consider the so-called "actuator-disc" theory, which cannot be accepted on any rational basis - unless the rows are widely separated from each other - we must admit interaction from row to row and also unsteadiness. An obvious feature of the scheme is that through-flow averages out this unsteadiness as well as any peculiar feature of cascade-like flow. A moment's reflection shows that the basic (local) length scales for cascade flow are e and $l°$ in the directions of θ and z, while r is expected to act as a parameter. On the other hand the basic (macro) length scales for the through-flow are D and L, for the directions θ and $z°$, when we assume that in a proximity $O(\kappa)$ of $z = z°$, the geometry of the turbo-machine has a quasi-periodic structure which is determined by the configuration of a two-row stage, repeated by periodicity. As a result we expect a two-scale structure in z. Indeed, a small portion of the turbo-machine is converted into a fictitious rigorously periodic system. In mathematical form the flow, which depends on z (and κ), cannot be expanded in powers of κ unless we accept that z and $\zeta = z/\kappa$, enter separately! Of course this is formal in that, for the real flow, z and ζ are related by the relation $z = \kappa\zeta$. To the scale of ζ the rows have a thickness of order one, they are infinite in number and the "turbo-machine" is endless. We note that in the present two-scale approach we consider the following situation:

$$\frac{e}{l°} = O(1),\ \frac{e}{\delta} = O(1),\ \frac{l°}{L} \ll 1. \tag{6.147}$$

Expanding again in powers of κ, but with: $\tau = (1/\varepsilon\mu)t$, χ, ζ, r, z fixed, and substituting in the basic equation [derived from the dimensionless matrix equation (6.112)]:

$$\frac{\partial \Gamma_k}{\partial \chi} + 2\pi r\left[\frac{1}{\mu}\frac{\partial T_k}{\partial \tau} + \frac{\partial Z_k}{\partial \zeta}\right] + 2\pi\kappa r N_k = 0, \tag{6.148}$$

with

$$N_k = \frac{\partial Z_k}{\partial z} + \frac{\partial R_k}{\partial r} + \frac{H_k}{r},$$

where the significant variables are τ, χ, ζ, r and z, we get, in place of the hierarchy (6.122), the following new hierarchy of equations:

$$\frac{\partial \Gamma_{k,0}}{\partial \chi} + 2\pi r\left[\frac{1}{\mu}\frac{\partial T_{k,0}}{\partial \tau} + \frac{\partial Z_{k,0}}{\partial \zeta}\right] = 0, \quad (6.149a)$$

$$\frac{\partial \Gamma_{k,1}}{\partial \chi} + 2\pi r\left[\frac{1}{\mu}\frac{\partial T_{k,1}}{\partial \tau} + \frac{\partial Z_{k,1}}{\partial \zeta}\right] + 2\pi r N_{k,0} = 0, \quad (6.149b)$$

....

Again there is a degeneracy in that, to zeroth order, the dependence on r and z remains arbitrary. This indeterminancy should be resolved when going to next order through the condition that the term:

$$U_{k,1}(\tau, \chi, \zeta, r, z) = [u_{k,1}, v_{k,1}, w_{k,1}, p_{k,1}]^T, \quad (6.150)$$

remains bounded when τ and ζ go to infinity, and satisfies the second equation of the hierarchy, (6.149b), as far as χ is concerned. But now it is necessary to consider more deeply the structure of the first equation, (6.149a). By using the Cartesian coordinates, in the plane related to periodic cascade-like flow, and the associated velocity components, we get, in fact, in place of this matrix equation, the system of inviscid incompressible unsteady equations in two dimensions. We stress that the coordinates in these equations correspond to a short length scale of the flow. As a consequence, the matrix equation (6.148), with the term $2\pi \kappa r N_k$, corresponds to a cascade-like flow which looks like the flow over a periodic array of alternately fixed and moving cascades with source terms which express the coupling with through-flow. It is seen that the component of velocity in the direction defined by the leading edges of the blades is transported by convection, again with a source term which arises from coupling, all along the array.

Now, suppose we known the right hand sides, then we have to solve a very difficult problem of an unusual type. The geometry shows periodicity in time and in two space variables in the plane for the cascade-like flow, but the flow cannot be periodic in the variable in the direction of the tangent to the trace of the blades in this plane because of the wakes! Indeed, the interaction between non-contiguous cascades is very weak, and any approximate scheme which will take into account at most *three* cascades is expected to be accurate enough. Even this is not a simple matter and we don't know computations which have used this scheme. But, even assuming that we have a device for computing this cascade-like flow, it is evident from the equations derived from (6.149) that the way in which this flow depends on r and z will remain undetermined.

Let us come back to the matrix equation (6.148) - we stress that $N_k = 0$, may be identified with the equations of axially symmetric through-flow. The geometry of cascade-like flow involves periodicity, and if the flow were periodic we could derive a compatibility condition by integrating over a domain of periodicity. In doing this the integral over the boundary of this domain vanishes. But, while the flow is not periodic we expect that the result holds! As a matter of fact, if we integrate over *K* periods, then we integrate on one hand over K^3 cells, while on the other hand we integrate on a boundary which involves only K^2 two-dimensional cells. Unless some cancellation occurs, as *K* may be as large as we want, it is necessary that the boundary integral vanishes. The integral over a cell leads to an average, and applying this averaging process to equation (6.148) we get an equation [see, for instance, in the Guiraud and Zeytounian (1974) paper, the matrix equation (41) and also the analysis of §6] where the local variables, τ, χ, ζ have been averaged out and as such are equations governing the through-flow. Finally, the cascade-like flow is treated as a small perturbation of the through-flow and has to be computed, locally, as the two dimensional unsteady flow around an array of pairs of cascades alternately fixed and in motion. The array is constructed by developing the cross-section of the compressor on a plane and continuing, by periodicity, the pair of cascades, so obtained, at each location. The coupling between through-flow and cascade-like flow is part of the analysis carried out by Guiraud and Zeytounian (1974). It arises from the way that the equations of through-flow are obtained through an averaging process, completed on a domain of periodicity of the array of cascades, while the through-flow appears, locally, as an unperturbed flow for the linearized problem defining the cascade-like flow. The three-dimensional nature of the complete flow is built in by the coupling itself, as is visualized by the occurence of source terms in each of the two sets of equations describing through-flow and cascade-like flow. As a conclusion: *For the inclusion of the scheme of cascade-like flow within the computation of a mean through-flow, the classical concept of cascade flow* *should be revisited and reassessed as one of unsteady flow around an array of cascades.*

Indeed, the last paper related to the asymptotic theory of turbomachinery flow [Guiraud and Zeytounian (1978)], concerns also the coupling between the cascade and through-flow ("cascade and through-flow theories as inner and outer expansions"). But, in this case we consider only one row and the situation is such that

$$\kappa \to 0 \text{ while keeping } \frac{l^{\circ}}{\kappa D} = O(1). \tag{6.151}$$

It is then fairly obvious that, under this limiting process (6.151), the row shrinks to a disc and the blades shrink to infinitely many segments. Using cylindrical polar coordinates r, θ, z we assume that $z = 0$ is the plane containing the actuator disc. As a consequence of our assumption, before passing to the limit, the row is located within an $O(\kappa)$ neighbourhood of $z = 0$, if we non-dimensionalize all lengths by taking $D/2$ as a unit.

If

$$\kappa \to 0 \text{ with } r,\ \theta,\ z \text{ fixed,} \tag{6.152a}$$

we obtain the so-called outer limit process and expect to generate the through-flow approximation and improvements on it.

If

$$\kappa \to 0 \text{ with } r,\ \theta,\ \zeta = \frac{z}{\kappa} \text{ fixed,} \tag{6.152b}$$

we obtain the so-called inner limit process or boundary-layer limit process which is expected to generate the cascade-flow approximation and improvements on it.

Matching between the two limit process expansions will provide the links between the two kinds of approximations. But, this over-simplified scheme must be corrected - for instance, it appears that for the downstream through-flow, a technique of multiple scales is necessary, in the *vibrating case,* in order to deal with the unsteady wakes, generated by the vibrating blades, and slowly modulated downstream by the steady part of the through-flow. Indeed, these irrotational shear waves are excited downstream of the cascade, and these are convected downstream by the through-flow. As a matter of fact, the shear waves have a wavelength in the axial direction which is of order κ, and as such shrinks to zero when κ tends to zero. In order to obtain a well defined limit we must change the definition of the outer limit process in such a way that the phase of one shear wave remains fixed to order one when $\kappa \to 0$. Measuring this phase on one particular vortex sheet, we may set it equal to $(1/\kappa)\, G(r, z)$ where the function G has to be found from the process of building the expansion by elimination of secular terms in higher approximations. More precisely, in Guiraud and Zeytounian (1978), where through-flow theory forms the basis for the outer expansion while cascade theory forms the basis for the inner one: matching provides boundary conditions for both flows, what we propose to do may be stated in words as follows. Considering incompressible flow through a one-row machine, assuming that there is a great number of blades and that the corresponding cascades have a chord to spacing ratio of order one, we want to show that the first few terms of an asymptotic representation of the 3D flow may be guessed, having the form of an inner and outer multiple scale expansion. We confirm our guess, as is usually done with problems not amenable to a mathematically rigorous analysis, by an internal consistency argument: we show that each term of the expansion, up to the order considered, may be computed by solving well posed problems. We describe the process by which these problems may be extracted from the definition of the original 3D problem, and each one of these sub-problems is a classical one and a number of works are devoted to their solutions, either analytically or numerically so that we do not comment on this point. There is an exception with the description of the irrotational shear waves in the downstream through-flow for which we give a full solution because there seems to be none in the literature. Stated more concisely: we start from known solutions to partial 2D problems, we glue them together in an asymptotic representation; then going the other way we apply matched asymptotic expansion and multiple scale methods to this representation in order to show, first that the representation itself is not inconsistent, and second that the partial problems were the right ones to solve. Naturally, for engineering applications it would have been very useful to find, as partial problems cascade flow theory as well as through-flow theory across a thick row. We have been unable (with J.P. Guiraud) to find any asymptotic process leading to such a scheme. As a matter of fact the obvious way to proceed leads to only two significant degeneracies: One is the through-flow of Guiraud and Zeytounian (197la, b; 1974) which leaves no room for cascade flow, the other is the one considered in Guiraud and Zeytounian (1978) which leads to cascade flow but leaves no room for through-flow including a thick row. This conclusion, is disappointing because there is no way to embed Wu's (1952) technique within an asymptotic framework!

The concept of mean through-flow was elaborated by Lorenz (1905), who imagined replacing the action of row blades on the fluid by an axisymmetric field of volume forces distributed in the whole space swept by the row. This idea was taken us again and much developed (in 1950)) by Wu (1951). In Veuillot (1976), according to the above asymptotic theory [see the equations (6.145) with (6.146a, b, h)], an iterative method determines the meridian stream function, the circulation, and the density. The equations are discretized on an orthogonal mesh and solved by classical finite difference techniques. The calculation of the steady transonic blade-to-blade flow is achieved by a time marching method using the MacCormack scheme and various applications are presented. Naturally, it would be of interest to find out whether or not the above asymptotic approach can be extended to flows with shock waves; this appears doubtful if the calculated through flow is found to contain a shock. However, the case when the through- flow is entirely subsonic or mixed supersonic-subsonic without shock (whereas the blade-to-blade flow is transonic with shocks) could be physically meaningful.

In the review paper by Hawthorne and Novak (1969) some aspects of the fluid mechanics of turbo-machinery the axial compressor are discussed.

In the paper by Horlock and Marsh (1971) some flow models for turbomachines are derived - in particular, averaged equations for the flow through a real cascade have been compared with the corresponding equations for three hypothetical flows, the many bladed cascade, an axisymmetric flow and flow on a mean stream surface.

In the recent Doctoral thesis by Depriester (1996) the reader can find more recent references concerning three-dimensional flows in turbo-machine (mainly using the two stream functions).

6.7. THE TRANSONIC MODEL EQUATION

6.7.1. gas dynamics problems and the Mach number M

In (inviscid compressible) gas dynamics problems, the Mach number M is the more important dimensionless parameter and we have, in fact, five main cases:

hyposonic ($M << 1$), subsonic ($M < 1$), supersonic ($M > 1$),
transonic ($M \sim 1$) and hypersonic ($M >> 1$),

here the case corresponding to a very small Mach number (a very low characteristic fluid flow velocity) is called “hyposonic”. In most applications, the bodies of interest are thin and streamlined, so that generally the 'shape' parameter δ, in the dimensionless equation of the profile see, for instance, (2.58a, b)), is a small parameter $(\delta << 1)$. We note here only that the classical linear equations for the subsonic/supersonic theory (see, below, the equation (6.165)), are *invalid* when:

$$\frac{M^2-1}{\delta^{3/2}} = O(1) \text{ - transonic similarity,} \tag{6.153a}$$

$$\delta M = O(1) \text{ - hypersonic similarity,} \tag{6.153b}$$

$$\eta\delta = O(1) \text{ - far field } (\eta \sim \infty, M \text{ fixed) similarity,}$$

where $\eta = x + [M^2 - 1]^{1/2} y$, is a characteristic coordinate. (6.153c)

6.7.2. The Steichen equation

For an isentropic flow, with: $p = k^\circ \rho^\gamma$, where $k^\circ = exp\,(S^\circ/C_v) = const$, and in this case the perfect gas is a piezotropic (or more precisely polytropic, since $\gamma > 1$) fluid, we have the following (dimensional) isentropic Euler equations, for the velocity vector $\boldsymbol{u}$ and enthalpy h,

$$\frac{D(\log h)}{Dt} + (\gamma - 1)(\nabla \cdot \boldsymbol{u}) = 0\,; \tag{6.154a}$$

$$\frac{D\boldsymbol{u}}{Dt} + \nabla h = \boldsymbol{0}\,, \tag{6.154b}$$

where: $h = k^\circ[\gamma/(\gamma - 1)]\,\rho^{\gamma-1}$, when we assume that the body force is zero.

For a compressible, isentropic and irrotational Eulerian fluid flow, when $\boldsymbol{u}$ is derived from a velocity potential Φ, according to: $\boldsymbol{u} = \nabla\Phi$, we have, first, the following Bernoulli (compressible) integral

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + \frac{a^2}{\gamma - 1} = B^\circ\,, \tag{6.155}$$

when we assume that the potential - U, of the gravitational acceleration $\boldsymbol{g}$ is zero and when we introduce the square of the speed of sound $a^2 = \gamma\, p/\rho \equiv (\gamma - 1)\, h$, for the isentropic fluid flow. Now, differentiating (6.155) with respect to t we obtain the following relation between Φ and ρ:

$$\frac{\partial^2 \Phi}{\partial t^2}+\frac{\partial \Phi}{\partial x_k}\frac{\partial^2 \Phi}{\partial t \partial x_k}+a^2\frac{\partial \log \rho}{\partial t}=0. \qquad (6.156a)$$

Then, from (6.154a), we obtain a second relation between Φ and ρ:

$$\frac{\partial(\log \rho)}{\partial t}+\frac{\partial \Phi}{\partial x_L}\frac{\partial(\log \rho)}{\partial x_L}+\Delta\Phi=0. \qquad (6.156b)$$

Finally, a third relation between Φ and ρ is derived from (6.154b), if we take the scalar product with $\nabla\Phi$, namely:

$$\frac{\partial \Phi}{\partial x_k}\frac{\partial^2 \Phi}{\partial t \partial x_k}+\frac{\partial \Phi}{\partial x_k}\frac{\partial \Phi}{\partial x_j}\frac{\partial^2 \Phi}{\partial x_j \partial x_k}+a^2\frac{\partial \Phi}{\partial x_k}\frac{\partial(\log \rho)}{\partial x_k}=0. \qquad (6.156c)$$

A simple arrangement of the above three relations:

a^2 (6.156b) - (6.156a) - (6.156c)

gives the following, so-called Steichen (1909), equation for the velocity potential $\Phi(t, x_i)$:

$$a^2\Delta\Phi-\frac{\partial^2 \Phi}{\partial t^2}=2\frac{\partial \Phi}{\partial x_k}\frac{\partial^2 \Phi}{\partial t \partial x_k}+\frac{\partial \Phi}{\partial x_k}\frac{\partial \Phi}{\partial x_j}\frac{\partial^2 \Phi}{\partial x_j \partial x_k}. \qquad (6.157)$$

Indeed, the above Steichen equation (6.157) is a single equation for $\Phi(t, x_i)$, only because it is possible to express the square of the speed of sound, a^2, as a fonction of $\Phi(t, x_i)$. For this we use the Bernoulli integral (6.155), where the constant $B°$ is given by: $B° = (a°)^2/(\gamma - 1) + (1/2)(U°)^2$, when we assume the existence of an uniform flow region - for example, far upstream of an obstacle disturbing the given unifom flow, with constant velocity magnitude $U°$ and constant thermodynamic functions, $p°$, $\rho°$ and $a°$. In this case:

$$a^2=(a°)^2-(\gamma-1)\left\{\frac{\partial \Phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x_k}\right)^2-U^{\circ 2}\right]\right\}. \qquad (6.158)$$

Finally, with (6.158), we obtain the following single partial differential equation for $\Phi(t, x_i)$, in place of (6.157):

$$\frac{\partial^2 \Phi}{\partial t^2} - (a^\circ)^2 \Delta\Phi = (1-\gamma)\left\{\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial x_k}\right)^2 - U^{\circ 2}\right]\right\}\Delta\Phi$$
$$- 2\frac{\partial \Phi}{\partial x_k}\frac{\partial^2 \Phi}{\partial t \partial x_k} - \frac{\partial \Phi}{\partial x_k}\frac{\partial \Phi}{\partial x_j}\frac{\partial^2 \Phi}{\partial x_j \partial x_k}, \qquad (6.159)$$

written with dimensions, which is an hyperbolic second-order nonlinear equation.

6.7.3. The 2D steady case

In the two-dimensional steady case, if we introduce dimensionless quantities, we can write in place of (6.159) the following *dimensionless* equation for the 2D steady velocity potential $\varphi(x, y; M)$:

$$\left[a^2 - M^2\left(\frac{\partial \varphi}{\partial x}\right)^2\right]\frac{\partial^2 \varphi}{\partial x^2} - 2M^2\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y}\frac{\partial^2 \varphi}{\partial x \partial y} +$$
$$\left[a^2 - M^2\left(\frac{\partial \varphi}{\partial y}\right)^2\right]\frac{\partial^2 \varphi}{\partial y^2} = 0, \qquad (6.160a)$$

where

$$a^2 = 1 + \frac{\gamma - 1}{2}M^2\left\{1 - \left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2\right]\right\}. \qquad (6.160b)$$

If now we consider a fluid flow around a profile which have as dimensionless equation, in the (x, y) plane: $y = \delta H(x), x \in [0, 1]$, where δ is the maximum value of y^*/L° of the profile and $H(x^*/L^\circ)$ is a function of order unity (where y^* and x^* are the dimensional variables), then for the equation (6.160a), with (6.160b), we can write the following boundary conditions:

$$\frac{\partial \varphi}{\partial y} = \delta\frac{dH(x)}{dx}\frac{\partial \varphi}{\partial x}, \text{ on } y = \delta H(x), x \in [0, 1], \qquad (6.161a)$$

$$\varphi \to x, \text{ at } x \to \infty. \tag{6.161b}$$

An interesting case, from the point of view of asymptotic methods, is the so-called *transonic case* when:

$$M \to 1 \text{ and } \delta \to 0. \tag{6.162}$$

In this case, it is necessary to write a *similarity relation* between M and δ, for the derivation of a significant (consistent) transonic limit equation.

This similarity relation is, in fact, of the following form [see, for example, the book by Cole (1968, Chapter 5, §5.1)]:

$$\frac{M^2 - 1}{\nu(\delta)} = K_t = O(1), \tag{6.163}$$

which is the ratio of two small parameters, and the choice of the gauge $\nu(\delta) \downarrow 0$, with $\delta \to 0$, is dictated by the fact that the transonic asymptotic expansion:

$$\varphi = x + \alpha(\delta)\,\varphi_t(x, \zeta; K_t) + \ldots, \tag{6.164}$$

must yields the *least degenerate limit transonic equation.*

In $\varphi_t(x, \zeta; K_t)$, we have the 'short' variable:

$$\zeta = \beta(\delta)\,y, \text{ and } \alpha(\delta) \text{ and } \beta(\delta), \text{ both } \downarrow 0 \text{ with } \delta \to 0.$$

We note that it is necessary to introduce this strained variable ζ, if we want to derive another (nonlinear) equation, in place of the classical linear equation:

$$\frac{\partial^2 \varphi'}{\partial y^2} = \left[M^2 - 1\right]\frac{\partial^2 \varphi'}{\partial x^2}, \text{ for } \varphi'(x, y; M) = \lim_{\delta \to 0}\left[\frac{\varphi - x}{\delta}\right], \tag{6.165}$$

which is appropriate *only* for the *subsonic* ($M < 1$) or *supersonic* ($M > 1$) cases.

First, from the slip condition (6.161a) we obtain the following relation:

$$\alpha(\delta)\beta(\delta)\frac{\partial \varphi_t}{\partial y} + \ldots = \delta\,\frac{dh(x)}{dx} + \ldots, \text{ on } \zeta = 0,$$

and as a consequence

$$\beta(\delta)=\frac{\delta}{\alpha(\delta)}. \tag{6.166a}$$

On the other hand, from the equation (6.160a) with (6.160b), we derive, according to (6.163), (6.164) and (6.166a), the following 'dominant' equation for $\varphi_t(x, \zeta; K_t)$:

$$\alpha(\delta)\nu(\delta)K_t\frac{\partial^2\varphi_t}{\partial x^2}+\ldots+\alpha(\delta)\beta(\delta)^2\frac{\partial^2\varphi_t}{\partial\zeta^2}$$
$$-(\gamma+1)\alpha(\delta)^2\left(\frac{\partial\varphi_t}{\partial x}\right)\frac{\partial^2\varphi_t}{\partial x^2}+\ldots=0,$$

and as a consequence it is necessary that (for the derivation of a more degenerate limit equation for φ_t):

$$\nu(\delta)=\beta(\delta)^2 \text{ and } \nu(\delta)=\alpha(\delta). \tag{6.166b}$$

Finally, from (6.166a) and (6.166b) we obtain:

$$\beta(\delta)^3=\delta \Rightarrow \beta(\delta)=\delta^{1/3},\ \alpha(\delta)=\nu(\delta)=\delta^{2/3}. \tag{6.167}$$

Hence, we derive the following "transonic" equation:

$$\left[K_t+(\gamma+1)\frac{\partial\varphi_t}{\partial x}\right]\frac{\partial^2\varphi_t}{\partial x^2}-\frac{\partial^2\varphi_t}{\partial\zeta^2}=0, \tag{6.168}$$

from the above dominant equation for φ_t when we take into account (6.167).

This above equation (6.168) is significant, when $\delta\to 0$, for:

$$M^2=1+K_t\,\delta^{2/3}. \tag{6.169}$$

In the particular case when $K_t = 0$ in (6.169), we derive, in place of (6.168) the following "sonic" equation:

$$\frac{\partial^2\varphi_t}{\partial\zeta^2}-(\gamma+1)\frac{\partial\varphi_t}{\partial x}\frac{\partial^2\varphi_t}{\partial x^2}=0. \tag{6.168a}$$

In the book by Cole and Cook (1986) the reader can find a very valuable account of the transonic aerodynamics. Problems in asymptotic theory for transonic aerodynamics are considered in Cook (1993). It is interesting to note that the above transonic equation (6.168) can be rewritten in the form of two "divergent" equations:

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi_t}{\partial \zeta}\right)+\frac{\partial}{\partial \zeta}\left(-\frac{\partial \varphi_t}{\partial x}\right)=0; \tag{6.170a}$$

$$\frac{\partial}{\partial x}\left[K_t\frac{\partial \varphi_t}{\partial x}+\frac{\gamma+1}{2}\left(\frac{\partial \varphi_t}{\partial x}\right)^2\right]+\frac{\partial}{\partial \zeta}\left(-\frac{\partial \varphi_t}{\partial \zeta}\right)=0. \tag{6.170b}$$

As boundary conditions we have:

$$\frac{\partial \varphi_t}{\partial \zeta}=\frac{dH(x)}{dx},\ \text{on } \zeta=0, \tag{6.170c}$$

$$\varphi_t \to 0,\ \text{at } x \to \infty. \tag{6.170d}$$

Note that the mathematical problem posed by the equations (6.170a, b), with the boundary conditions (6.170c, d), is still nonlinear. The particular cases:

$$\delta \text{ fixed, } M \to 1 \text{ and then } \delta \to 0 \text{ is equivalent to } K_t \to 0, \tag{6.171a}$$

and

$$M \text{ fixed, } \delta \to 0 \text{ and then } M \to 1 \text{ is equivalent to } K_t \to \infty. \tag{6.171b}$$

From (6.171b), we deduce that:

For large $\mathbf{K_t}$ *the results of the supersonic theory must be derived from the transonic solution!*

From the 'transonic divergent equations', (6.170a, b) we can automatically write the following jump condition across a shock:

$$\left[\left[\frac{\partial\varphi_t}{\partial\zeta}\right]\right]^2 = K_t\left[\left[\frac{\partial\varphi_t}{\partial x}\right]\right]^2 + \frac{\gamma+1}{2}\left[\left[\left(\frac{\partial\varphi_t}{\partial x}\right)^2\right]\right]\left[\left[\frac{\partial\varphi_t}{\partial x}\right]\right]. \quad (6.172)$$

Finally, concerning the similarity relation: $[M^2 - 1]/\delta^{2/3} = K^t = O(1)$, we precise that for supersonic flow, $M > 1$, the asymptotic solution of the equation (6.160a) with (6.160b), and (6.161a, b), has the following form:

$$\varphi(x,y;M,\delta) = x - \delta\left[\frac{1}{(M^2-1)^{1/2}}\right]H(\xi)$$
$$+\delta^2\left[\frac{\gamma+1}{8}\frac{M^4}{(M^2-1)^2}\left(\frac{dH}{d\xi}\right)^2\eta + G(\xi)\right] + \ldots$$

where $\xi = x - (M^2 - 1)^{1/2}\,y$, $\eta = x + (M^2 - 1)^{1/2}\,y$, and $G(\xi)$ is an arbitrary function, such that the slip condition on the profile, (6.161a), at the order δ^2 is satisfied by a judicious choice of this function. But, we know, that if this above supersonic solution is a good limiting asymptotic approximation for φ, the solution of the problem [6.160a) with (6.160b), and (6.161a, b)], then it is necessary that the ratio:

$$\frac{\delta^2\left[\frac{\gamma+1}{8}\frac{M^4}{(M^2-1)^2}\left(\frac{dH}{d\xi}\right)^2\eta + G(\xi)\right]}{\delta\left[\frac{1}{(M^2-1)^{1/2}}\right]H(\xi)} << 1\,!$$

This condition assumes that:

$$\left[\frac{(\gamma+1)\delta M^4}{(M^2-1)^{3/2}}\right]\eta << 1, \quad (6.173)$$

and obviously, this condition is satisfied for the transonic flow (M near 1 and in this case η is in fact x), if for x fixed:

$$\frac{\delta}{\left(M^2-1\right)^{3/2}}<<1.$$

In the case considered above, when we assume (6.169), the linear theory is not valid. But it is necessary to note that the resulting flow via the above transonic equations (6.168) or (6.170a, b) with (6.170c, d), (6.172), is different from the result obtained numerically when the full compressible Euler equations are used in place of the Steichen equation, and a challenging problem is the uniqueness of the numerically computed transonic solutions!

6.8. THE HYPERSONIC MODEL EQUATIONS

A second particularly interesting case is the hypersonic flow around of a thin profile with a pointed nose. In this case $U^\circ >> a^\circ$ and the characteristic Mach number, $M = U^\circ/a^\circ >> 1$, in the Steichen equation (6.160a), with (6.160b), is a large non-dimensional parameter. But, this Steichen equation is *not valid* for a hypersonic flow since, in this case, the flow is not isentropic because of the shocks!

Again we consider the perfect gas flow around a thin symmetrical body (aerofoil)

$$y^* = H^\circ H\left(\frac{x^*}{L^\circ}\right) \text{ and } \delta = \frac{H^\circ}{L^\circ} << 1.$$

As boundary condition we have:

$$p^* = p_\infty,\ \rho^* = \rho_\infty,\ u^* = U_\infty \text{ and } v^* = 0 \text{ at } x \to -\infty, \qquad (6.174a)$$

and

$$\delta \frac{dH\left(\frac{x^*}{L^\circ}\right)}{d\left(\frac{x^*}{L^\circ}\right)} = \frac{v^*}{u^*}, \text{ on the airfoil.} \qquad (6.174b)$$

where U_∞ is the uniform constant velocity, p_∞ is the pressure and ρ_∞ the density, at upstream infinity, and u^*, v^* the horizontal (along the x^* axis) and vertical (along the y^* axis) velocity components. Indeed, for very high Mach numbers, when

$$M_\infty = \frac{\left(\dfrac{\gamma p_\infty}{\rho_\infty}\right)^{1/2}}{U_\infty} >> 1,$$

even with reasonably small δ, strong shock waves will appear in the flow field, and our original assumption of isentropic flow [which leads to the dimensionless Steichen equation (6.160a), with (6.160b), for a steady 2D flow] will no longer be valid. In this case of *high Mach* number flows, where the shock waves remain close to the aerofoil and the entire disturbance field is sharply limited in lateral extent, the hypersonic regime is characterized by the condition that:

$$\delta << 1 \text{ and } \frac{1}{M_\infty} << 1, \tag{6.174c}$$

are of the same order of magnitude, that is: $\delta M_\infty = K_H = O(1)$.

For a steady 2D inviscid compressible gas flow, the governing Eulerian compressible nonlinear equations are (with dimensions, for u^*, v^*, p^* and ρ^*):

$$(U_\infty + u^*)\frac{\partial \rho^*}{\partial x^*} + \rho^* \frac{\partial u^*}{\partial x^*} + \frac{\partial \rho^* v^*}{\partial y^*} = 0,$$

$$(U_\infty + u^*)\frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} + \frac{1}{\rho^*}\frac{\partial p^*}{\partial x^*} = 0, \tag{6.175a}$$

$$(U_\infty + u^*)\frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} + \frac{1}{\rho^*}\frac{\partial p^*}{\partial y^*} = 0,$$

$$\left[(U_\infty + u^*)\frac{\partial}{\partial x^*} + v^* \frac{\partial}{\partial y^*}\right]\frac{p^*}{\rho^{*\gamma}} = 0. \tag{6.175b}$$

But, here, it is necessary to use also the shock relations for a curved shock wave.

The bow (attached curved) shock wave equation is assumed written in the following form:

$$y^* = G^\circ G\left(\frac{x^*}{L^\circ}\right). \tag{6.176a}$$

If

$$\tan\beta = G^\circ \frac{dG\left(\frac{x^*}{L^\circ}\right)}{dx^*}, \tag{6.176b}$$

is the *slope of the bow wave* and θ is the *deflection angle of the stream,* where:

$$\tan\theta = \frac{v^*}{u^*}, \tag{6.176c}$$

the classical *Rankine-Hugoniot relations* for shock waves [Rankine (1870), Hugoniot (1887,1889)] may be written, after some algebraic manipulations, in the following form:

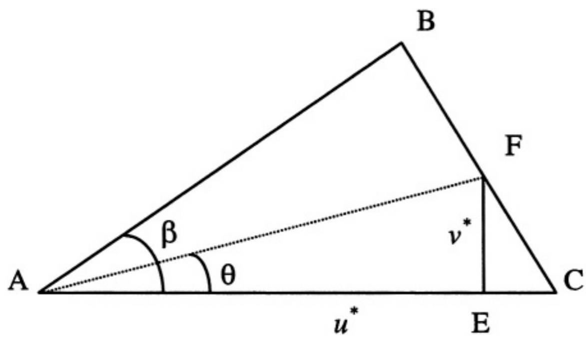
$$\frac{p^*}{p_\infty} = \frac{2\gamma}{\gamma+1}(M_\infty \sin\beta)^2 - \frac{\gamma-1}{\gamma+1}, \tag{6.177a}$$

$$\frac{\rho^*}{\rho_\infty} = \frac{(\gamma+1)(M_\infty \sin\beta)^2}{2+(\gamma-1)(M_\infty \sin\beta)^2}, \tag{6.177b}$$

$$\frac{u^*}{U_\infty} = 1 - \frac{2}{\gamma+1}\frac{1}{M_\infty^{\;2}}\left[(M_\infty \sin\beta)^2 - 1\right], \tag{6.177c}$$

$$\frac{v^*}{U_\infty} = \frac{2}{\gamma+1}\frac{\cos\beta}{M_\infty}\left[\frac{(M_\infty \sin\beta)^2 - 1}{M_\infty \sin\beta}\right], \tag{6.177d}$$

where the subscript ∞ denotes conditions ahead of the shock wave. Concerning these above shock conditions, the reader can consult the book by Anderson (1982). The two first relations (6.177a, b) are, in fact, derived from the corresponding normal shock relations, in a such a way that in place of M_1 we write simply $M_\infty \sin\beta$. The relations (6.177c, d) are the consequence of the similarity of the triangles CEF and ABC (see figure).



Next, it is necessary to note that, when $\delta << 1$ and $M_\infty >> 1$, for a slender pointed body with an attached shock wave, we have that:

$$\theta < \beta \text{ and } \beta \approx \delta. \tag{6.178}$$

Now, according to above relations, we may admit the following estimates:

$$\frac{v^*}{U_\infty} \approx \delta, \ \frac{u^*}{U_\infty} - 1 \approx \delta^2, \qquad \text{(6179.a, b)}$$

$$\frac{p^*}{\gamma p_\infty} \approx \delta^2 M_\infty^{\ 2}, \ \frac{\rho^*}{\rho_\infty} \approx 1. \qquad \text{(6.179c, d)}$$

If now we take into account the above relations (6.179a, b, c, d), then the solution of the 2D steady Euler equations (6.175), may be written as:

$$\frac{u^*}{U_\infty} = 1 + \delta^2 u + \dots,$$

$$\frac{v^*}{U_\infty} = \delta v + \dots, \tag{6.180}$$

$$\frac{p^*}{\gamma p_\infty} = M_\infty{}^2\left(\delta^2 p + \ldots\right),$$

$$\frac{\rho^*}{\rho_\infty} = \rho + \ldots,$$

where the dimensionless functions u, v, p and ρ are dependent on the dimensionless horizontal coordinate $x = x^*/L^\circ$ and the strained vertical coordinate:

$$\eta = \frac{y}{\delta} \equiv \frac{y^*}{H^\circ}, \text{ with } y = \frac{y^*}{L^\circ}. \tag{6.181}$$

In this case, we can write

$$\eta = H(x), \text{ and } \eta = s^\circ G(x), \text{ with } s^\circ = \frac{G^\circ}{H^\circ} = O(1). \tag{6.182}$$

Finally, when

$$\delta \to 0, \text{ with } \gamma, K_H \text{ fixed}, \tag{6.183}$$

we obtain the following limiting model equations for the hypersonic 2D steady flow:

$$\frac{\partial \rho}{\partial x} + \frac{\partial \rho v}{\partial \eta} = 0, \tag{6.184a}$$

$$\rho\left(\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial \eta}\right) + \frac{\partial p}{\partial \eta} = 0, \tag{6.184b}$$

$$\left(\frac{\partial}{\partial x} + v\frac{\partial}{\partial \eta}\right)\frac{p}{\rho^\gamma} = 0; \tag{6.184c}$$

and

$$\rho\left(\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial \eta}\right) + \frac{\partial p}{\partial x} = 0. \tag{6.184d}$$

For the three equations (6.184a, b, c), for v, p and ρ, we have the following boundary conditions:

$$v = \frac{dH(x)}{dx}, \text{ on } \eta = H(x), \tag{6.185a}$$

$$\rho = 1, \; p = \frac{1}{\gamma K_H^{\,2}}, \; v = 0, \text{ at } x \to -\infty, \tag{6.185b}$$

and also the shock conditions (6.177a, b, d), written on the shock wave (in fact: on $\eta = s°G(x)$). Concerning $u(x, \eta)$, instead of the equation (6.184d), one may use the energy equation for defining u as follows:

$$2u + v^2 = \frac{2}{\gamma + 1}\left[\frac{1}{K_H^{\,2}} - \gamma\frac{p}{\rho}\right]. \tag{6.186}$$

in the limit $\delta \to 0$.

The three above equations (6.184a, b, c) together with equation (6.186), condition (6.185a) on $\eta = H(x)$, shock conditions on $\eta = s°G(x)$, and conditions (6.185b) at $x \to -\infty$, suffice for the determination of a 2D steady hypersonic flow field including the effects of entropy changes through the shock wave. Furthermore, it is possible to find a similarity rule provided γ remains the same for the two fields. The rule states: If two bodies have the same thickness distributions with deflections characterized respectively by δ and δ', the flow patterns at Mach number $\mathbf{M}_\infty$ and $\mathbf{M}_\infty'$ will be similar if

$$\delta M_\infty = \delta' M_\infty'. \tag{6.187}$$

This result was first obtained by Tsien (1946) through consideration of a potential equation (!) - the fact that Tsien's results apply to flow with shock waves and rotation (vorticity) was established first by Hayes (1947). On the other hand, an interesting analogy between the hypersonic equations and the nonlinear equations of unsteady flow in one less dimension was also pointed out by Hayes. This Hayes', so-called "piston analogy" is very well presented in Guiraud Notes (1963) devoted to hypersonic flow theory and published in Russian (see Guiraud (1965)). Finally, according to Van Dyke (1951), it appears that supersonic and hypersonic rules can be unified for slender bodies that differ from one another only by a uniform expansion or contraction of all dimensions normal to the free stream direction. The above asymptotic hypersonic theory and Hayes' piston analogy are also derived for the general case of a 3D unsteady hypersonic flow, and in this case the 3D operator: $\partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ is replaced simply by the 2D operator, $\partial/\partial t + v\partial/\partial y + w\partial/\partial z$, reducing the 3D unsteady flow problem to a 2D unsteady one, the longitudinal dimension having disappeared, generating thus again the Hayes' piston analogy [see, for instance, in Guiraud (1963), pp. 30 to 44]. We mention also the very well documented book by Hayes and Probstein (1966) devoted to hypersonic inviscid flow theory.

As a conclusion, it is necessary to stress that there are dramatic physical and mathematical differences between subsonic and supersonic (when $M_\infty > 1$) flows. The temperature, pressure, and density of the flow increase almost explosively across the shock wave. As $M_\infty > 5$ is increased to *higher* supersonic (hypersonic) speeds ($M_\infty >> 1$), these increases become more severe and at the same time, the oblique shock wave moves closer to the surface. For example, for the values of $M_\infty > 5$ (hypersonic flow), the shock wave is very close to the surface, and the flow field between the shock and the body (the shock layer) becomes very hot. Indeed, hot enough to dissociate or even ionize the gas, and in such a case it is necessary to work with the Boltzmann equation!

6.9. ASYMPTOTIC MODELLING OF ROLLED-UP VORTEX SHEETS

6.9.1. Some properties of the vortex sheet

By definition, a free vortex sheet means a regular, discontinuity surface in the fluid domain across which the tangential velocity, but not the normal velocity component, is discontinuous. The connection with vorticity, $\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}$, is clarified by the following. Let the two sides of a vortex sheet Σ be distinguished by subscripts "+" and "-" and let $\boldsymbol{n}$ denote the unit normal on Σ toward the "+" side, and define

$$\boldsymbol{\Omega} = \boldsymbol{n} \wedge (\boldsymbol{u}_+ - \boldsymbol{u}_-), \tag{6.188}$$

then $\boldsymbol{\Omega}$ is the surface density of vorticity on the vortex sheet Σ. This a tangential vector field on Σ, and since $(\mathbf{u}_+ - \mathbf{u}_-)$ is also tangential to Σ, the rule on the triple product gives

$$\boldsymbol{u}_+ - \boldsymbol{u}_- = \boldsymbol{\Omega} \wedge \boldsymbol{n}. \tag{6.189}$$

For any path $\boldsymbol{P} \subset \boldsymbol{\Sigma}$, let s and $\boldsymbol{\tau}$ denote respectively the arc length and the unit tangent vector in the sense of the orientation of P.

Then with $\boldsymbol{\nu} = \boldsymbol{n} \wedge \boldsymbol{\tau}$,

$$\int_P (\boldsymbol{u}_+ - \boldsymbol{u}_-) \cdot d\boldsymbol{x} = \int_P (\boldsymbol{\Omega} \cdot \boldsymbol{\nu}) ds, \tag{6.190}$$

by (6.189), and this is a limiting form of Stokes' theorem in which the left hand side represents the circulation of the circuit $\boldsymbol{P}_+ - \boldsymbol{P}_-$, and the right hand side, the flux of $\boldsymbol{\Omega}$ through the limit of a surface which spans the circuit with surface normal $\boldsymbol{\nu}$ related by the right handed screw rule to the orientation of the circuit. The vector field $\boldsymbol{\Omega}$ on $\boldsymbol{\Sigma}$ is accordingly called, also, the vortex strength (per unit area), and its trajectories are called the vortex lines of the sheet. It follows that: *a surface across which the tangential velocity changes abruptly is a vortex sheet.*

The principal aerodynamic application of vortex sheets is to the surface and wake of a wing. When: $\boldsymbol{u} = \nabla\phi \Leftrightarrow \boldsymbol{\omega} = \nabla \wedge \boldsymbol{u} = 0$, the flow is irrotational (for, example, as a consequence of $\nabla \cdot \boldsymbol{u} = 0$, frictionless-dynamic viscosity $\mu = 0$, and upstream uniform constant flow assumptions), but the frictionless assumption is not valid immediately adjacent to the wing surface, and fluid particles gain vorticity there from frictional processes. As a consequence, the flow about a lifting finite wing is almost irrotational everywhere, except in the *wake,* where vorticity must be present. It must be emphasized that the fluid friction, or the viscosity, only acts as an agent for the existence of the drag force on the wing, but the magnitude of the drag due to the vortex wake is independent of the viscosity of the fluid. The thickness of this vortex wake, however, depends on the viscosity. For the air, the viscosity is very small and the vortex wake is extremely thin, and is generally referred to as *vortex sheet.*

By definition,

$$[\boldsymbol{u}] = \boldsymbol{u}_+ - \boldsymbol{u}_-, \tag{6.191}$$

is the jump of $\boldsymbol{u}$ across the vortex sheet $\boldsymbol{\Sigma}$, and we can also define

$$\boldsymbol{u}_m = \frac{1}{2}(\boldsymbol{u}_+ + \boldsymbol{u}_-). \tag{6.192}$$

In this case we can write for the jump of the material derivative of $\boldsymbol{u}$,

$$\left[\frac{Du}{Dt}\right]=\frac{D_m}{Dt}[u]+([u]\cdot\nabla)u_m, \tag{6.193}$$

with, $D_m/Dt = \partial/\partial t + u_m \cdot \nabla$. The right hand side of (6.193) is well defined, since $n \cdot [u] = 0$, where n is the above unit normal to Σ pointed towards Σ_+. Now we can apply (6.193) to the Eulerian fluid flow equations:

$$\frac{Du}{Dt}+\frac{1}{\rho}\nabla p=0 \text{ and } \frac{Ds}{Dt}=0,$$

and derive the following two equations

$$\frac{D_m}{Dt}([u])+([u]\cdot\nabla)u_m+\left[\frac{1}{\rho}\nabla p\right]=0, \tag{6.194a}$$

$$\frac{D_m[s]}{Dt}+[u]\cdot\nabla[s]=0. \tag{6.194b}$$

Next, it is adequate to write for any arbitrary vector V, *defined on the vortex sheet* Σ, the following decomposition: $V = V_T + V_n n$, with $n \cdot V_T = 0$. As consequence, since $[\nabla p] = 0$, we obtain on Σ:

$$\left\{\frac{D_m}{Dt}([u])+([u]\cdot\nabla)u_m\right\}_T+\left[\frac{1}{\rho}\right]\nabla_T p_\Sigma=0. \tag{6.195}$$

In particular, for an irrotational flow, with $[s] \equiv constant$, the equation for $[s]$ is unnecessary and, in fact, since $\nabla s = 0$ and $\omega = \nabla \wedge u = 0$ in place of (6.195) we can write:

$$\left\{\frac{D_m}{Dt}([u])\right\}_T=-(\nabla_T u_m)\cdot[u]-\nabla_T[h]=0. \tag{6.196}$$

On the other hand, since the flow is irrotational on both sides of the vortex sheet, Σ, we can also write the following relation:

$$\int_C [\boldsymbol{u}] \cdot d\boldsymbol{x} = 0 \text{ and } [\boldsymbol{u}] = \nabla_T \Gamma\,, \tag{6.197}$$

where $\Gamma = [\phi]$ is defined only on Σ and ϕ is the velocity potential of the irrotational flow defined on both sides of Σ. But if we have $[s] = 0$, then also for the enthalpy we have $[h] = 0$ and

$$\frac{D_m \Gamma}{Dt} = 0\,. \tag{6.198}$$

We see that Γ is *convected along the trajectories of the velocity field* $\boldsymbol{u}_m$ *on* Σ.

In the recent paper by Brenier (1997) the reader can find a 'homogenized' model for vortex sheets, and in Marchioro and Pulvirenti (1994, Chapter 6) book, various rigorous results are given concerning justification of the vortex sheet equation and also existence (only for a short time!) and behaviour of the solution of the initial-value problem associated with the vortex sheet equation. These above authors derive also the so-called Moore approximate equation for the vortex sheet - this equation is a good approximation for the vortex sheet equation when the initial data are small, and the basic tool for the analysis is a theorem of the Cauchy-Kowaleski type. Actually, there are recent results proving a global existence theorem for the weak solution of the Euler equation with a vortex sheet as initial datum. However, among these solutions, obtained by compactness methods, it is difficult to isolate those which correspond to the vortex sheets!

It is also reasonable to conjecture that the analytic solution (the vortex sheet equation has an analytic solution for bounded time t° and this existence time becomes larger as the initial datum gets smaller) would be characterized, for a short time, as the vanishing viscosity limit of the Navier solution which can be uniquely and classically constructed, globally in time, for positive viscosity coefficient, with initial datum given by an analytical profile. It seems that no interesting results are known in this direction. Finally, we note that numerical simulations show that a periodic vortex sheet *rolls up into a wound spiral* in a time of the order of the critical time and this is a motivation for the below GZ theory!

6.9.2. The Guiraud and Zeytounian asymptotic theory: incompressible case

As shown above, we can start from the simple idea that a vortex sheet is an infinitely narrow region carrying "infinite" vorticity, a concept which may be given a precise meaning with the aid of distribution theory - the vorticity of the sheet at a given point is then a Dirac delta function times $\boldsymbol{n} \wedge [\boldsymbol{u}]$, and accordingly, $\boldsymbol{\Omega}$ ($= \boldsymbol{n} \wedge [\boldsymbol{u}]$) is the "*vorticity of the sheet*". In the below mathematical dimensionless formulation, we consider a time-dependent, incompressible, irrotational flow. We use t for time, $\boldsymbol{x}$ for vector position, $\boldsymbol{u}$ for velocity vector and p for pressure, the density being unity. We introduce a function $\chi(t, \boldsymbol{x})$ such that the whole sheet is in fact given by: $\chi = constant$.

The set of basic equations (not independent) that we use are:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{6.199a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = 0, \; \nabla \wedge \boldsymbol{u} = 0 \tag{6.199b, c}$$

$$\frac{\partial \chi}{\partial t} + \boldsymbol{u} \cdot \nabla \chi = 0 \text{ on both sides of the sheet,} \tag{6.199d}$$

$$[p] = \nabla \chi \cdot [\boldsymbol{u}] = 0 \text{ across the sheet,} \tag{6.199e}$$

where $[f]$, as above, stands for the discontinuity in f across the sheet counted from lower to higher values of χ. Consider a slender vortex filament (a vortex tube whose cross-section is of infinitesimal maximum dimensions), according to Guiraud and Zeytounian (1977a), we intend to find an asymptotic representation for the core of a vortex sheet winding tightly around the filament (a so-called rolled-up vortex sheet). The flow is *irrotational* but it has *vorticity concentrated on the sheet.* The vortex filament with vorticity continuously spread over the tubular region is a physical model of the rolled vortex sheet. Let h be the distance between two consecutive turns, if $\boldsymbol{n}$ is the unit vector normal to the sheet, a physical argument suggest that:

$$\boldsymbol{\Omega} = \frac{\boldsymbol{n} \wedge [\boldsymbol{u}]}{h}, \tag{6.200}$$

should be a good approximation for the vorticity spread over the vortex filament which models the rolled sheet. One only needs to take care that $[\boldsymbol{u}]$ is counted on crossing the sheet in the sense defined by $\boldsymbol{n}$. This argument (first, 'imagined' by J.P. Guiraud) has been formalized by Guiraud and Zeytounian (1977a), to which I shall refer as GZ in all that follows. Below, I summarize this work and the interested reader will find a presentation of the work, with emphasis on the physical meaning, in Guiraud and Zeytounian (1977b). Since we consider a flow with very many, closely spaced, vortex sheets, each carrying a weak vorticity, then the overall equations of the many sheets are:

$$\chi(t, \boldsymbol{x}) = (2k + 1)\pi, \text{ with } k = \dots -2, 0, 1, 2 \dots,$$

and state the closeness assumption as:

$$\left|\frac{\partial \chi}{\partial t}\right| >> 1 \text{ and } |\nabla\chi| >> 1. \tag{6.201}$$

Naturally, the whole sheet is given by $\chi = const$, but it happens that, when the sheet is rolled-up around a filament, the function $\chi(t, \boldsymbol{x})$ is multiple valued and this explains why the constant takes several values. As a consequence of the introduction of the function χ, we see that making a traverse of the flow perpendicular to $\chi = const$, we get, for any of the flow variables, a modulated saw tooth like graph. Considering the velocity, for example, we may write:

$$\boldsymbol{u} = \boldsymbol{u}^* + \boldsymbol{u}_1 Y(\chi) + \dots, \tag{6.202}$$

where $Y(\chi)$ is the exact saw-tooth-like:

$$Y(\chi) = \chi, \text{ for } |\chi| < \pi \text{ and } Y(\chi) - 2\pi \text{ periodic.} \tag{6.203}$$

Then, we observe that

$$[\boldsymbol{u}] = 2\pi \boldsymbol{u}_1, \tag{6.204}$$

if we make the convention that we cross the sheet in the direction of increasing χ. Now, if L is a characteristic length of the flow such that, for example,

$$|\Omega| = O\left(\frac{|\boldsymbol{u}|}{L}\right), \tag{6.205}$$

then, we get:

$$|\boldsymbol{u}_1| = O(C)\ |\boldsymbol{u}|, \qquad (6.206)$$

where $C = L/h$, is the closeness parameter and provided $C << 1$, (6.202) looks like an expansion with respect to closeness. How one can derive such an expansion from the problem (6.199) is explained in full detail in GZ and here I merely state the result stopping the expansion at the step indicated in (6.202). One more term has been computed in GZ but I shall not need it here.

Our main GZ result is stated below. First, we start from a rotational solution $(\boldsymbol{u}^*, p^*)$ of the inviscid, incompressible Euler equations, then we compute a function $\chi(t, \boldsymbol{x})$ such that:

$$\frac{\partial \chi}{\partial t} + \boldsymbol{u}^* \cdot \nabla \chi = 0 \text{ and } \boldsymbol{\Omega}^* \cdot \nabla \chi = 0, \qquad (6.207\text{a, b})$$

where $\boldsymbol{\Omega}^* = \nabla \wedge \boldsymbol{u}^*$, is the vorticity associated with $\boldsymbol{u}^*$. Then, we ask that χ be non uniform, with the normalization and the structure which has been explained previously. Now, let us define, as an order of magnitude relation:

$$\frac{|\boldsymbol{\Omega}^*|}{|\boldsymbol{u}^*||\nabla \chi|} \approx C << 1, \qquad (6.208)$$

then the following formulae:

$$\boldsymbol{u} = \boldsymbol{u}^* + \frac{\nabla \chi \wedge \boldsymbol{\Omega}^*}{|\nabla \chi|^2} Y(\chi) + \ldots, p = p^* + \ldots \qquad (6.209)$$

determine the first two terms of a double-scale expansion with respect to the closeness parameter C of an irrotational flow, having embedded in it a rolled vortex sheet: $\chi(t, x) = (2k + 1)\pi$, $k = \ldots - 2, 0, 1, 2 \ldots$. We see , therefore, that our (pseudo) mathematical picture relies on an algorithm which allows us to relate (in a quite convincing but not strictly rigorous way):

A continuous rotational flow with vorticity spread over a fluid region to a corresponding irrotational discontinuous flow with vorticity concentrated on a rolled sheet.

Of course, the correspondence holds only in the *asymptotic limit* that the turns of the sheet are *infinitely close* to each other. In GZ it is stated also that the following must hold, for the jump in velocity potential across the sheet (∇_T is the gradient operator along it):

$$\nabla_T \Gamma = -2\pi \frac{\nabla\chi \wedge \boldsymbol{\Omega}^*}{|\nabla\chi|^2}, \tag{6.210a}$$

and

$$\frac{\partial \Gamma}{\partial t} + \boldsymbol{u}^* \cdot \nabla \Gamma = 0, \tag{6.210b}$$

in order that the next terms of the expansion might be computed (compatibility conditions). Of course (6.210b) is a well-known condition in the theory of vortex sheets (see equation (6.198)) and it is not all surprising that we recover it. In Guiraud (1977), the reader can find a formal proof that (6.210a) may always be solved for Γ and that (6.210b) holds automatically. We observe that at least two small parameters may be built into the problem of a rolled-up sheet: one is the *slenderness* parameter, which for the leading-edge conical sheet is the distance to the focus of the spiral, while the other is the reciprocal of the number of turns or the distance between turns, which we may call the *closeness* parameter. For the *conical leading edge* vortex sheet it turns out that the second small parameter is of the order of the square of the first if we adopt the useful convention that logarithms are of order one. It is through the slendemess parameter that the core expansion is influenced by the exterior solution and thus has to take into account the departure from circular symmetry. For the application of our, GZ, theory to the core of a leading edge vortex and also to the so - called *Kaden's problem,* see Guiraud and Zeytounian (1977a, pp. 100-109). Here we note only that:

An important problem is to find under what condition the process we have just described works, when one starts from a given pair $(\boldsymbol{u}^*, p^*)$ *which solves the Euler equations!*

The problem is to find a χ satisfying the three requirements listed above and such that the Γ computed from $\boldsymbol{\Omega}^*$ and χ is convected without change by the velocity field $\boldsymbol{u}^*$. From very rough arguments we expect that this will hold whenever the velocity field $\boldsymbol{u}^*$ has the structure of a *slender vortex filament.* The key of the argument is that the particles are forced by the velocity field $\boldsymbol{u}^*$ to spiral around the filament in an almost circular path with small pitch, then a moving surface generated by these trajectories will have the structure of a tightly wounded sheet and will be a convenient surface $\chi = contant$. Now, $\boldsymbol{\Omega}^*$ is directed almost along the axis of the filament while $\nabla\chi$ will be directed almost perpendicular to it so that the more slender the filament, the better it will satisfy the second requirement and it is evident that the closeness parameter will small. Without actually computing χ and Γ we don't know of any argument which might suggest that Γ will be convected without change by $\boldsymbol{u}^*$. What we can safely say is that the set of rotational flows $(\boldsymbol{u}^*, p^*)$ meeting all the requirements is not empty. As a matter of fact, as is noted above, two configurations have been shown to satisfy these requirements, with a special consideration for the relation between slenderness and closeness. For the *delta leading edge problem,* the third term in the closeness expansion is of the same order as the second term in the slenderness expansion and the reader will find in Guiraud and Zeytounian (1977a) both expansions up to this precise order, taking for $\boldsymbol{u}^*$ the axisymmetric conical steady solution. But the numerical computations with the delta wing problem have shown that the sheet shows considerable departure from a nearly circular spiral in that the sheet rolls up close to an ellipse rather than to a circle. In order to fit the theory with this behaviour one would have to start with a $\boldsymbol{u}^*$ without rotational symmetry! Unfortunately, there is no known solution without rotational symmetry. A first step would be to start from a $\boldsymbol{u}^*$ which is an asymmetrical perturbation to Hall's (1961) solution - this seems feasible but has not, yet, been carried out. We note also that, from the knowledge of $\boldsymbol{u}^*$, according to Hall (1961), the two terms of Mangler and Weber's (1967) expansion can be derived without the need of any new computation. Indeed, it is necessary to note that, in fact, our, GZ, double-scale technique for the derivation of an asymptotic representation of the core of highly rolled-up vortex sheet, is a direct consequence of a careful examination of the Mangler and Weber (1967) asymptotic representation for the core - while the analysis leading to this, M-W, representation is rather cumbersome, the result is fairly well understood and may easily be grasped [see, for instance, Guiraud and Zeytounian (1976)]. Concerning *Kaden's problem* we have computed five terms - the first term is the Kaden's (1931) leading term, while the second term is the term computed by Moore (1975), as asymptotic value for the shape of the spiral. The reason why we have computed only five terms in the slenderness expansion, is that only these terms are consistent with two terms in the closeness expansion - the sixth term in the slenderness expansion would contribute to the closeness expansion at a higher order that the first two terms in the closeness expansion. As a matter of fact, the closeness expansion is known, in general terms, up to three terms and it would be possible to carry the process a little bit further. Indeed, the main result is on the emphasis which has been put the fact that the difficulty is not in the handling of the nonlinear condition on the sheet as one might expect. Through the mechanism of the closeness expansion this difficulty is resolved in an almost trivial manner. The main trouble is rather in finding an appropriate rotational continuous solution involving particles spiralling with small pitch in such a way that a function χ might be computed meeting the requirements for consistency of the closeness expansion. A very fruitful area of research towards this goal would be to use as $\boldsymbol{u}^*$ the slenderness expansions which are now becoming popular in the theory of vortex filaments as discussed by Ting (1971), Fraenkel (1972) and Widnall (1975). Concerning the dynamics of rolled vortex sheets tightly wound around slender vortex filaments in inviscid incompressible flow, see Guiraud (1977) and also Guiraud and Zeytounian (1982).

6.9.3. The Guiraud and Zeytounian asymptotic theory: rotational compressible case

Finally, in Guiraud and Zeytounian (1980) the previous work on irrotational incompressible inviscid flow, is extended to a *rotational and compressible Euler flow*. A formal proof is given that, within the core, one may avoid computing with the sheet by defining an equivalent continuous flow. One shows, also, how the vorticity and the entropy gradient between the turns of the sheet are transported along trajectories of the equivalent flow. All this may seem to be rather transparent from a physical point of view and the main interest of our, GZ, theory stands in the formal proof that this is consistent with a systematic scheme of (asymptotic-double-scale) expansion with respect to a small closeness parameter. The main new feature is the occurence of entropy gradients and of two kinds of vorticity. There is first the distributed vorticity and then the one concentrated on the sheets. Both are summed up in the equivalent continuous flow, and the analytical algorithm must be able to separate them out. The same will be done with the entropy gradient. Indeed, provided we know an asymptotic representation of the core at some initial time, consistent with the double-scale structure, then we are able to compute the evolution of this asymptotic representation. We need only compute the evolution of a continuous flow in the core region and transport the rolled sheet together with the corresponding discontinuities of the velocity and the entropy along the trajectories of the continuous flow. As a consequence, once the discontinuities associated with the sheet have been transported, no new effort is necessary in order to detail the saw-tooth structure of the velocity and entropy. In fact, corresponding signatures are transported without change, only their amplitudes being changed during the transport process. A check on the above discussed theory has been obtained, in Guiraud and Zeytounian (1980), by deriving the Brown and Mangier (1967) solution from Brown's (1965) axisymmetric compressible solution for an inviscid homentropic leading-edge vortex. This last one is obtained under the slenderness assumption. We note that Brown and Mangier (1967) have extended the work of Mangier and Weber (1967) to compressible, barotropic, irrotational flow. Of course, the main new features brought into the scheme by rotationality and entropy gradient, which are absent in their analysis.

6.9.4. Some complementary comments

The computation of rolled vortex sheets has a very long history that I do not intend to evoke, even very briefly, here. Above, I have only reported the GZ theory on the asymptotic modelling of that part of the flow which corresponds to the region where the contiguous branches of the rolled sheet are so close to each other that they are very difficult to capture by a numerical simulation. One encounters a small parameter which is built into the structure of the flow, not at all in the equations - the ratio of length scales, namely one for the distance separating two consecutive turns of the sheet, and the other for the diameter of the filament-like region within which the rolled sheet is embedded. The result obtained by the GZ theory is quite simple and may be considered as typical of the capabilities of asymptotic modelling. As a matter of fact an algorithm, rather simple to work out, is provided which does exactly what one is temped to ask with such an approach. Imagine that one has in hand a solution with vorticity spread over a filament-like region, the algorithm gives a way of detecting whether a rolled vortex sheet, with closely spaced turns, may be embedded into it, and, when the answer is positive it gives a way of finding the geometrical shape of the rolled sheet.

Asymptotic modelling is substituted for numerical simulation precisely when the simulation becomes so stiff,

due to the close spacing, that it is getting increasingly difficult to continue it with vorticity concentrated on the sheet - an application of this may be found in Huberson (1980). Of course, the rolled sheet may fail to exist as a stable flow system, it is obvious that asymptotic modelling is not suited for settling such a question and again this example is very significant with respect to the capabilities and the limits of asymptotic modelling. With some amount of “caricature” one could say that asymptotic modelling consists

in a set of rules applied in a more or less systematic manner to some guess about the structure of the flow, it is unable to decide whether or not the guess is a good one!

Guiraud (1977) has worked out asymptotic modelling one step further. Provided the diameter of the filament-like region is small in comparison to the other length scales of the overall flow, asymptotic modelling may be substituted for numerical simulation for getting the flow with vorticity spread over the filament. One could apply numerical simulation for finding the motion of the filament and the rest of the flow away from it, while asymptotic modelling would be applied both for getting the structure of the vorticity distribution within the filament and for getting the way in which any sheet approaching the filament and winding around it is engulfed into its core. A review is given in Guiraud and Zeytouian (1982). For a review of numerical codes relying strongly on vorticity concepts, see Leonard (1980).

CHAPTER 7

BOUNDARY-LAYER MODELS FOR HIGH-REYNOLDS NUMBERS

After a short discussion concerning the 'vanishing viscosity' singular problem, in the §7.1, the reader can find a consistent derivation of 'dominant equations' from the NS-F steady equations, valid near of the wall of a 3D solid body, which is the first step in the derivation of the Prandtl steady boundary-layer (BL) equations. The process of matching with the Euler equations is discussed in §7.2 and the second-order BL equations, for steady 2D flows, and their infuence on the inviscid outer flow are considered in the §7.3. But, curiously, the BL equations are singular in the vicinity of initial time $t = 0$ (as a consequence of the constancy of the pressure in the thickness of the BL) and, in §7.4, this initialization problem, via the so-called Rayleigh-Howarth equations, is discussed. In the theory of the BL the more familiar problem is the basic Blasius problem related with a steady (incompressible) flow past a solid flat plate placed edgewise in a uniform stream, and in §7.5 we consider this Blasius problem, but for a slightly compressible fluid flow. The next §7.6 is devoted to a Navier flow with variable viscosity, which leads to a three-layer asymptotic model. Finally, the modelling of fluid flow within the Taylor shock layer is considered in §7.7.

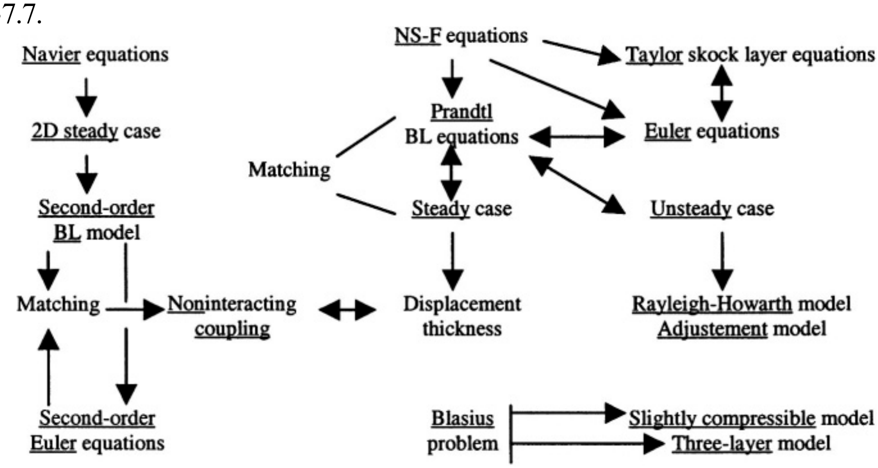


Fig. 7.1 Various models related with the high-Reynolds numbers asymptotics

Curiously, the d'Alembert theorem (paradox) states that an object moving with velocity $\boldsymbol{U}_\infty$ in a potential field does not feel any force (neither drag nor lift). Obviously, this result is in sharp contrast with experience. For instance, aeroplanes can fly! In fact, suppose that, initially, the aeroplane and the fluid (air) are both at rest. Then the aeroplane begins to move. Since vorticity cannot be produced, the potential flow around the aeroplane cannot produce any lift, so that flight is impossible. Such a paradox can be avoided if vorticity is present. However, the problem remains of understanding how vorticity can be created in the system. The assumption of conservation of vorticity in an inviscid, 2D incompressible fluid flow, while reasonable far from the obstacle, is too drastic near the boundary. A more accurate description of the interaction among the particles of the fluid and the obstacle leads us to introduce "viscous equations" which are a correction to the inviscid Euler equations of motion. Such new equations can explain effects, such as vorticity production, which are relevant near the boundary. However, to obtain a well-posed boundary value problem for fixed positive viscosity, one must also (following Stokes) replace the *slip boundary condition* for the Eulerian velocity, by the more stringent condition of *no-slip.*

Concerning this *no - slip boundary condition,* it is interesting to note that, in his 1904 lecture to the ICM Prandtl stated:

"The physical processes in the boundary layer (BL - Grenzschicht) between fluid and solid body can be calculated in a sufficiently satisfactory way if it is assumed that the fluid adheres to the walls, so that the total velocity there is zero - or equal to the velocity of the body. If the viscosity is very small and the path of the fluid along the wall not too long, the velocity will again have its usual value very near to the wall (outside of the thin transition layer). In the transition layer (Übergangsschicht) the sharp changes of velocity, in spite of the small viscosity coefficient, produce noticeable effects".

Prandtl not only mentioned the existence and nature of the boundary layer and its connection with frictional drag, but derived heuristically (on the basis of a comparison of magnitudes of various terms of the Navier equations in a thin layer near the wall of the solid body) the boundary-layer equations [the so-called, "Prandtl equations" - see the equations (4.41a, b,c) derived in §4.4, Chapter 4], valid in a thin viscous layer close to the wall of the solid body.

But, it is necessary, also, to take into account the important investigations of Lanchester (1907) in England, concerning the nature of the boundary layer and the explanation of the separation (independent of Prandtl).

A fundamental mathematical problem arises quite naturally, in relation to the behaviour of the viscous solutions in the vanishing viscosity limit $\nu_0 \to 0$. It is a famous question whether as $\nu_0 \to 0$, the solution of the viscous problem, $\boldsymbol{u}_\nu$, for a viscous (for example, Navier incompressible) fluid flow, tends to the solution, $\boldsymbol{u}_e$, of the corresponding problem for the Euler (incompressible) nonviscous equations, in which the viscous term is omitted, and the no-slip boundary condition is weakened to the Euler slip condition.

In the absence of a boundary it is easy to prove that:

$$\boldsymbol{u}_\nu \to \boldsymbol{u}_e \text{ in } L^2 \text{ (} L^2 \text{ convergence) when } \nu_0 \to 0,$$

where $\boldsymbol{u}_\nu$ and $\boldsymbol{u}_e$, are both associated with the fixed initial value (for $t = 0$) $\mathbf{u}^0$.

In fact, it is necessary to introduce the concept of the *inviscid limit,* such that:

the time t is fixed and the control parameter (Reynolds number, Re) tends to infinity.

Let us focus on the Navier equation (written with dimensionless quantities),

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \frac{1}{Re} \Delta \boldsymbol{u} ,$$

where the Reynolds number is assumed very large (high Reynolds numbers asymptotics). For the case $\Omega = \boldsymbol{R}^3$, and for an interval of time depending on the initial value, $\boldsymbol{u}^0$, but not on the viscosity, the problem was solved, first, by Kato (1972) - see also Kato (1975, 1986) - and by Marchioro and Pulvirenti (1984), Delort (1991), Constantin and Wu (1994), Shelukhin (1998). In the paper by Marchioro and Pulvirenti (1984), the proof is based on the convergence of stochastic processes describing the Navier flow. But the problem with boundaries has remained a famous challenge - obviously, the behaviour of the solution in the vanishing viscosity limit when boundaries are present is much more complicated (this problem is, in fact, singular!) and not yet completely mathematically understood. A rigorous mathematical theory of the boundary layer problem is far from being achieved and an analysis of the existing results is beyond the scope of the present discussion. It seems that the key to the analysis is to construct solutions of the Navier equations incorporating an appropriate boundary approximation of Prandtl type (as in the case of the formal asymptotic derivation). This is an important step in the rigorous justification of classical boundary-layer theory, but the problem still remains open for general boundaries!

On the other hand, according to a short but very pertinent paper by Constantin (1995; pp. 661-662), the first question is: what are the limiting equations? In realistic closed systems where the boundary effects are important, unstable boundary-layers drive the system: the inviscid limit is not well understood.

In the case of no boundaries (periodic solutions or solutions decaying at infinity) the issue becomes one of smoothness and rates of convergence. Indeed in 2D, if the initial data are very smooth, then the limit is the Euler incompressible 2D equation and the difference between Navier solutions and corresponding Euler solutions is optimally small ($O(1/Re)$). However, if the initial data are not that smooth, for instance in the case of vortex patches, then the situation changes. Vortex patches are solutions whose vorticity (antisymmetric part of the gradient) is a step function. They are the building blocks for the phase space of an important statistical theory. When one leaves the realm of smooth initial data, the inviscid limit becomes more complicated: internal transition layers form because the smoothing effect present in the Navier solution is absent in the Eulerian solution. In the case of vortex patches with smooth boundaries, the inviscid limit is still the Euler equation, but there is a definite price to pay for rougher data: the difference between solution (in L^2) is only $O((1/Re)^{1/2})$. This drop in rate of convergence actually occurs - there exist exact solutions providing lower bounds. The question of the inviscid limit for the whole phase space of the statistical theory is open! If the initial data are more singular, then even the classic notion of weak solutions for the Euler equation might need revision, except when the vorticity is of one sign.

In many interesting flows the generation of vorticity occurs on very small sets, typically at the surface of an obstacle, and if the viscosity is small, as time goes on the support of the vorticity remains small, as is the case for wakes or detached boundary layers. It is therefore natural to study the Navier equation with an initial vorticity concentrated on a set of Lebesgue measure zero.

However, when the viscosity tends to zero we have to deal with a singular perturbation problem. The *a priori* estimates available for the

Navier equation obviously fail to be uniform with respect to the viscosity. If one has available a model for generating vorticity in the limiting case of vanishing viscosity and expects that the Euler equations provide a good model for the evolution of this vorticity, the question of the well- posedness of these equations for singular initial data arises. In particular if one starts with a vorticity field concentrated on a curve, it is important to know whether more severe singularities can develop.

This problem has recently received a lot of interest from Applied Mathematicians because it is a fundamental question in fluid dynamics and requires sophisticated tools from applied analysis, and in Cottet (1992) the reader can find some essential features of the concept of concentration and the keys steps in the proof of global existence (in 2D incompressible fluid flows) for a positive vortex sheet. In the case of a 3D fluid flow and smooth initial data, the inviscid limit is the Euler equation as long as the corresponding solution to the Euler equation is smooth [see, for example, Constantin (1986)]. This might be a true limitation because of the possibility of finite-time blow-up. In fact, the blow-up problem for the Euler ($1/Re = 0$) equation is:

Do smooth data (with smooth, rapidly decaying initial velocity) guarantee smooth solutions for all time?

The answer is known to be yes only for 2D, not known for 3D!The main difference between 2D and 3D is now clear. In the right hand side of the vorticity equation, derived from the full compressible Euler equations, when the exterior forces (represented by $\boldsymbol{g}$) are conservative:

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\boldsymbol{u} - \omega(\nabla \cdot \boldsymbol{u}) + \nabla p \wedge \nabla\left(\frac{1}{\rho}\right),$$

the second and the third terms are zero, for an incompressible flow, but the first term, for a 3D flow, is different from zero!It is important to note that a valid blow-up scenario requires the formation of a singular object in physical space. Such an object might be as simple as conical "*elbows*" in a pair of vortex tubes, and such structures might have been observed in numerical simulations [see, for instance, the paper by Kerr (1993)].

In the recent Note by Caflisch and Sammartino (1997), the authors are concerned with the construction of the solution of the incompressible 2D unsteady Navier equations outside a circular domain in the zero viscosity limit. Under suitable hypotheses and by an explicit construction of the solution, it is proved that, when viscosity goes to zero, the solution converges to the Euler solution ($\boldsymbol{u}_e$) outside the boundary layer and to the Prandtl solution ($\boldsymbol{u}_p$) inside the boundary layer (the Prandtl solution decays exponentially outside the boundary layer). It is stated that the existence and uniqueness of the Euler and Prandtl solutions can be shown using an abstract Cauchy-Kovalevskaya theorem. The Euler and Prandtl solutions together with a small correction term [proportional to $(1/Re)^{1/2}$] are used to construct the Navier solution (as a composite asymptotic expansion):

$$\boldsymbol{u} = \boldsymbol{u}_e + \boldsymbol{u}_p + \left(\frac{1}{Re}\right)^{1/2} \boldsymbol{w},$$

and it turns out that the norm of $\boldsymbol{w}$, in the appropriate function space, remains bounded for a time t^* which is independent of the viscosity. In fact, these above authors, imposing analyticity on the initial data, solve two problems on a half-space: (1) to prove existence and uniqueness for the solution $\boldsymbol{u}_p$ and, (2) to prove that the Euler and Prandtl equations correctly describe the behaviour of the solution of the Navier equations in the zero viscosity limit. Vanishing viscosity in the initial boundary value problem (for the 2D unsteady flow in a bounded domain) for the Navier equations is also considered in the short paper by Khatskevich (1996). Under some assumptions (that include, in particular, the case of potential external forces) a positive answer is obtained, by the above author, to the problem of whether solutions to the initial boundary value problem for the Navier equations converge to the solution of the corresponding initial boundary value problem for the Euler equations as the viscosity coefficient tends to zero. The inviscid limit of 2D incompressible fluids with bounded vorticity in the whole space is analysed in the paper by Chemin (1996). In the recent paper of Temam and Wang (2000), the author consider the boundary layer for a flow in a channel with permeable walls and observing that the Prandtl equation can be solved almost exactly in this case the author derive rigorously a number of results (the stability issue) concerning the boundary layer, the convergence of the Navier equations to the Euler equations and the validity of the asymptotic expansion. Concerning the behaviour of the solutions of the Navier equations at vanishing viscosity, see, also the paper by Temam and Wang (1998).

Finally, we note that in the theory of viscous flows the double passage to the limit: $\nu_0 \to 0$ and $t \to \infty$ is quite intriguing and its elucidation would shed light on many important questions. Naturally, the order in which the limits are taken is crucial. In general the inviscid limit ($\nu_0 \to 0$) and the temporal limit ($t \to \infty$) do not commute!The appropriate order for closed systems is taking the temporal limit first, but taking the inviscid limit before the temporal limit might be appropriate for systems in which the forcing fluctuates rapidly in time. If we consider, for simplicity, a linear model [as in Guiraud and Zeytounian (1985)], then due to the linearity of the model, there is no boundary layer on the steady limiting solution and, accordingly, we cannot understand from this simplified model how the growth of the thickness of the boundary layer settles down to a finite $O(\nu_0^{1/2})$ value, as it should for a true flow problem. On the other hand, on a sufficiently long time scale: $t = O(1/\nu_0)$, diffusion has invaded the whole domain and that it is only through its pervading action that, ultimately, the steady solution is established. For a true flow situation, nonlinearity changes the situation drastically but we may safely conjecture that the pervading action of viscosity contributes significantly to the ultimate setting of the steady solution, on a time scale: $t = O(1/\nu_0)$. The main difference between linear and nonlinear models is that, probably due to nonlinearity, the steady solution is built up much more rapidly in the nonlinear case and is probably almost completed by the time when the pervading effect of viscosity is felt. Accordingly, this ultimate behaviour should be controlled by linear equations. In the short paper by Constantin (1995), cited above, the reader can also find a pertinent discussion concerning the temporal limit [note that in Constantin (1995) the case in which the infinite time limit is taken first is referred as the "temporal limit" and the case in which the control parameter (the Reynolds number, *Re*) is taken to infinity first is referred to as the "inviscid limit"]. The recent book by Oleinik and Samokhin (1999) adresses the mathematical theory of the Prandtl equations themselves.

7.1. FROM NS-F TO EULER AND PRANDTL MODEL EQUATIONS VIA 'DOMINANT', VISCOUS AND THERMALLY CONDUCTING, EQUATIONS FOR HIGH REYNOLDS NUMBERS NEAR THE WALL

The practical usefulness of the inviscid, Eulerian fluid concept, considered in the Chapter 6, derives from the fact that the most commonly encountered fluids/liquids (e.g., water and atmospheric air) have small viscosity. Their behaviour should therefore be approximated by that of inviscid fluids.

However, the paradoxes (for instance: d'Alembert's and Jeffrey's paradoxes) will have made plain that the *successful approximation of real fluid motion by inviscid fluid motion cannot be at all straigthforward.* The precise meaning of small viscosity/high Reynolds number must be: $Re \to \infty$. Indeed, in the most common physical circumstances the Reynolds numbers encountered lie between 10^4 and 10^8, and it is clearly desirable to describe such motions in terms of their limit as $Re \to \infty$. In this limit the NS-F equations reduce to the inviscid form and the limiting solutions of the NS-F equations should therefore be solutions of the Euler system analyzed in Chapter 6.

There is an important exception, however. If $|\nabla \cdot [2\mu \boldsymbol{D}]|$ is very large compared with $|S D\boldsymbol{u}/Dt|$, then the limiting NS-F equation of motion (2.56b) may differ radically from the equation (6.3b) for $\boldsymbol{u}_E$, in the Eulerian system, *even though* the parameter: $\varepsilon^2 \equiv 1/Re$ is *very small*! The proper conclusion is therefore that the viscous term distinguishing (2.56b) from the Eulerian equation (6.3b) is negligible in motions at large Reynolds number, unless: $|\nabla \cdot [2\mu \boldsymbol{D}]| \to \infty$ as $Re \to \infty$. Conversely, we must generally anticipate that $|\nabla \cdot [2\mu \boldsymbol{D}]| \to \infty$ somewhere in the fluid domain as $Re \to \infty$, for otherwise the terms proportional to $(1/Re)$ would indeed become negligible in the NS-F equations (2.56b, c) so that the NS-F solution would tend to a solution of the Euler equations, which would be incompatible with the following classical Postulate:

The boundary condition at a solid surface is zero relative fluid velocity

and from the point of view of singular perturbations techniques,

slightly viscous fluid motion is a "singular" perturbation of (inviscid) Eulerian fluid motion,

mainly because the operator $S\,D/Dt = S\,\partial/\partial t + \boldsymbol{u} \cdot \nabla$ is of first order, but the terms $\nabla \cdot [2\mu \boldsymbol{D}]$ and $\nabla \cdot [k \nabla T]$ are of second order.

7.1.1. The Euler (outer) limit and Euler limit equations

The outer (external) Eulerian flow was defined, in fact, as the limit of the 'full exact' NS-F solution as $Re \to \infty$. It was thereby tacitly understood that in taking the (outer) limit, the position of each point in 3D space, $\boldsymbol{P}(\boldsymbol{x})$, and time t, in the fluid motion, was *fixed* (according to the main principle of asymptotic modelling). If

$$U = [\boldsymbol{u}, p, \rho, T], \tag{7.1}$$

is the solution of the dimensionless NS-F equations, then we define the following Euler, outer, limit (when $Bo = 0$):

$$Lim^E = [Re \to \infty, t, P(\boldsymbol{x}), S, M, Pr, \gamma, \text{fixed} = O(1)], \tag{7.2}$$

and $U_E = Lim^E\, U$, is a limit, outer, Euler solution.

It is obvious that for $U_E = [u_E, p_E, \rho_E, T_E]$ one obtains the system of Euler equations (6.3a, b, c), but with $Bo = 0$, and $S\, D/Dt = S\, \partial/\partial t + u_E \cdot \nabla$. In fact, the Euler, outer, limit equations (6.3a, b, c) can be derived from the following outer (x and t fixed) asymptotic expansion:

$$U = U_E + \varepsilon^a\, U_1 + \ldots, \text{ when } \varepsilon \to 0 \text{ with } x \text{ and } t \text{ fixed}, \tag{7.3}$$

where $0 < a < 2$, as leading-order, outer, equations. In this case, for the second term $U_1 = [u_1, p_1, \rho_1, T_1]$ in (7.3), we derive the second-order linearized system of Euler equations (6.25), since $a < 2$. But, for the moment the scalar "a", in the outer expansion (7.3), remains undetermined!Only from the matching, with the Prandtl (inner) limit equations, we determine the appropriate value of "a" (see the Section 7.2.2 in §7.2).

For the derivation of Prandtl, inner, boundary-layer equations, which will be valid near of the wall of the 3D solid body Ω (denoted by $\Sigma = \partial\Omega$), it is convenient, first, to introduce an orthogonal cuvilinear coordinates system, made up of straight lines normal to Σ and curves parallel to Σ. This is the main step in a formalized asymptotic derivation of the (BL) equations.

7.1.2. The 'dominant' NS-F steady equations for high Reynolds numbers near the wall Σ of a 3D solid body

Consider the NS-F steady flow over a solid body with a regular surface Σ and any point M of the flow. Through M let us draw the normal to the surface Σ meeting it in P_Σ. Then the position of M, in 3D space near to Σ, can be expressed in terms of the position of P_Σ and the distance $z = P_\Sigma M$ along the normal to the surface Σ. We assume that n is the unit normal vector to Σ (in the direction of the fluid region) and $s = (s_1, s_2)$ define a system of Lagrangian curvilinear coordinates on the surface Σ (indeed, we can assume that the intrinsic surface coordinates (s_1, s_2) are aligned with the principal directions of curvature at each point of Σ and parametrize Σ). In fact, from (x_1, x_2, x_3) we pass to a local dimensionless coordinates system (s_1, s_2, z). In the steady case, if M is the position vector of M in 3D space then, for the point near the surface Σ, we write:

$$M = P_\Sigma(s) + z\, n(s), \tag{7.4}$$

so that s_1, s_2, z may be taken as curvilinear coordinates of M. Near the surface Σ it is convenient, in the case of high Reynolds numbers asymptotics, to introduce (in place of z)

$$\zeta = Re^{\alpha} z,\ \alpha > 0,\ Re \equiv \frac{1}{\varepsilon^2}. \qquad (7.5)$$

Finally, near the surface Σ of the given solid body, we work with the system of curvilinear coordinates (s_1, s_2, ζ) and the wall Σ is defined by the position vector $\boldsymbol{P}_\Sigma = \boldsymbol{P}_\Sigma(s_1, s_2)$ in the case of a steady fluid flow at high Reynolds numbers. It is now this system of coordinates that will be used, below, to form *steady Prandtl boundary-layer* equations (when $\varepsilon \to 0$, with s_1, s_2, ζ fixed) from the steady NS-F equations written with $(\boldsymbol{s}, \zeta)$ in place of Cartesian coordinates (x_1, x_2, x_3). For the derivation of Prandtl limiting steady approximate equations it is more convenient to introduce the following representation for the velocity vector $\boldsymbol{u}$:

$$\boldsymbol{u} = \boldsymbol{v} + w\,\boldsymbol{n},\ \boldsymbol{v}\cdot\boldsymbol{n} = 0, \qquad (7.6)$$

and in this case, we have the following no- slip condition:

$$\boldsymbol{v} = 0 \text{ and } w = 0 \text{ on } \Sigma. \qquad (7.7)$$

Again, near the surface Σ, it is convenient to introduce a new vertical component of the velocity, in place of w:

$$w^* = Re^{\beta} w,\ \beta > 0. \qquad (7.8)$$

Now we consider the field $U^* = [\boldsymbol{v}^*, w^*, p^*, \rho^*, T^*] = U^*(\boldsymbol{s}, \zeta; \varepsilon)$ and in this case a simple calculation gives, when in the steady case the Strouhal number $S = 0$:

$$S\frac{D}{Dt} \equiv S\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla = \boldsymbol{v}^*\cdot\boldsymbol{D} + \varepsilon^{2(\beta-\alpha)} w^* \frac{\partial}{\partial \zeta} + O\left(\varepsilon^{2\alpha}\right), \qquad (7.9a)$$

where $\boldsymbol{D}$ is the gradient operator on the surface Σ and it is necessary to define $\boldsymbol{D}$.

First, for a function $f(\boldsymbol{M}) = f^E(\boldsymbol{s}, z)$ (in an Eulerian representation) we can write (in the steady case):

$$\frac{\partial f^E}{\partial s_\alpha}=\left[\frac{\partial \boldsymbol{P}_\Sigma}{\partial s_\alpha}+z\frac{\partial \boldsymbol{n}}{\partial s_\alpha}\right]\cdot\nabla f^E,\ \alpha=1,2,\ \frac{\partial f^E}{\partial z}=\boldsymbol{n}\cdot\nabla f^E. \qquad (7.9b)$$

On the other hand, by definition of the vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ we have the following relations:

$$\frac{\partial \boldsymbol{P}_\Sigma}{\partial s_i}\cdot\boldsymbol{E}_j=\delta_{ij};\ \left[\left(\frac{\partial \boldsymbol{P}_\Sigma}{\partial s_i}+z\frac{\partial \boldsymbol{n}}{\partial s_i}\right)\cdot\boldsymbol{H}\right]\boldsymbol{E}_j=\delta_{ij}.\ i,j=1,2. \qquad (7.9c,\ d)$$

As a consequence, we write the following expression for the gradient operator on the surface $\boldsymbol{\Sigma}$

$$\boldsymbol{D}f^E=\left(\frac{\partial f^E}{\partial s_1}\right)\boldsymbol{E}_1+\left(\frac{\partial f^E}{\partial s_2}\right)\boldsymbol{E}_2, \qquad (7.9e)$$

and in this case we derive the following formula for the spatial gradient vector:

$$\nabla f=\boldsymbol{H}\cdot\boldsymbol{D}f^E+\boldsymbol{n}\frac{\partial f^E}{\partial z}. \qquad (7.10)$$

In the above relations (7.9c, d) and (7.10),

$$\boldsymbol{H}=\boldsymbol{1}+z\boldsymbol{H}_1+O(z^2) \qquad (7.11)$$

is a linear operator acting on the vectors ($\boldsymbol{E}_1$ and $\boldsymbol{E}_2$) of the tangent plane to $\boldsymbol{\Sigma}$ at the point $\boldsymbol{P}_\Sigma$ and $\boldsymbol{1}$ is the unit operator.

If, now, we work with the above formulae (7.9a, e), (7.10) and (7.11), and take into account (7.5) and (7.8), it can be shown that:

$$\nabla\cdot\boldsymbol{u}=\boldsymbol{D}\cdot\boldsymbol{v}^*+\varepsilon^{2(\alpha-\beta)}\frac{\partial w^*}{\partial\zeta}+O\left(\varepsilon^{2\alpha}\right), \qquad (7.12a)$$

$$(\nabla \boldsymbol{u})+(\nabla \boldsymbol{u})^{T}-\frac{2}{3}(\nabla\cdot\boldsymbol{u})\boldsymbol{I}=\frac{1}{\varepsilon^{2\alpha}}\left[\left(\frac{\partial \boldsymbol{v}^*}{\partial\zeta}\otimes\boldsymbol{n}\right)+\left(\boldsymbol{n}\otimes\frac{\partial \boldsymbol{v}^*}{\partial\zeta}\right)\right]$$
$$+\boldsymbol{O}(1)+\boldsymbol{O}\left(\varepsilon^{2(\alpha-\beta)}\right) \quad (7.12b)$$

where the dyadic product $(\boldsymbol{f}\otimes\boldsymbol{g})$ of two vectors in (7.12b) generates a second- order tensor and $\boldsymbol{O}(\varepsilon^{2(\beta-\alpha)})$ is a "small" second-order tensor which tends to zero as $\varepsilon\to 0$. In conclusion, with the above relations, we derive the following 'dominant' NS-F steady equations (with $Bo = 0$), valid for high Reynolds number *steady flows* near a 3D solid body surface Σ:

$$\boldsymbol{v}^*\cdot\boldsymbol{D}\rho^*+\rho^*\boldsymbol{D}\cdot\boldsymbol{v}^*+\varepsilon^{2(\beta-\alpha)}\left[\rho^*\frac{\partial w^*}{\partial\zeta}+w^*\frac{\partial\rho^*}{\partial\zeta}\right]=O\left(\varepsilon^{2\alpha}\right), \quad (7.13a)$$

$$\rho^*\left(\boldsymbol{v}^*\cdot\boldsymbol{D}\right)\boldsymbol{v}^*+\varepsilon^{2(\beta-\alpha)}\rho^* w^*\frac{\partial \boldsymbol{v}^*}{\partial\zeta}-\frac{1}{\gamma M^2}\boldsymbol{D}p^*$$
$$-\varepsilon^{2(1-2\alpha)}\frac{\partial}{\partial\zeta}\left(\mu^*\frac{\partial \boldsymbol{v}^*}{\partial\zeta}\right)=\boldsymbol{O}\left(\varepsilon^{2\alpha}\right)+\boldsymbol{O}\left(\varepsilon^{2\beta}\right)+\boldsymbol{O}\left(\varepsilon^{2(1-\alpha)}\right), \quad (7.13b)$$

$$\frac{\partial p^*}{\partial\zeta}=\boldsymbol{O}\left(\varepsilon^{2\alpha}\right)+\boldsymbol{O}\left(\varepsilon^{2}\right), \quad (7.13c)$$

$$\rho^*\boldsymbol{v}^*\cdot\boldsymbol{D}T^*+(\gamma-1)p^*\left[\boldsymbol{D}\cdot\boldsymbol{v}^*+\varepsilon^{2(\beta-\alpha)}\frac{\partial w^*}{\partial\zeta}\right]+\varepsilon^{2(\beta-\alpha)}\rho^* w^*\frac{\partial T^*}{\partial\zeta}$$
$$-\varepsilon^{2(1-2\alpha)}\left[\frac{\gamma}{\gamma-1}M^2\mu^*\left|\frac{\partial \boldsymbol{v}^*}{\partial\zeta}\right|^2+\frac{\gamma}{Pr}\frac{\partial}{\partial\zeta}\left(k^*\frac{\partial T^*}{\partial\zeta}\right)\right]$$
$$=\boldsymbol{O}\left(\varepsilon^{2\alpha}\right)+\boldsymbol{O}\left(\varepsilon^{2(1-\alpha)}\right), \quad (7.13d)$$

$$p^*-T^*\rho^*=0. \quad (7.13e)$$

From the above 'dominant' NS-F steady equations (7.13) we can now elucidate the Prandtl degeneracy of these steady 'dominant' equations in relation to the two positive scalars: α and β.

7.1.3. *The Prandtl (inner) limit and Prandtl (BL) equations*

First, from the equation of continuity (7.13a), it is clear that the significant degeneracy corresponds to:

$$\alpha = \beta. \tag{7.14a}$$

Next, from the equations (7.13b) and (7.13d) we derive the viscous and thermally conducting significant degeneracy if and only if

$$\alpha = 1/2. \tag{7.14b}$$

The local degeneracy corresponding to:

$$\alpha = \beta = 1/2, \tag{7.14c}$$

is the so-called "Prandtl degeneracy" and as a consequence we derive the following *Prandtl, inner, boundary-layer equations:*

$$\boldsymbol{D}\cdot(\rho_P \boldsymbol{v}_P) + \frac{\partial(\rho_P w_P)}{\partial \zeta} = 0,$$

$$\rho_P\left[(\boldsymbol{v}_P\cdot\boldsymbol{D})\boldsymbol{v}_P + w_P\frac{\partial \boldsymbol{v}_P}{\partial \zeta}\right] + \frac{1}{\gamma M^2}\boldsymbol{D}p_e = \frac{\partial}{\partial \zeta}\left(\mu_P\frac{\partial \boldsymbol{v}_P}{\partial \zeta}\right),$$

$$\frac{\partial p_P}{\partial \zeta} = 0,\ p_P \equiv p_e(\boldsymbol{s}), \tag{7.15}$$

$$\rho_P\left[\boldsymbol{v}_P\cdot\boldsymbol{D}T_P + w_P\frac{\partial T_P}{\partial \zeta}\right] + (\gamma-1)p_P\left[\boldsymbol{D}\cdot\boldsymbol{v}_P + \frac{\partial w_P}{\partial \zeta}\right]$$
$$= \gamma(\gamma-1)M^2\mu_P\left|\frac{\partial \boldsymbol{v}_P}{\partial \zeta}\right|^2 + \frac{\gamma}{Pr}\frac{\partial}{\partial \zeta}\left(k_P\frac{\partial T_P}{\partial \zeta}\right),$$

$$p_e = T_P\,\rho_P,$$

where $p_e(\boldsymbol{s})$ is given by matching with the Euler outer flow (see below, in §7.2 the process of matching between (7.15) and the steady inviscid Euler equations). In fact, the above Prandtl, inner, boundary-layer, limiting steady equations (7.15) can be derived from the system of 'dominant' NS-F equations (7.13) by the following *Prandtl, inner* **(s,** ζ ***fixed***), *steady limit:*

$$Lim^P = [\varepsilon \to 0; \mathbf{s}, \zeta, S, M, Pr, \gamma, \text{fixed} = O(1)] \tag{7.16}$$

and in this case

$$U_P = [\mathbf{v}_P, w_P, p_P, \rho_P, T_P] = Lim^P\, U^*(\mathbf{s}, \zeta; \varepsilon), \tag{7.17}$$

and $\mu_P = \mu(T_P)$, $k_P = k(T_P)$. More precisely, in the boundary layer, we consider the following inner (with s and $\zeta = z/\varepsilon$ fixed) asymptotic expansions:

$$\mathbf{v} = \mathbf{v}_P + O(\varepsilon), w = \varepsilon w_P + o(\varepsilon), (p, \rho, T) = (p_P, \rho_P, T_P) + O(\varepsilon), \tag{7.18}$$

where $[\mathbf{v}_P, w_P, p_P, \rho_P, T_P] = U_P$ are functions of $\mathbf{s}$ and $\zeta = z/\varepsilon$ and they satisfy the Prandtl, leading-order boundary-layer equations (7.15). We note that:

$$O(\varepsilon) \to 0 \text{ as } \varepsilon \to 0 \text{ and } \frac{o(\varepsilon)}{\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{7.19}$$

7.1.3a. The Prandtl BL equations in a system of curvilinear coordinates locally orthogonal on the wall Σ

For many practical purposes it is convenient to choose the curvilinear co-ordinates s_1, s_2 on the surface Σ to be orthogonal. It should be noted that the three surfaces $s_1 = const$, $s_2 = const$ and $\zeta = const$ are *not* necessarily orthogonal everywhere in space. Of course they are *locally orthogonal* at Σ but only when the curves $s_1 = const$ and $s_2 = const$ on Σ coincide with the *lines of curvatures* of Σ do the *three families of surfaces become orthogonal everywhere*. However the boundary layer is a local phenomenon on the surface of Σ and it is quite sufficient for our purpose to have the system of curvilinear coordinates locally orthogonal at Σ [see Rosenhead (1963); pp.412-414)]. When therefore the curves: $s_1 \equiv \xi = const$ and $s_2 \equiv \eta = const$ are orthogonal on Σ we write:

$$\mathbf{v}_P = u_P\, \mathbf{e}_1 + v_P\, \mathbf{e}_2, \tag{7.20a}$$

where

$$\boldsymbol{e}_1 = \frac{1}{h_1}\frac{\partial \boldsymbol{P}_\Sigma}{\partial \xi},\ \boldsymbol{e}_2 = \frac{1}{h_2}\frac{\partial \boldsymbol{P}_\Sigma}{\partial \eta},\ h_1 = \left|\frac{\partial \boldsymbol{P}_\Sigma}{\partial \xi}\right|,\ h_2 = \left|\frac{\partial \boldsymbol{P}_\Sigma}{\partial \eta}\right|. \qquad (7.20b)$$

In (7.20a, b) $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit tangential vectors respectively to the two parametric curves on $\boldsymbol{\Sigma}$ and $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{n})$ forms an orthogonal triad of unit vectors. Since $\boldsymbol{\Sigma}$ is supposed to be *smoothly curved,* it follows that h_1, h_2, their derivatives with respect to ξ and η and the derivatives of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{n}$ are all $O(1)$.

If we assume that $\boldsymbol{P}_\Sigma$ is independent of time: $\boldsymbol{P}_\Sigma = \boldsymbol{P}_\Sigma(\xi, \eta)$, then all these quantities are, of course, functions of ξ and η only. In the boundary-layer equations (7.15), with (7.20a, b), the gradient operator on $\boldsymbol{\Sigma}$ takes the form:

$$\boldsymbol{D} = \frac{1}{h_1}\frac{\partial}{\partial \xi}\boldsymbol{e}_1 + \frac{1}{h_2}\frac{\partial}{\partial \eta}\boldsymbol{e}_2, \qquad (7.21a)$$

and

$$\boldsymbol{v}_P \cdot \boldsymbol{D} + w_P\frac{\partial}{\partial \zeta} \equiv \frac{u_P}{h_1}\frac{\partial}{\partial \xi} + \frac{v_P}{h_2}\frac{\partial}{\partial \eta} + w_P\frac{\partial}{\partial \zeta}. \qquad (7.21b)$$

Finally, according to (7.20a, b) and (7.21a, b), in terms of the local variables ξ, η, ζ we derive, in place of the system (7.15), the following boundary-layer equations for u_P, v_P, w_P, ρ_P, T_P,

$$\frac{1}{h_1 h_2}\frac{\partial}{\partial \xi}(\rho_P u_P h_2) + \frac{1}{h_1 h_2}\frac{\partial}{\partial \eta}(\rho_P v_P h_1) + \frac{\partial}{\partial \zeta}(\rho_P w_P) = 0, \qquad (7.22a)$$

$$\frac{u_P}{h_1}\frac{\partial u_P}{\partial \xi} + \frac{v_P}{h_2}\frac{\partial u_P}{\partial \eta} + w_P\frac{\partial u_P}{\partial \zeta} + \left[\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \eta}\right]u_P v_P$$
$$-\left[\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}\right](v_P)^2 = -\frac{1}{\gamma M^2}\frac{1}{\rho_P h_1}\frac{\partial p_e}{\partial \xi} + \frac{1}{\rho_P}\frac{\partial}{\partial \zeta}\left(\mu_P\frac{\partial u_P}{\partial \zeta}\right), \qquad (7.22b)$$

$$\frac{u_P}{h_1}\frac{\partial v_P}{\partial \xi}+\frac{v_P}{h_2}\frac{\partial v_P}{\partial \eta}+w_P\frac{\partial v_P}{\partial \zeta}+\left[\frac{1}{h_1h_2}\frac{\partial h_2}{\partial \xi}\right]u_Pv_P$$

$$-\left[\frac{1}{h_1h_2}\frac{\partial h_1}{\partial \eta}\right](u_P)^2=-\frac{1}{\gamma M^2}\frac{1}{\rho_P h_2}\frac{\partial p_e}{\partial \eta}+\frac{1}{\rho_P}\frac{\partial}{\partial \zeta}\left(\mu_P\frac{\partial v_P}{\partial \zeta}\right), \quad (7.22c)$$

$$\rho_P\left[\frac{u_P}{h_1}\frac{\partial T_P}{\partial \xi}+\frac{v_P}{h_2}\frac{\partial T_P}{\partial \eta}+w_P\frac{\partial T_P}{\partial \zeta}\right]$$

$$+(\gamma-1)p_e\left[\frac{1}{h_1h_2}\left\{\frac{\partial}{\partial \xi}(u_Ph_2)+\frac{\partial}{\partial \eta}(v_Ph_1)\right\}+\frac{\partial w_P}{\partial \zeta}\right]$$

$$=\gamma(\gamma-1)M^2\mu_P\left[\left(\frac{\partial u_P}{\partial \zeta}\right)^2+\left(\frac{\partial v_P}{\partial \zeta}\right)^2\right]+\frac{\gamma}{Pr}\frac{\partial}{\partial \zeta}\left(k_P\frac{\partial T_P}{\partial \zeta}\right), \quad (7.22d)$$

$$p_e = T_P\,\rho_P, \quad (7.22e)$$

where $p_P \equiv p_e(\xi,\eta)$.

Unlike the two dimensional case ($v_P = 0$ and $\partial/\partial\eta = 0$) these above boundary-layer equations (7.22a, b, c, d, e) contain terms $\partial h_1/\partial\eta$ and $\partial h_2/\partial\xi$ depending explicitly on the *curvatures* of the coordinates system. We may notice that if it is possible to choose a coordinate system such that both $\partial h_1/\partial\eta$ and $\partial h_2/\partial\xi$ *vanish everywhere on the surface* then the *Gausssian curvature K* of the surface Σ which is given by

$$K=-h_1h_2\left\{\frac{\partial}{\partial \xi}\left[\frac{1}{h_1}\frac{\partial h_2}{\partial \xi}\right]+\frac{\partial}{\partial \eta}\left[\frac{1}{h_2}\frac{\partial h_1}{\partial \eta}\right]\right\}, \quad (7.23)$$

vanishes and so the surface Σ must be *developable*. Conversely, see Howarth (1959, pp. 309 and 310), if the surface Σ is developable the parametric curves can be chosen so that: $h_1 = h_2 = 1$, and the curvature terms *disappear* from the equations of motion. This a point of some practical importance.

7.2. BOUNDARY CONDITIONS AND MATCHING

7.2.1. More on matching

In order to relate the outer, Euler, solution and the inner, Prandtl, solution to each other, one presupposes the existence of a region of *overlapping,* where the two asymptotic expansions are valid. Kaplun and Lagerstrom (1957) admitted that there is no *a priori* reason for the region of validity of the inner and outer asymptotic expansions to overlap!Therefore, the results should be assumed to be *a priori* asymptotically correct. The simplest asymptotic matching principle due to Prandtl states:

$$\lim_{z\to 0}\,[\text{Euler solution rewritten in the variables } (\xi,\eta,z)]$$
$$= \lim_{\zeta\to\infty}\,[\text{Prandtl solution in the variables } (\xi,\eta,\zeta)] \qquad (7.24\text{a})$$

where $\zeta = z/\varepsilon$, and $\varepsilon^2 = 1/Re$.

One more general asymptotic matching principle which involves the intermediate limits is that due to Kaplun. Another widely used asymptotic matching principle is that due to Van Dyke, which states:

[The *m*-term inner expansion of (the *n*-term outer expansion)]
= The *n*-term outer expansion of (the *m*-term inner expansion)], (7.24b)

where m and n are any two integers.

Thus in order to find the left hand side, one writes n terms of the outer expansion in terms of the inner variable $(z = \varepsilon\zeta)$ expands it for small ε keeping the inner variable ζ fixed, and truncates the resulting expansion after m terms - and similarly for the right hand side. But, a variant form of asymptotic matching principle due to Shivamoggi (1978) is more expeditious in this context (see §3.2, in the Chapter 3). Note that this principle puts a less stringent restriction on the domain of validity of the outer solution in that the latter is required to extend merely to the neighbourhood of the inner boundary whereas the basic principle due to Prandtl requires the domain of validity of the outer solution to extend right up to the inner boundary - a probable source of the difficulties the latter method develops in higher order problems. This may also be the reason why Shivamoggi's (1978) principle succeeds where Prandtl's principle fails.

Concerning the *intermediate matching principle,* see for instance the book by Kevorkian and Cole (1981, p. 9). According to this intermediate matching we assume that: the outer (Euler) expansion, which was constructed under the assumption, $\varepsilon \to 0$ with z fixed $\neq 0$, is actually valid in the extended sense, such that:

$\varepsilon \to 0$, $z^* = z/\eta(\varepsilon)$ fixed for some class of gauge functions, $1 >> \eta(\varepsilon) << \lambda(\varepsilon)$.

Similarly, the inner (Prandtl) expansion which was constructed under the assumption, $\varepsilon \to 0$ with $\zeta = z/\varepsilon$ fixed $\neq \infty$, is actually valid in the extended sense, such that

$\varepsilon \to 0$, $z^* = z/\eta(\varepsilon)$ fixed for some class of gauge functions, $\eta(\varepsilon) >> \mu(\varepsilon)$.

The extended domains of validity of the inner, $E_n(z)$, and outer, $P_n(\zeta = z/\varepsilon)$, expansions overlap in the following sense:

$$\begin{gathered}\text{For each } m = 0, 1, 2, \ldots, \text{ there exist integers}\\ p, \text{ and } q, \text{ and functions } \lambda(\varepsilon) \text{ and } \mu(\varepsilon)\\ \text{with } \mu(\varepsilon) << \lambda(\varepsilon) \text{ such that:}\\ Lim \{\Sigma_{p\ terms} E_n(z^*\eta)\,\varepsilon^n - \Sigma_{q\ terms} P_n(z^*\eta/\varepsilon)\,\varepsilon^n\}/\varepsilon^m = 0,\\ \text{when } \varepsilon \to 0 \text{ and } z^* = z/\eta(\varepsilon) \text{ fixed, for all } \mu << \eta << \lambda.\end{gathered} \tag{7.24c}$$

With (7.24c), the matching occurs at $O(\varepsilon^m)$. For a deeper discussion of domains of validity, overlap and matching, see the book by Lagerstrom (1988; Section 1.4) devoted to MMAE.

7.2.2. Euler-Prandtl matching

In order to complete the formulation of the system of Prandtl boundary-layer equations (7.22) it is necessary to specify the appropriate boundary conditions. Obviously at the wall $\zeta = 0$ (which is the inside of the domain of validity of the boundary-layer equations), assumed impermeable (and neglecting slip effects),

$$u_P = v_P = w_P = 0 \text{ and } T_P = T_W \text{ on } \zeta = 0, \tag{7.25}$$

where $T_W > 0$ is a known function (prescribed wall temperature) of (ξ, η).

A permeable wall means that w_P takes a value (usually prescribed) there.

Slip effects are strictly outside the scope of continuum mechanics and are related to so-called Knudsen layer. The appropriate boundary conditions as

$\zeta \to \infty$ require a little care! According to (7.18) and (7.20a), in the boundary layer, we consider the following inner (with: ξ, η and $\zeta = z/\varepsilon$ fixed) asymptotic expansions:

$$(u, v) = (u_P, v_P) + O(\varepsilon), w = \varepsilon w_P + o(\varepsilon), (p, \rho, T) = (p_P, \rho_P, T_P) + O(\varepsilon),$$

where u_P, v_P, w_P, p_P, ρ_P, T_P, are functions of ξ, η and $\zeta = z/\varepsilon$.

In the inviscid Euler flow, we consider the following outer (with ξ, η and z fixed) asymptotic expansions:

$$(u, v, w, p, \rho, T) = (u_E, v_E, w_E, p_E, \rho_E, T_E) + O(\varepsilon),$$

where u_E, v_E, w_E, p_E, ρ_E, T_E are functions of ξ, η and z.

These functions (with subscript "$_E$") satisfy the Euler equations written in the curvilinear co-ordinates, ξ, η, z.

Now, according to the Prandtl asymptotic, matching principle (7.24a) it is necessary, first, to calculate the limiting value of: u_E, v_E, w_E, p_E, ρ_E and T_E when $z \to 0$. As a consequence, we obtain the values of the inviscid flow "above" (just outside) the boundary-layer

$$\lim_{z \to 0} (u_E, v_E, w_E, p_E, \rho_E, T_E) = (u_e, v_e, w_e, p_e, \rho_e, T_e)$$

and these limiting values $(u_e, v_e, w_e, p_e, \rho_e, T_e)$ are functions of ξ and η. But, as a consequence of the above inner asymptotic expansion for $w\ (= \varepsilon w_P + o(\varepsilon))$, it is obvious that:

$$w_e = 0 \Rightarrow w_e = 0 \text{ on } z = 0. \tag{7.26a}$$

This boundary condition (7.26a) is the steady slip condition on the wall, $z = 0$, for the inviscid Euler equations (written in the curvilinear coordinates, ξ, η, z).

In this case, in expansion (7.3), necessarily: $a = 1$, and as a consequence of the outer expansion for $w\ (= w_e + \varepsilon w_1 + \ldots)$ we obtain:

$$w_P \to w_1 |_{z=0} \text{ as } \zeta \to \infty, \tag{7.26b}$$

where w_1 is the second-order vertical velocity (in the direction of the normal $\boldsymbol{n}$ to the body wall Σ) term in the outer expansion (a function of ξ, η and z).

The matching condition (7.26b) gives an 'interaction' relation between the Prandtl boundary-layer solution and the second-order linear Eulerian outer flow.

Since the pressure $p_P = p_e$ in the boundary layer is independent of ζ, then it is determined by its value at $z = 0$ according to the inviscid solution; namely:

$$\frac{u_e}{h_1}\frac{\partial u_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial u_e}{\partial \eta}+\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \eta}u_e v_e-\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}v_e^{\,2}=-\frac{1}{h_1\gamma M^2}\frac{1}{\rho_e}\frac{\partial p_e}{\partial \xi}, \tag{7.27a}$$

$$\frac{u_e}{h_1}\frac{\partial v_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial v_e}{\partial \eta}+\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial \xi}u_e v_e-\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial \eta}v_e^{\,2}=-\frac{1}{h_2\gamma M^2}\frac{1}{\rho_e}\frac{\partial p_e}{\partial \eta} \tag{7.27b}$$

with

$$\rho_P\, T_P = T_e\, \rho_e. \tag{7.27c}$$

Thus, at the "infinity", for the Prandtl boundary-layer equations (7.22a, b, c, d), with (7.22e), for u_P, v_P, T_P, ρ_P, we write the following conditions (as consequence of matching with the Euler solution):

$$(u_P, v_P, T_P, \rho_P) \to (u_e, v_e, T_e, \rho_e) \text{ as } \zeta \to +\infty. \tag{7.28}$$

We note that, if:

$$(u_P, v_P, T_P, \rho_P) = (u_P, v_P, T_P, \rho_P)^{\infty} + exp(\zeta),\ \zeta \to +\infty, \tag{7.29}$$

where

$$\frac{exp(\zeta)}{O\left(\frac{1}{\zeta}\right)} \to 0 \text{ when } \zeta \to +\infty,$$

then from the matching principle (7.24c), with $m = 0$, $p = q = 1$ and the condition: $\varepsilon << \eta << 1$, we derive again the above matching conditions

(7.28). The slip condition (7.26a) for the Euler equations is also a consequence of the intermediate matching condition (7.24c). Finally, we note that from the Euler equation for T_e we derive the following "compatibility" equation:

$$\left\{\frac{u_e}{h_1}\frac{\partial}{\partial\xi}+\frac{v_e}{h_2}\frac{\partial}{\partial\eta}\right\}\left[log\left(\frac{T_e}{(\gamma-1)\rho_e^{(\gamma-1)}}\right)\right]=0. \tag{7.30}$$

If, now, we take into account the following behaviour (at infinity) for the vertical velocity component in the boundary layer (as consequence of the continuity equation (7.22a)):

$$w_P = w_{P,1}\zeta+(w_P)^\infty+exp(\zeta), \text{ when } \zeta\to\infty, \tag{7.31}$$

then the limit boundary-layer functions $(u_P,\ v_P,\ T_P,\ \rho_P)^\infty$, in (7.29), satisfy also an equation analogous to above equation (7.30).

Again, from the continuity equation for the Prandtl boundary layer and (7.31) we derive the following relation:

$$(\rho_P w_P)^\infty = \boldsymbol{D}\cdot\int_0^\infty[(\rho_P\boldsymbol{v}_P)^\infty-\rho_P\boldsymbol{v}_P]d\zeta, \tag{7.32}$$

where $\boldsymbol{v}_P = u_P\,\boldsymbol{e}_1 + v_P\,\boldsymbol{e}_2$ and $\boldsymbol{D}$ is defined by (7.21a). This last relation (7.32) is closely linked with the displacement thickness of the leading-order (Prandtl) boundary layer and in the next §7.3, we shall explicitly take into account its effect on the outer, second-order, flow. In fact, in the next §7.3, for the steady 2D incompressible (Navier) flow, we derive the first (leading) and second-order equations for the outer and inner flows and we obtain formally the corresponding matching conditions and the interaction relation between the leading-order inner (boundary layer) flow and second-order outer (linearized Euler) flow. For this we utilize the Shivamoggi's (1978) matching principle.

7.3. SECOND-ORDER STEADY 2D BOUNDARY-LAYER EQUATIONS AND NON-INTERACTING EULER-PRANDTL COUPLING

The development of higher-order boundary-layer theory has been marked by two prolonged controversies [Van Dyke (1969; p.268)]. One revolves around the extra terms that must be included in the boundary-layer equations to describe the effects of surface curvature. Obviously, centrifugal force produces a pressure gradient across the boundary layer on a curved wall, which has a second-order effect, of relative order $(1/Re)^{1/2} \equiv \varepsilon$. In addition, longitudinal curvature contributes other terms to the second-order equations and boundary conditions. Several investigators have omitted essential terms, whereas many others have worked with equations that are unnecessarily complicated. To clarify this point, according to Van Dyke treatment, we consider below the Navier equations for steady plane, 2D incompressible and viscous fluid flow. With dimensionless variables, with s measured along and n normal to the curved wall, and u and v the corresponding velocity components, the full divergence-free condition and steady Navier equation:

$$\nabla \cdot \boldsymbol{u} = 0, \text{ and } (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \frac{1}{Re} \Delta \boldsymbol{u},$$

may be written concisely, in dimensionless form, as:

$$\frac{\partial u}{\partial s} + \frac{\partial}{\partial n}(hv) = 0, \tag{7.33a}$$

$$u\frac{\partial u}{\partial s} + v\frac{\partial}{\partial n}(hu) + \frac{\partial p}{\partial s} = \frac{h}{Re}\frac{\partial}{\partial n}\left\{\frac{1}{h}\left[\frac{\partial}{\partial n}(hu) - \frac{\partial v}{\partial s}\right]\right\}, \tag{7.33b}$$

$$u\frac{\partial v}{\partial s} + hv\frac{\partial v}{\partial n} - Ku^2 + h\frac{\partial p}{\partial n} = -\frac{h}{Re}\frac{1}{h}\frac{\partial}{\partial s}\left\{\frac{1}{h}\left[\frac{\partial}{\partial n}(hu) - \frac{\partial v}{\partial s}\right]\right\}, \tag{7.33c}$$

where $h = 1 + K(s)\, n$, and $K(s)$ is the positive curvature for a convex curved wall. The boundary conditions at a solid body wall $(n = 0)$ are:

$$u = v = 0, \text{ on } n = 0, \tag{7.34}$$

if slip (from the Knudsen layer) is neglected. Far upstream the flow is to approach prescribed (possibly non-uniform) velocity and pressure fields: (u_∞, v_∞) and p_∞. We assume, for simplicity, that conditions far upstream are independent of Reynolds number - the oncoming stream is then represented by a solution of the inviscid $(Re \equiv \infty)$ equations.

7.3.1. Outer Eulerian equations

Following the usual procedure (MMAE) in the singular perturbation analysis, we first make an outer expansion of our variables by the (outer or Euler) limit:

$$s \text{ and } n \text{ fixed, } \left(\frac{1}{Re}\right)^{\frac{1}{2}} \equiv \varepsilon \to 0\,; \tag{7.35}$$

thus we write

$$(u, v, p) = (u_1, v_1, p_1) + \varepsilon(u_2, v_2, p_2) + \dots \,. \tag{7.36}$$

For the leading terms (u_1, v_1, p_1) we derive the classical incompressible two- dimensional steady Euler equations written in the (s, n) coordinates:

$$\frac{\partial u_1}{\partial s} + \frac{\partial}{\partial n}(hv_1) = 0\,, \tag{7.37a}$$

$$u_1\frac{\partial u_1}{\partial s} + v_1\frac{\partial (hu_1)}{\partial n} + Ku_1v_1 + \frac{\partial p_1}{\partial s} = 0\,, \tag{7.37b}$$

$$u_1\frac{\partial v_1}{\partial s} + hv_1\frac{\partial v_1}{\partial n} - Ku_1^{\,2} + h\frac{\partial p_1}{\partial n} = 0\,. \tag{7.37c}$$

The next order outer terms give the following small-perturbation form of the Euler equations, for the second-order terms (u_2, v_2, p_2):

$$\frac{\partial u_2}{\partial s} + \frac{\partial}{\partial n}(hv_2) = 0\,, \tag{7.38a}$$

$$u_1\frac{\partial u_2}{\partial s} + v_1\frac{\partial}{\partial n}(hu_2) + u_2\frac{\partial u_1}{\partial s} + v_2\frac{\partial}{\partial n}(hu_1) + \frac{\partial p_2}{\partial s} = 0\,, \tag{7.38b}$$

$$u_1\frac{\partial v_2}{\partial s} + hv_1\frac{\partial v_2}{\partial n} + u_2\frac{\partial v_1}{\partial s} + hv_2\frac{\partial v_1}{\partial n} - 2Ku_1u_2 + h\frac{\partial p_2}{\partial n} = 0\,, \tag{7.38c}$$

Viscous terms appear in the outer equations beginning only at third order. The outer expansion (7.36) is invalid at the body surface, where the no-slip condition (7.34) must be given up. In fact, it appears that all boundary conditions at the body wall must be dropped except that on the normal component of velocity in the leading approximation:

$$v_1 = 0, \text{ on } n = 0. \tag{7.39}$$

More precisely, this above slip condition, for the system of outer Euler equations (7.37), is a direct consequence of the matching between outer (7.36) and inner (see below (7.42)) expansions according to (7.47a).

7.3.2. Inner Prandtl and second-order boundary - layer equations

The outer expansion (7.36) violates the condition on velocity at the wall. It is therefore invalid within the boundary layer, whose dimensionless thickness is [for finite s and $K(s)$] of order $(1/Re)^{1/2} \equiv \varepsilon$. Following Prandtl, we magnify the normal coordinate accordingly by introducing the boundary-layer vertical variable

$$N = \frac{n}{\varepsilon}. \tag{7.40}$$

The normal velocity is likewise small in the boundary layer, and must be magnified similarly. Then Prandtl's boundary-layer approximation is obtained by letting (inner limit - obviously the outer horizontal variable s also plays the role of an inner variables):

$$\varepsilon \to 0 \text{ with the inner variables } s \text{ and } N \text{ fixed.} \tag{7.41}$$

Repeated application of this inner (or Prandtl) limit, in conjuction with an appropriate sequence of functions of Re (gauge functions ε^n, $n = 0, 1, 2,...$) produce the inner expansion. For an analytic semi-infinite body free of separation this sequence is believed to consist again of negative half powers of Re. Thus the inner expansion is:

$$u = U_1(s, N) + \varepsilon U_2(s, N) +..., \tag{7.42a}$$

$$v = \varepsilon [V_1(s, N) + \varepsilon V_2(s, N) +...], \tag{7.42b}$$

$$p = P_1(s, N) + \varepsilon P_2(s, N) + \dots \tag{7.42c}$$

Substituting into the Navier equations the new variable (7.40), and the inner expansion (7.42) gives, via the inner limit (7.41), for the first approximation $[U_1(s, N), V_1(s, N), P_1(s, N)]$ the following first-order boundary-layer equations:

$$\frac{\partial U_1}{\partial s} + \frac{\partial V_1}{\partial N} = 0, \tag{7.43a}$$

$$U_1 \frac{\partial U_1}{\partial s} + V_1 \frac{\partial U_1}{\partial N} + \frac{\partial P_1}{\partial s} = \frac{\partial^2 U_1}{\partial N^2}, \tag{7.43b}$$

$$\frac{\partial P_1}{\partial N} = 0, \tag{7.43c}$$

which are identical to Prandtl's classical BL equations showing that curvature has no explicit influence to this order, since:

$$h = 1 + \varepsilon K(s) N \to 1 \text{ as } \varepsilon \to 0, \text{ with } N \text{ fixed.}$$

As a consequence the next order terms give the following second-order linear BL equations:

$$\frac{\partial U_2}{\partial s} + \frac{\partial V_2}{\partial N} = -K(s) \frac{\partial}{\partial N}[N V_1], \tag{7.44a}$$

$$U_1 \frac{\partial U_2}{\partial s} + V_1 \frac{\partial U_2}{\partial N} + U_2 \frac{\partial U_1}{\partial s} + V_2 \frac{\partial U_1}{\partial N} = -\frac{\partial P_2}{\partial s}$$
$$+ \frac{\partial^2 U_2}{\partial N^2} + K(s)\left\{ N\left[U_1 \frac{\partial U_1}{\partial s} + \frac{\partial P_1}{\partial s} \right] + \frac{\partial U_1}{\partial N} - U_1 V_1 \right\}, \tag{7.44b}$$

$$\frac{\partial P_2}{\partial N} = K(s)(U_1)^2. \tag{7.44c}$$

There results were given by Van Dyke (1969) and Narasimha and Ojha (1967) and these second-order boundary-layer equations for the secondary terms of the inner expansion (7.42), U_2, V_2 and P_2, are linearized equations relative to the first order BL functions U_1, V_1 and P_1. The requirement of zero velocity at the surface $N = 0$ gives the boundary conditions:

$$U_1 = V_1 = U_2 = V_2 = 0, \text{ on } N = 0. \tag{7.45}$$

The upstream conditions will not in general be satisfied by the inner expansion.

7.3.3. Matching

Insufficient boundary conditions are available. The missing conditions are supplied by matching the inner and outer expansions (we hope that the MMAE is a good method for this singular perturbation problem). Here it suffices to apply Shivamoggi's (1978) matching principle. For instance, matching of horizontal velocity components (outer and inner) u gives:

$$u_1(s,0) + \varepsilon N\left(\frac{\partial u_1}{\partial n}\right)_{n=0} + \varepsilon u_2(s,0) + O(\varepsilon^2)$$
$$= U_1(s,\infty) + \varepsilon U_2(s,\infty) + O(\varepsilon^2)$$

and as a consequence we derive the following two matching conditions:

$$U_1(s, \infty) = u_1(s, 0); \tag{7.46a}$$

$$U_2(s,N) \sim N\left(\frac{\partial u_1}{\partial n}\right)_{n=0} + u_2(s,0), \text{ as } N \to \infty. \tag{7.46b}$$

Matching of vertical velocities (outer and inner) v, in the same way gives the following matching conditions; first:

$$lim_{n\to 0}\, v_1(s, n) = v_1(s, 0) = 0, \tag{7.47a}$$

which is just the usual slip condition (7.39), for the inviscid Euler equations (7.37), and then the following condition is derived:

$$v_2(s,0) = lim_{N\to\infty}\left[V_1(s,N) - N\left(\frac{\partial v_1}{\partial n}\right)_{n=0}\right]. \qquad (7.47b)$$

Finally, the matching of pressures p gives:

$$P_1 \equiv P_1(s) = p_1(s,\ 0), \qquad (7.48a)$$

$$P_2(s,N) \sim N\left(\frac{\partial p_1}{\partial n}\right)_{n=0} + p_2(s,0), \text{ as } N \to \infty. \qquad (7.48b)$$

But from the outer Euler equation (7.37c) we derive also the following relation:

$$\left(\frac{\partial p_1}{\partial n}\right)_{n=0} = K[u_1(s,0)]^2, \qquad (7.48c)$$

and, in place of the relation (7.48b), for $P_2(s,\ N)$, we obtain the following behaviour far from the wall:

$$P_2(s,N) \sim NK[u_1(s,0)]^2 + p_2(s,0), \text{ as } N \to \infty. \qquad (7.48d)$$

As a consequence of the relation (7.48a), in the boundary-layer equations (7.43b) we have, for the term $\partial P_1/\partial s$, the relation:

$$\frac{\partial P_1}{\partial s} \equiv \frac{\partial p_{1e}}{\partial s} = -u_{1e}\frac{\partial u_{1e}}{\partial s}, \qquad (7.49)$$

where $(p_{1e},\ u_{1e}) \equiv (p_1,\ u_1)_{n=0}$, and $u_{1e}(s)$ is the surface speed (since, according to slip condition: $v_{1e} \equiv v_1(s,\ 0) = 0$) and $p_{1e}(s)$ is the surface pressure from the outer Euler solution. In the second-order boundary-layer equations (7.44), according to (7.44c) and (7.48d), we have for the term $\partial P_2/\partial s$ the following remarkable relation:

$$\frac{\partial P_2}{\partial s} = \frac{d}{ds}\left\{p_{2e} + K\int_0^N (U_1)^2\,dN + K\int_0^\infty \left[(u_{1e})^2 - (U_1)^2\right]dN\right\}, \qquad (7.50)$$

where $p_{2e}(s) \equiv p_2(s, 0)$.

Finally, in place of (7.47b) for $v_2(s, 0)$, it is possible to write a more appropriate boundary condition for the system of second-order linearized Euler equations (7.38).

7.3.4. Displacement thickness and the condition for $v_2(s, 0)$

Indeed, an important feature of the Prandtl boundary layer is the displacement thickness and its effect on the outer Eulerian flow, which is, in aerodynamics applications, the more interesting flow.

This displacement thickness can be determined from the first-order velocity distribution $U_1(s, N)$ through the well known relationship:

$$\delta^*(s) = \varepsilon h(s), \text{ with: } h(s) = \int_0^\infty \left[1 - \frac{U_1(s,N)}{u_{1e}(s)}\right] dN . \qquad (7.51)$$

Now, from the BL continuity equation (7.43a) we derive for $V_1(s,N)$ the following relation:

$$V_1(s,N) = -\int_0^N \frac{\partial U_1}{\partial s} dN = -N\frac{du_{1e}(s)}{ds} + \int_0^N \left[\frac{du_{1e}(s)}{ds} - \frac{\partial U_1}{\partial s}\right] dN$$
$$= -N\frac{du_{1e}(s)}{ds} + \frac{d}{ds}[u_{1e}(s)h(s)] - \int_N^\infty \left[\frac{du_{1e}(s)}{ds} - \frac{\partial U_1}{\partial s}\right] dN . \qquad (7.52)$$

As a consequence, with (7.51), we obtain, from (7.47b) and (7.52):

$$v_2(s,0) = lim_{N\to\infty}\left[V_1(s,N) - N\left(\frac{\partial v_1}{\partial n}\right)_{n=0}\right] = \frac{d}{ds}[u_{1e}(s)h(s)], \qquad (7.53)$$

since, according to the Euler continuity equation, (7.37a), we have $-(\partial v_1/\partial n)_{n=0} = du_{1e}(s)/ds$. Therefore, the effect of dispacement thickness on the normal component of the second-order outer flow velocity $v_2(s, 0)$ is given by:

$$v_2(s,0)=\frac{d}{ds}\left[\frac{u_{1e}(s)\delta^*(s)}{\varepsilon}\right]. \tag{7.54}$$

Thus:

The Prandtl boundary-layer displaces the outer inviscid flow like a solid surface described by the equation:

$$n = \varepsilon h(s) = \delta^*(s), \tag{7.55}$$

and the matching condition (7.54) is, in fact, the *linearized thin-body* (relative to the body, $n = 0$, *augmented* on account of the thin BL "thickness") approximation, transferred to the surface $n = 0$. From a slightly different point of view, we see that:

The second-order component of normal velocity in the outer flow, v_2, *evaluated at the surface* $n = 0$, *is required to be equal to the slope of the displacement of the thin body* $n = \varepsilon h(s)$.

7.3.5. Noninteracting coupling

In conclusion, we see that it is necessary to resolve: In the first place, the classical Euler equations (7.37a, b, c), for (u_1, v_1, p_1), with the slip boundary condition (7.39) [or matching condition (7.47a)] and the conditions far upstream:

$$(u_1, v_1, p_1) \to (u_\infty, v_\infty, p_\infty). \tag{7.56}$$

Then, with the wall values:

$$(p_{1e}, u_{1e}) \equiv (p_1, u_1)_{n=0}$$

and the relation (7.49), we can resolve the Prandtl boundary-layer equations (7.43a, b) for: $U_1(s, N)$ and $V_1(s, N)$ according to boundary conditions in N:

$$U_1(s, 0) = V_1(s, 0) = 0 \text{ and } U_1(s, \infty) = u_1(s, 0) = u_{1e}(s). \tag{7.57}$$

Next, we obtain a "well-posed problem" for the second-order, small-perturbation form (linearized) of the Euler equations (7.38a, b, c), if we assume for these equations the boundary condition (7.54), for $v_2(s, 0)$, which takes into account the effect of displacement thickness generated by the Prandtl boundary-layer.

For these equations (7.38a, b, c) it is also necessary to assume the following conditions far upstream:

$$(u_2, v_2, p_2) \to 0. \tag{7.58}$$

Finally, for the second-order boundary-layer equations (7.44a) and (7.44b), for $U_2(s, N)$ and $V_2(s, N)$, we have the following boundary conditions in N:

$$U_2(s, 0) = V_2(s, 0) = 0, \tag{7.59a}$$

and

$$U_2(s,N) \sim u_2(s,0) + N\left(\frac{\partial u_1}{\partial n}\right)_{n=0}, \text{ as } N \to \infty, \tag{7.59b}$$

when we take into account the relation (7.50) for $\partial P_2/\partial s$.

It is important to note that the Prandtl boundary-layer steady problem for $U_1(s, N)$ and $V_1(s, N)$ is well-posed only if we prescribe an initial profile for $U_1(s, N)$ at the point $s = s°$, namely:

$$U_1(s°, N) = U^*(N), \tag{7.60}$$

where $U^*(N)$ is assumed to be known.

Below, in the schematic diagram (Fig.7.2) we demonstrated a "non-interacting" coupling of outer (Euler) and inner (BL) model flows.

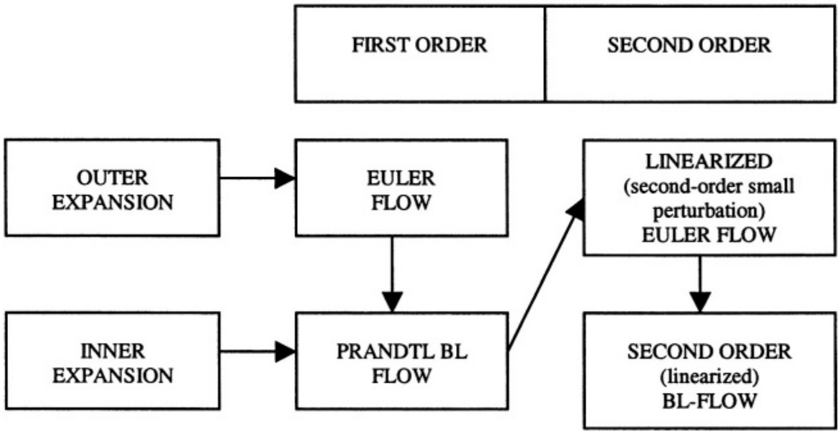


Fig. 7.2. Schematic diagram of the non-interacting coupling of Euler (inviscid) and Prandtl (BL) flows for Re >> 1.

Curiously, because of a coincidence peculiar to the parabola, the above non-interacting, outer - inner, coupling gives no second- order contribution for the semi-infinite plate. For the finite plate, on the other hand, the above procedure gives significant result (see §12.2, in Chapter 12, devoted to the triple-deck model).

7.4. THE ASYMPTOTIC STRUCTURE OF THE UNSTEADY NS-F EQUATIONS AT *RE* >> *1*

As a consequence of the constancy of the pressure in the normal (to the wall) direction within the thickness of the boundary layer (near the wall of a body) the associated *unsteady Prandtl equations break down near the initial time.*

This singularity is *very strong* for an *unsteady compressible* flow, and curiously, it seems that this singular problem has not been carefully considered until now.

In Zeytounian's (1980) short Note, the reader can find a phenomenological approach to this singular problem. It is true that this singular problem, which is related to an *unsteady gas dynamics adjustment problem,* pose a difficult matching problem for the *initialization* of the unsteady compressible boundary-layer equations.

In this §7.4, we give only some preliminary results of an asymptotic analysis of the unsteady NS-F equations, at high Reynolds numbers, *near the initial time* and in the *vicinity of a wall* of a solid body when the velocity vector of the material point on the wall (dependent on time) is taken into account. In particular, this asymptotic analysis makes it possible to derive in a consistent way the equations usually used for the classical compressible Rayleigh problem related to the evaluation of the unsteady flow which arises when an infinite flat plate, submerged in a viscous and thermally conducting, originally quiescent fluid, is impulsively started moving in its own plane with constant velocity. Indeed, these Rayleigh-Howarth (according to Howarth (1951) paper) viscous layer equations are a *local significant degeneracy of full unsteady NS-F equations* only near the wall and close to initial time in the limit of high Reynolds numbers. Below we are concerned exclusively with external aerodynamics, and we assume again that the Boussinesq number $Bo = 0$. The pressure p', density ρ' and temperature T' (the primes ' meaning dimensional quantities), each tends to its constant value at infinity and we make the choice that the velocity $\boldsymbol{u}'$ is zero at infinity. The compressible and heat conducting fluid (a perfect gas) is set into motion by the displacement of some bounded body Ω whose boundary is denoted by Σ. The motion of the boundary, Σ, is characterized by the position vector $\boldsymbol{P}'(t'/\Delta t_c, s'/L_c)$ and in this case we write

$$\frac{\partial \boldsymbol{P}'}{\partial t'} = \boldsymbol{U}'_\Sigma = U_c \Phi\left(\frac{t'}{\Delta t_c}\right)\boldsymbol{u}'_\Sigma\left(\frac{s'}{L_c}\right),$$

and $\boldsymbol{U}'_\Sigma$ is the velocity of the material point $\boldsymbol{P}' \in \Sigma$, while Δt_c is a short transient characteristic time, L_c a characteristic length related to the solid bounded body Ω and U_c the reference velocity for the motion of its boundary Σ.

As a consequence, with dimensionless quantities (as in §2.3 of Chapter 2) we write (without the primes '):

$$\frac{\partial \boldsymbol{P}}{\partial t} = \boldsymbol{U}_\Sigma \equiv \Phi\left(\frac{t}{\omega}\right)\boldsymbol{u}_\Sigma(\boldsymbol{s}), \text{ with } \omega = \frac{\Delta t_c}{t_c} << 1. \tag{7.61}$$

Later, the parameter ω will be related to the gauge for the time, σ, which is introduced in the Section 7.4.1 below.

For instance, if the dimensionless time $\tau = t/\sigma$, is taken into account, then we obtain for the time-function Φ in (7.61): $\Phi((\sigma/\omega)\tau)$ and we note that $\Phi(\infty) \equiv 1$ (steady limit case). When the solid body starts from rest then: $\Phi((t/\omega) \leq 0^-) \equiv 0$.

7.4.1. 'Dominant' NS-F unsteady equations close to initial time and near the wall $\Sigma(t)$ in motion

In the unsteady case the system of dimensionless *Lagrangian* curvilinear coordinates $s = (s_1, s_2)$ on the boundary $\Sigma(t)$ in motion is very appropriate and in this unsteady case from the system of variables (t, x_1, x_2, x_3) we pass to the local dimensionless system of variables (t, s_1, s_2, z), when to avoid unnecessary additional notation for the time in the new system of variables.

Below we denote by $f^L(t, s, z)$, a function in a Lagrangian representation and we work with this Lagrangian representation which is more appropriate in the unsteady case. The motion of the boundary $\Sigma(t)$ is characterized by the velocity vector U_Σ, according to (7.61). In the unsteady case, if again M is the position vector of M in 3D space then, for the point near the boundary Σ, we write (in place of (7.4)):

$$M = P(t, s) + z\, n(t, s), \tag{7.62}$$

Near the boundary $\Sigma(t)$ it is convenient again, in the case of high Reynolds numbers asymptotics, to introduce (in place of z) the new coordinate: $\zeta = z/\delta$, where δ is an arbitrary small gauge which tends to zero as Re tends to infinity. But, in the unsteady case, for the derivation of dominant equations valid for high Reynolds number it is more convenient to introduce the following representation for the velocity vector u:

$$u = U_\Sigma + v + w\, n,\; v \cdot n = 0, \tag{7.63}$$

and in this case, we have again the no-slip condition (7.7) on $\Sigma(t)$.

With the new vertical component of the velocity (in place of w), $w^* = w/\Delta$, where Δ is a second arbitrary gauge, we consider now a new unsteady field

$$V^* = [v^*, w^*, p^*, \rho^*, T^*] = V^*(\tau, s, \zeta)$$

with $\tau = t/\sigma$, and close to initial time this third gauge $\sigma << 1$.

Now, if we assume that the Strouhal number $S \equiv 1$ (in a such case $t_c = L_c/U_c$ and, in (7.61), $\omega = U_c \Delta t_c/L_c$, is the inverse of a Strouhal number based on Δt_c), a simple calculation give in the unsteady case (in place of (7.9a)):

$$S\frac{D}{Dt} \equiv \frac{1}{\sigma}\frac{\partial}{\partial \tau} + \boldsymbol{v}^* \cdot \boldsymbol{D} + \frac{\Delta}{\delta} w^* \frac{\partial}{\partial \zeta} + O(\delta) + O(\Delta), \qquad (7.64a)$$

where $\boldsymbol{D}$ is the gradient operator on the boundary $\boldsymbol{\Sigma(t)}$ defined in the Section 7.1.2.

But, in the unsteady case, in place of (7.9b), we write for a function $f = f^L(t, \boldsymbol{s}, z)$ - in a Lagrangian representation, the following relations:

$$\frac{\partial f^L}{\partial s_\alpha} = \left[\frac{\partial \boldsymbol{P}}{\partial s_\alpha} + z\frac{\partial \boldsymbol{n}}{\partial s_\alpha}\right] \cdot \nabla f^E, \ \alpha = 1,2, \ \frac{\partial f^L}{\partial z} = \boldsymbol{n} \cdot \nabla f^E, \qquad (7.64b)$$

$$\frac{\partial f^L}{\partial t} = \frac{\partial f^E}{\partial t} + \left(\boldsymbol{U}_\Sigma + z\frac{\partial \boldsymbol{n}}{\partial t}\right) \cdot \nabla f^E. \qquad (7.64c)$$

As a consequence, in the unsteady case we work with the following formulae:

$$\frac{\partial f^E}{\partial t} = \frac{\partial f^L}{\partial t} - \left[\boldsymbol{U}_\Sigma + z\frac{\partial \boldsymbol{n}}{\partial t}\right] \cdot \left(\boldsymbol{H} \cdot \boldsymbol{D} f^L + \boldsymbol{n}\frac{\partial f^L}{\partial z}\right), \qquad (7.65a)$$

$$\nabla f^E = \boldsymbol{H} \cdot \boldsymbol{D} f^L + \boldsymbol{n}\frac{\partial f^L}{\partial z}, \qquad (7.65b)$$

where $\boldsymbol{H}$ is given by (7.11) with $\boldsymbol{1}$ the identity operator and $\boldsymbol{O(z^2)}$ an operator which tends to zero, with z, as z^2. Finally, in the unsteady case (in place of (7.12a)) we obtain:

$$\nabla \cdot \boldsymbol{u} = \boldsymbol{D} \cdot \boldsymbol{U}_\Sigma + \boldsymbol{D} \cdot \boldsymbol{v}^* + \frac{\Delta}{\delta}\frac{\partial w^*}{\partial \zeta} + O(\delta) + O(\Delta), \qquad (7.66)$$

Usually, in applications it is assumed that: $\boldsymbol{D} \cdot \boldsymbol{U}_\Sigma = 0$ and below we make use of such a constraint on $\boldsymbol{U}_\Sigma$. In Germain and Guiraud (1966, §2) the reader can find a detailed derivation of formulae equivalent to (7.65a, b).

In conclusion, with the above relations, we derive the ‘dominant’ NS-F unsteady equations (again with $Bo = 0$), valid for high Reynolds number flows near a 3D solid body boundary $\boldsymbol{\Sigma(t)}$, in motion with the velocity $\boldsymbol{U}_\Sigma$, and close to initial time. Below we write only the terms wich are necessary for our asymptotic structure analysis; namely

$$\frac{1}{\sigma}\frac{\partial \rho^*}{\partial \tau}+\boldsymbol{v}^*\cdot \boldsymbol{D}\rho^*+\rho^*\boldsymbol{D}^*\cdot \boldsymbol{v}+\frac{\Delta}{\delta}\left[\rho^*\frac{\partial w^*}{\partial \zeta}+w^*\frac{\partial \rho^*}{\partial \zeta}\right]=0, \quad (7.67a)$$

$$\frac{1}{\sigma}\rho^*\left[\frac{\partial}{\partial \tau}\left(\boldsymbol{U}_\Sigma+\boldsymbol{v}^*\right)\right]_T+\rho^*\left[\left(\boldsymbol{v}^*\cdot \boldsymbol{D}\right)\left(\boldsymbol{U}_\Sigma+\boldsymbol{v}^*\right)\right]_T+\frac{\Delta}{\delta}\rho^* w^*\frac{\partial \boldsymbol{v}^*}{\partial \zeta} \quad (7.67b)$$
$$+\frac{1}{\gamma M^2}\boldsymbol{D}p^*-\frac{\varepsilon^2}{\delta^2}\frac{\partial}{\partial \zeta}\left[\mu^*\left(T^*\right)\frac{\partial \boldsymbol{v}^*}{\partial \zeta}\right]=0,$$

$$\frac{\Delta}{\sigma}\rho^*\frac{\partial w^*}{\partial \tau}+\frac{\Delta^2}{\delta}\rho^* w^*\frac{\partial w^*}{\partial \zeta}+\Delta\left(\boldsymbol{v}^*\cdot \boldsymbol{D}\right)w^* \quad (7.67c)$$
$$+\frac{1}{\delta}\frac{1}{\gamma M^2}\frac{\partial p^*}{\partial \zeta}-\frac{\varepsilon^2\Delta}{\delta^2}\frac{\partial}{\partial \zeta}\left\{\left[\mu_v^*\left(T^*\right)+\frac{4}{3}\mu^*\left(T^*\right)\right]\frac{\partial w^*}{\partial \zeta}\right\}=0,$$

$$\frac{1}{\sigma}\rho^*\frac{\partial T^*}{\partial \tau}+\frac{\Delta}{\delta}\left[\rho^* w^*\frac{\partial T^*}{\partial \zeta}+(\gamma-1)p^*\frac{\partial w^*}{\partial \zeta}\right] \quad (7.67d)$$
$$+\left[\rho^*\boldsymbol{v}^*\cdot \boldsymbol{D}T^*+(\gamma-1)p^*\boldsymbol{D}\cdot \boldsymbol{v}^*\right]$$
$$-\frac{\varepsilon^2\Delta}{\delta^2}(\gamma-1)\gamma M^2\left[\mu_v^*\left(T^*\right)+\frac{4}{3}\mu^*\left(T^*\right)\right]\left|\frac{\partial w^*}{\partial \zeta}\right|^2$$
$$-\frac{\varepsilon^2}{\delta^2}\left\{(\gamma-1)\gamma M^2\mu^*\left(T^*\right)\left|\frac{\partial \boldsymbol{v}^*}{\partial \zeta}\right|^2-\frac{\gamma}{Pr}\frac{\partial}{\partial \zeta}\left[k^*\left(T^*\right)\frac{\partial T^*}{\partial \zeta}\right]\right\}=0$$

with

$$p^* - T^*\rho^* = 0. \quad (7.67e)$$

In equation (7.67b) the notation $[f]_T$ means that we take the projection of the vector f on the tangent to the boundary $\Sigma(t)$. We observe, again, that the 'dominant' unsteady equations (7.67a, b, c, d) are valid only near the boundary $\Sigma(t)$ and close to initial time. From this system of 'dominant' unsteady NS-F equations we can now elucidate the various significant degeneracies of these equations in accordance with the choice of three gauges: σ, Δ, δ and also ε.

But, first, we write the boundary conditions on the wall $\zeta = 0$, namely for the velocity we have the no-slip condition:

$$v^* = w^* = 0 \text{ on } \zeta = 0, \tag{7.68a}$$

while for the temperature we can write the following thermal condition:

$$T^* = 1 + \kappa\, \Xi(s) \text{ on } \zeta = 0, \tag{7.68b}$$

where $\kappa = \Delta T_c / T_c$ is a temperature parameter (see, for instance, (2.57b)).

7.4.2. Prandtl and Rayleigh-Howarth significant equations

According to discussion of the §7.1-§7.3, we know, first, that in the vicinity of the wall it is necessary to derive the Prandtl equations and in this case (as t is fixed in the Prandtl limit considered in Section 7.1.3) we have the following choice for the gauges, in the BL:

$$\sigma = 1, \Delta = \delta \equiv \varepsilon \Rightarrow \tau \equiv t \text{ and } \zeta = Z. \tag{7.69a}$$

But, from the unsteady equations (7.67a, b, c, d) we can also to derive an another set of model equations which are valid near the wall and close to initial time. Indeed, to do so it is necessary that:

$$\sigma = \delta \text{ and } \Delta = 1, \text{ with } \frac{\varepsilon^2}{\delta^2} = \frac{1}{\delta}$$

and in a such case we obtain the following choice for the gauges, in the so-called Rayleigh-Howarth viscous layer (RH-VL):

$$\sigma = \delta \equiv \varepsilon^2, \text{ with } \Delta = 1 \Rightarrow \tau \equiv \theta \text{ and } \zeta = \eta. \tag{7.69b}$$

Now, it is important to observe that only in the RH-VL it is possible to take into account simultaneously the boundary conditions (7.68a, b) and the time-dependent velocity $\boldsymbol{U}_\Sigma = \Phi(\theta)\, \boldsymbol{u}_\Sigma$ in the tangential equation of motion (7.67b). To do so it is necessary to make the choice:

$\omega \equiv \varepsilon^2$ according to (7.61) and (7.69b).

So, on the one hand, we have the Prandtl BL limit:

$$Lim^{Pr} = [\varepsilon \to 0;\ t, s,\ Z = \frac{z}{\varepsilon}, M, Pr, \gamma, \text{fixed} = O(1)], \qquad (7.70a)$$

with

$$v = v^{Pr} + \ldots,\ w = \varepsilon w^{Pr} + \ldots,\ (p, \rho, T) = (p^{Pr}, \rho^{Pr}, T^{Pr}) + \ldots., \qquad (7.70b)$$

and the limiting functions $(\boldsymbol{v}^{Pr}, w^{Pr}, p^{Pr}, \rho^{Pr}, T^{Pr})$ are dependent on BL variables (*t, s* and *Z*). With $\omega \equiv \varepsilon^2$ and $\sigma = 1$, these "Prandtl functions" are solutions of the unsteady BL equations; namely:

$$\frac{\partial \rho^{Pr}}{\partial t} + \boldsymbol{v}^{Pr} \cdot \boldsymbol{D}\rho^{Pr} + \rho^{Pr}\boldsymbol{D} \cdot \boldsymbol{v}^{Pr} + \rho^{Pr}\frac{\partial w^{Pr}}{\partial Z} + w^{Pr}\frac{\partial \rho^{Pr}}{\partial Z} = 0, \qquad (7.71a)$$

$$\rho^{Pr}\frac{\partial \boldsymbol{v}^{Pr}}{\partial t} + \left[\left(\boldsymbol{v}^{Pr} \cdot \boldsymbol{D}\right)\left(\boldsymbol{u}_\Sigma + \boldsymbol{v}^{Pr}\right)\right]_T + \rho^{Pr} w^{Pr}\frac{\partial \boldsymbol{v}^{Pr}}{\partial Z}$$
$$+ \frac{1}{\gamma M^2}\boldsymbol{D}p^{Pr} - \frac{\partial}{\partial Z}\left[\mu\left(T^{Pr}\right)\frac{\partial \boldsymbol{v}^{Pr}}{\partial Z}\right] = 0, \qquad (7.71b)$$

$$\frac{\partial p^{Pr}}{\partial Z} = 0, \qquad (7.71c)$$

$$\rho^{Pr}\frac{\partial T^{Pr}}{\partial t} + \left[\rho^{Pr} w^{Pr}\frac{\partial T^{Pr}}{\partial Z} + (\gamma - 1)p^{Pr}\frac{\partial w^{Pr}}{\partial Z}\right]$$
$$+ \left[\rho^{Pr}\boldsymbol{v}^{Pr} \cdot \boldsymbol{D}T^{Pr} + (\gamma - 1)p^{Pr}\boldsymbol{D} \cdot \boldsymbol{v}^{Pr}\right] - \frac{\gamma}{Pr}\frac{\partial}{\partial Z}\left[k\left(T^{Pr}\right)\frac{\partial T^{Pr}}{\partial Z}\right]$$
$$- (\gamma - 1)\gamma M^2 \mu\left(T^{Pr}\right)\left|\frac{\partial \boldsymbol{v}^{Pr}}{\partial Z}\right|^2 = 0 \qquad (7.71d)$$

with

$$p^{Pr} - \rho^{Pr} T^{Pr} = 0. \tag{7.71e}$$

Then, on the other hand, we consider the Rayleigh-Howarth VL limit

$$Lim^{RH} = [\varepsilon \to 0;\ \theta = \frac{t}{\varepsilon^2},\ s,\ \eta = \frac{z}{\varepsilon^2},\ M,\ Pr,\ \gamma,\ \text{fixed} = O(1)], \tag{7.72a}$$

with

$$v = v^{RH} + \ldots,\ w = w^{RH} + \ldots,\ (p, \rho, T) = (p^{RH}, \rho^{RH}, T^{RH}) + \ldots, \tag{7.72b}$$

and the limit functions $(\boldsymbol{v}^{RH}, w^{RH}, p^{RH}, \rho^{RH}, T^{RH})$ are dependent on VL variables (θ, s and η). These "Rayleigh-Howarth functions" are solutions of the unsteady VL equations; namely:

$$\frac{\partial \rho^{RH}}{\partial \theta} + \rho^{RH} \frac{\partial w^{RH}}{\partial \eta} + w^{RH} \frac{\partial \rho^{RH}}{\partial \eta} = 0, \tag{7.73a}$$

$$\rho^{RH} \left\{ [\boldsymbol{u}_\Sigma]_T \frac{d\Phi(\theta)}{d\theta} + \frac{\partial \boldsymbol{v}^{RH}}{\partial \theta} \right\} + \rho^{RH} w^{RH} \frac{\partial \boldsymbol{v}^{RH}}{\partial \eta} - \frac{\partial}{\partial \eta}\left[\mu(T^{RH}) \frac{\partial \boldsymbol{v}^{RH}}{\partial \eta} \right] = 0, \tag{7.73b}$$

$$\rho^{RH} \frac{\partial w^{RH}}{\partial \theta} + \rho^{RH} w^{RH} \frac{\partial w^{RH}}{\partial \eta} + \frac{1}{\gamma M^2} \frac{\partial p^{RH}}{\partial \eta} - \frac{\partial}{\partial \eta}\left\{ \left[\mu_v(T^{RH}) + \frac{4}{3}\mu(T^{RH}) \right] \frac{\partial w^{RH}}{\partial \eta} \right\} = 0, \tag{7.73c}$$

$$\rho^{RH}\frac{\partial T^{RH}}{\partial\theta}+\rho^{RH}w^{RH}\frac{\partial T^{RH}}{\partial\eta}+(\gamma-1)p^{RH}\frac{\partial w^{RH}}{\partial\eta}$$

$$-(\gamma-1)\gamma M^2\left\{\left[\mu_v\left(T^{RH}\right)+\frac{4}{3}\mu\left(T^{RH}\right)\right]\left|\frac{\partial w^{RH}}{\partial\eta}\right|^2+\mu\left(T^{RH}\right)\left|\frac{\partial v^{RH}}{\partial\eta}\right|^2\right\}$$

$$-\frac{\gamma}{Pr}\frac{\partial}{\partial\eta}\left[k\left(T^{RH}\right)\frac{\partial T^{RH}}{\partial\eta}\right]=0 \qquad (7.73d)$$

with

$$p^{RH}-T^{RH}\rho^{RH}=0\,. \qquad (7.73e)$$

7.4.3. Intermediate matching equations

First, we shall elucidate the behaviour of the above RH-VL equations (7.73a, b, c, d)) for large values of the variables θ and η. Since we have:

$$\varepsilon^2\,\theta=t \text{ and } \varepsilon\,\eta=Z, \text{ as } \varepsilon\to 0,$$

then we can write, with the new unknown variables t^* and z^* fixed and the arbitrary gauge $\beta(\varepsilon)$:

$$\theta=\frac{t^*}{\beta} \text{ and } \eta=\frac{z^*}{\beta^{1/2}},\ 0<\beta=\beta(\varepsilon)\to 0 \text{ with } \varepsilon\to 0, \qquad (7.74a)$$

and

$$lim_{\beta(\varepsilon)\to 0}\left(v^{RH},\ \frac{w^{RH}}{\beta^{1/2}},\ p^{RH},\ \rho^{RH},\ T^{RH}\right)=\left(v^I,\ w^I,\ p^I,\ \rho^I,\ T^I\right), \qquad (7.74b)$$

where all intermediate functions (with a superscript 'I') are dependent on time-space variables $(t^*,\ s,\ z^*)$. We observe that the intermediate gauge $\beta \ll \varepsilon^2$, and in this case, from the RH-VL equations (7.73a, b, c, d) we derive the following intermediate matching model equations for, v^I, w^I, p^I, ρ^I, and T^I:

$$\frac{\partial\rho^I}{\partial t^*}+\frac{\partial}{\partial z^*}\left(\rho^I w^I\right)=0\,,$$

$$\rho^I \frac{\partial v^I}{\partial \theta} + \rho^I w^I \frac{\partial v^I}{\partial z^*} = \frac{\partial}{\partial z^*}\left[\mu(T^I)\frac{\partial v^I}{\partial z^*}\right],$$

$$\frac{\partial p^I}{\partial z^*} = 0, \qquad (7.75)$$

$$\rho^I\left\{\frac{\partial T^I}{\partial t^*} + w^I \frac{\partial T^I}{\partial z^*}\right\} + (\gamma - 1)p^I \frac{\partial w^I}{\partial z^*}$$

$$= \frac{\gamma}{Pr}\frac{\partial}{\partial z^*}\left[k(T^I)\frac{\partial T^I}{\partial z^*}\right] + (\gamma - 1)\gamma M^2 \mu(T^I)\left|\frac{\partial v^I}{\partial z^*}\right|^2$$

with

$$p^I = \rho^I T^I.$$

Obviously, the above limit intermediate matching model equations, (7.75), are those derived when carrying out on the RH-VL equations (7.73a, b, c, d) the approximations of the classical Prandtl boundary-layer theory. In particular, the pressure p^I is independent of z^* and is determined by matching.

Now, we can investigate the behaviour of the Prandtl boundary-layer equations (7.71a, b, c, d), when t and Z both $\to 0$. We write (again with: t^* and z^* fixed):

$$t = \alpha t^*, Z = \alpha^{1/2} z^*, 0 < \alpha = \alpha(\varepsilon) \to 0, \text{ with } \varepsilon \to 0, \qquad (7.76a)$$

and

$$lim_{\alpha(\varepsilon)\to 0}(v^{Pr}, \alpha^{1/2} w^{Pr}, p^{Pr}, \rho^{Pr}, T^{Pr}) = (v^I, w^I, p^I, \rho^I, T^I). \qquad (7.76b)$$

Since, in this case, from the boundary-layer equations (7.71a, b, c, d) we derive again the intermediate matching model equations, (7.75), for the limit functions v^I, w^I, p^I, ρ^I and T^I dependent on the variables (t^*, s, z^*).

Finally, from the compatibility of (7.74a) with (7.76a), we obtain the following relation between the gauges α and β:

$$\alpha\beta = \varepsilon^2, \qquad (7.77)$$

and as a consequence it is necessary to consider the following changes of variables:

$$t = \alpha t^*, z = \varepsilon\alpha^{1/2} z^* \text{ and } w = \frac{\varepsilon}{\alpha^{1/2}} w, \tag{7.78}$$

which characterizes the intermediate matching region. The gauge $\alpha(\varepsilon)$ is, in this intermediate matching region, such that: $\alpha(\varepsilon) \to 0$, as $\varepsilon \to 0$, and the precise localization of this intermediate matching region between RH-VL and BL, is not possible in this stage of the analysis. But, we observe that, on the one hand, when

$$\alpha = O(\varepsilon^2), \text{ then } t^* = \theta \text{ and } z^* = \eta,$$

and we are situated again in the RH-VL region, and on the other hand, if

$$\alpha = O(1), \text{ then } t^* = t \text{ and } z^* = Z,$$

and we are situated in the BL region.

In this intermediate matching region, the limit equations, for the high Reynolds numbers, when $\varepsilon \to 0$, are those which are derived by carrying out on the BL equations the simplifications leading to the RH equations, when we consider the full NS-F unsteady equations. These same equations, valid in this intermediate matching region, are also derived from the RH equations if we carry out on these equations the classical simplifications of BL theory. This was pointed out first by Zeytounian (1980).

7.4.4. The adjustment problem to the unsteady Prandtl BL equations

Indeed, the unsteady adjustment region associated to the BL region (characterized by the variables: t, s and $Z = z/\varepsilon$, where the BL equations (7.71) are valid), is a close-initial-time *inviscid* region, which is governed by the one-dimensional unsteady inviscid gas dynamics equations, since in this case it is necessary to introduce an initial layer associated with the Prandtl boundary layer. Namely, it is necessary to consider for the 'dominant' equations (7.67a, b, c, d) the following choice for the gauges σ, Δ and δ:

$$\sigma = \delta = \varepsilon, \text{ and } \Delta = 1, \tag{7.79}$$

and the following inviscid adjustment limit:

$$lim^{Ad} = [\varepsilon \to 0;\ t_a = \frac{t}{\varepsilon}, s,\ Z = \frac{z}{\varepsilon}, M, Pr, \gamma, \text{fixed} = O(1)], \quad (7.80a)$$

with

$$v = v^{Ad} + \dots,\ w = w^{Ad} + \dots,\ (p, \rho, T) = (p^{Ad}, \rho^{Ad}, T^{Ad}) + \dots. \quad (7.80b)$$

The limit functions $(v^{Ad}, w^{Ad}, p^{Ad}, \rho^{Ad}, T^{Ad})$ are dependent on adjustment variables (t_a, s and Z). These "unsteady inviscid adjustment functions" are the solution of the unsteady inviscid adjustment equations; namely:

$$\rho^{Ad} \frac{\partial v^{Ad}}{\partial t_a} + \rho^{Ad} w^{Ad} \frac{\partial v^{Ad}}{\partial Z} = 0, \quad (7.81a)$$

$$\frac{\partial \rho^{Ad}}{\partial t_a} + \rho^{Ad} \frac{\partial w^{Ad}}{\partial Z} + w^{Ad} \frac{\partial \rho^{Ad}}{\partial Z} = 0,$$

$$\rho^{Ad} \frac{\partial w^{Ad}}{\partial t_a} + \rho^{Ad} w^{Ad} \frac{\partial w^{RH}}{\partial Z} + \frac{1}{\gamma M^2} \frac{\partial \rho^{Ad}}{\partial Z} = 0, \quad (7.81b)$$

$$\rho^{Ad} \frac{\partial T^{Ad}}{\partial t_a} + \rho^{Ad} w^{Ad} \frac{\partial T^{Ad}}{\partial Z} + (\gamma - 1) p^{Ad} \frac{\partial w^{Ad}}{\partial Z} = 0,$$

with

$$p^{Ad} = T^{Ad} \rho^{Ad}.$$

We observe that in the unsteady inviscid adjustment region we derive, in fact, the one-dimensional unsteady inviscid gas dynamics equations, which are the degenerate form of the full compressible inviscid Euler equations, close to initial time and near the wall in a layer with thickness $O(\varepsilon)$ - the thickness of the boundary layer.

But, on the other hand, these same one-dimensional unsteady inviscid gas dynamics equations (7.81), are a limiting form of RH-VL equations (7.73a, b, c, d) when the viscous, heat-conducting and viscous dissipation terms are dropped. Unfortunately, these adjustment equations (7.81a, b) are a degenerate inviscid form of RH-VL equations (7.73a, b, c, d) and as a consequence we cannot write for these inviscid equations (7.81a, b) the conditions on the wall.

Indeed, we can resolve these unsteady adjustment equations (7.81a, b), only when the solution of the RH-VL equations (7.73) is known, by setting (according to matching):

$$lim^{RH}{}_{\theta\to\infty} = lim^{Ad}{}_{t_a\to 0} \text{ and } lim^{RH}{}_{\eta\to\infty} = lim^{Ad}{}_{Z\to 0}. \tag{7.82}$$

Finally it seems more appropriate, for the initialization of the unsteady compressible Prandtl BL equations, to resolve directly the RH-VL equations (7.73a, b, c, d) via the intermediate matching model equations (7.75).

Indeed, the Rayleigh-Howarth viscous layer region, where we have a compressible Rayleigh problem, which was been first seriously considered by Howarth (1951), is necessary because at the level of one-dimensional unsteady inviscid gas dynamics equations (7.8la, b), valid in the adjustment region, we cannot take into account the boundary conditions imposed (on the full NS-F equations) on the wall of the solid body in motion! For this it is necessary to introduce close to initial time $t = 0$ the new time $\theta = t/\varepsilon^2$ and near the wall the new normal coordinate $\eta = z/\varepsilon^2$, which characterize just the RH-VL region. In wiew of the above description it seems that, according to the spirit of the asymptotic modelling:

"The initialization of the unsteady compressible Prandtl BL equations is possible only if the behaviour of the solution of the companion Rayleigh-Howarth unsteady initial-viscous layer equations when θ and η both tend to infinity, match (is asymptotically compatible!) with the behaviour of the unsteady compressible Prandtl BL equations when t and Z tend to zero".

It can therefore be seen that:

"A priori we cannot be completely sure of prescribing the 'right' initial conditions for the unsteady compressible Prandtl equations - we must make sure that the above-mentioned compatibility has been carefully verified!

But, this matching, between the RH-VL and BL equations, is a very difficult problem and actually we do not have a completely clear idea of this matching problem, which makes it possibile to initialize in a consistent manner the unsteady, compressible BL equations.

As a conclusion, for the initialization of the Prandtl unsteady boundary-layer equations, it is necessary to resolve (numerically!) the Rayleigh-Howarth viscous layer equations (7.73a, b, c, d, e), with the boundary conditions:

$$v^{RH} = w^{RH} = 0,\ T^{RH} = 1 + \kappa\, \Xi(s) \text{ on } \eta = 0, \tag{7.83a}$$

and initial conditions

$$\theta = 0\text{: } \rho^{RH} = v^{RH} = w^{RH} = T^{RH} = 0, \tag{7.83b}$$

and also

$$\Phi(0) = 0. \tag{7.83c}$$

Naturally, it is necessary also to consider the matching. It seems that the RH-VL solution matches for $\eta \uparrow \infty$ with the solution of the adjustment equation (7.81a, b) close to initial time. On the other hand, for *large time* this RH-VL solution matches with the Prandtl unsteady boundary-layer solution and gives the initial condition for these BL equations via the intermediate matching region characterized by the equations (7.75).

7.4.5. The case of the low-Mach numbers

When, in the 'dominant' equations (7.67a, b, c, d), with (7.67e), the Mach number M is a small parameter, we can derive another system of model equations near the wall and close to initial time. On close inspection, we see that, for this case, it is necessary to assume the following choice for the gauges:

$$\sigma = \varepsilon^2 M^2,\ \Delta = 1,\ \delta = \varepsilon^2 M,\ \omega = \varepsilon^2 M^2. \tag{7.84}$$

With (7.84) we derive, from (7.67a, b, c, d), a set of approximate equations; namely:

$$\frac{\partial \rho^*}{\partial \tau} + M\frac{\partial}{\partial \zeta}\left(\rho^* w^*\right) + O\left(\varepsilon^2 M^2\right) = 0; \tag{7.85a}$$

$$\rho^*\left\{[\boldsymbol{u}_\Sigma]_T \frac{d\Phi(\tau)}{d\tau} + \frac{\partial \boldsymbol{v}^*}{\partial \tau}\right\} + M\rho^* w^* \frac{\partial \boldsymbol{v}^*}{\partial \zeta} + \left(\frac{\varepsilon^2}{\gamma}\right)\boldsymbol{D}p^*$$
$$-\frac{\partial}{\partial \zeta}\left[\mu^*\left(T^*\right)\frac{\partial \boldsymbol{v}^*}{\partial \zeta}\right] + O\left(\varepsilon^2 M^2\right) = 0; \tag{7.85b}$$

$$\rho^* \frac{\partial w^*}{\partial \tau} + M\rho^* w^* \frac{\partial w^*}{\partial \zeta} + \frac{1}{\gamma M}\frac{\partial p^*}{\partial \zeta} - \frac{\partial}{\partial \zeta}\left\{\left[\frac{4}{3}\mu^*(T^*)\right]\frac{\partial w^*}{\partial \zeta}\right\} + O(\varepsilon^2 M^2) = 0; \tag{7.85c}$$

$$\rho^* \frac{\partial T^*}{\partial \tau} + M\left[\rho^* w^* \frac{\partial T^*}{\partial \zeta} + M(\gamma - 1)p^* \frac{\partial w^*}{\partial \zeta}\right] - (\gamma - 1)\gamma M^2 \mu^*(T^*)\left\{\frac{4}{3}\left|\frac{\partial w^*}{\partial \zeta}\right|^2 + \left|\frac{\partial v^*}{\partial \zeta}\right|^2\right\} - \frac{\gamma}{Pr}\frac{\partial}{\partial \zeta}\left[k^*(T^*)\frac{\partial T^*}{\partial \zeta}\right] + O(\varepsilon^2 M^2) = 0, \tag{7.85d}$$

with

$$p^* = \rho^* T^*, \text{ and when } \mu^*_v(T^*) = 0. \tag{7.85e}$$

If now we assume the following two similarity relations:

$$\varepsilon^2 = \Lambda_0 M^2 \text{ and } \kappa = \Lambda_1 M^2, \tag{7.86}$$

with $\Lambda_0 = O(1)$ and $\Lambda_1 = O(1)$, then we can consider the following asymptotic expansions:

$$v^* = v_1 + ...;\ w^* = M[w_1 + ...];\ p^* = 1 + M^2 [p_1 + ...],$$

$$T^* = 1 + M^2 [T_1 + ...],\ \rho^* = 1 + M^2 [\rho_1 + ...],\ \mu^*(1 + M^2 [T_1 + ...]) = 1 + ...,$$
$$k^*(1 + M^2 [T_1 + ...]) = 1 + ...,$$

and for v_1, w_1, p_1, T_1, and ρ_1 we derive the following system of model equations:

$$\frac{\partial \rho_1}{\partial \tau} + \frac{\partial w_1}{\partial \zeta} = 0, \tag{7.87a}$$

$$[\boldsymbol{u}_{\Sigma}]_T \frac{d\Phi(\tau)}{d\tau} + \frac{\partial \boldsymbol{v}_1}{\partial \tau} = \frac{\partial^2 \boldsymbol{v}_1}{\partial \zeta^2},$$

$$\frac{\partial w_1}{\partial \tau} + \frac{1}{\gamma}\frac{\partial p_1}{\partial \zeta} = \frac{4}{3}\frac{\partial^2 w_1}{\partial \zeta^2}, \tag{7.87b}$$

$$\frac{\partial T_1}{\partial \tau} + M\left[\rho_1 w_1 \frac{\partial T_1}{\partial \zeta} - \frac{\gamma - 1}{\gamma}\frac{\partial p_1}{\partial \tau}\right] = \frac{1}{Pr}\frac{\partial^2 T_1}{\partial \zeta^2} + (\gamma - 1)\left|\frac{\partial \boldsymbol{v}_1}{\partial \zeta}\right|^2,$$

with $p_1 = T_1 + \rho_1$. With (7.87a, b) we recover the model equations considered by Howarth (1951) and Hanin (1960), for the case of the classical compressible Rayleigh problem.

7.4.5a. The compressible Rayleigh problem

In the case of Rayleigh's problem when we consider an infinite flat plate, (submerged in viscous, originally quiescent fluid) which is impulsively started moving in its own plane with constant velocity, then obviously we can write (see (7.61)) for the velocity $\boldsymbol{U}_{\Sigma}$:

$$\boldsymbol{U}_{\Sigma} \cdot \boldsymbol{n} = 0 \text{ and } \boldsymbol{U}_{\Sigma} = \boldsymbol{1}(\tau)\, \boldsymbol{e}, \tag{7.88}$$

where $\boldsymbol{1}(\tau)$ is the unit, Heaviside function $[\boldsymbol{1}(\tau \leq 0^-) = 0$ and $\boldsymbol{1}(\tau \geq 0^+) \equiv 1]$ and $\boldsymbol{e}$ a unit vector tangent to the flat plate in the direction of the constant velocity. Naturally, in this case, in equations (7.87), we have also $\Phi(\tau) \equiv 1$ and for the function, $[\boldsymbol{u}_{\Sigma}]_T + \boldsymbol{v}_1 = u_1\, \boldsymbol{e}$ we obtain, in place of the second (vectorial) equations in (7.87),

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial^2 u_1}{\partial \zeta^2}, \tag{7.89a}$$

with the conditions

$$u_1 = 1 \text{ on } \zeta = 0 \text{ and } u_1 = 0 \text{ for } \tau = 0 \text{ and } \zeta \uparrow \infty. \tag{7.89b}$$

We observe that the condition for $\zeta \uparrow \infty$ is, in fact, a matching condition with an outer inviscid solution which in the framework of Rayleigh's problem is the rest (zero flow). The solution of the above problem (7.89) is

$$u_1 = \frac{2}{\sqrt{\pi}} \int_{\frac{\zeta}{2\sqrt{\tau}}}^{\infty} exp\left(-z^2\right) dz . \tag{7.90}$$

So, it is necessary to solve for the functions: w_1, p_1, T_1, and ρ_1 the following system of three linear equations:

$$\frac{\partial \rho_1}{\partial \tau} + \frac{\partial w_1}{\partial \zeta} = 0 ;$$

$$\frac{\partial w_1}{\partial \tau} + \frac{1}{\gamma}\frac{\partial p_1}{\partial \zeta} = \frac{4}{3}\frac{\partial^2 w_1}{\partial \zeta^2} ; \tag{7.91}$$

$$\frac{\partial T_1}{\partial \tau} + M\left[\rho_1 w_1 \frac{\partial T_1}{\partial \zeta} - \frac{\gamma - 1}{\gamma}\frac{\partial p_1}{\partial \tau}\right] = \frac{1}{Pr}\frac{\partial^2 T_1}{\partial \zeta^2} + \frac{\gamma - 1}{\pi \tau} exp\left(-\frac{\zeta^2}{2\tau}\right),$$

with $p_1 = T_1 + \rho_1$, and conditions

$$\zeta = 0: w_1 = 0 \text{ and } T_1 = \Lambda_1\, \Xi_0, \tag{7.92a}$$

$$w_1 = 0,\ T_1 = 0, \text{ and } \rho_1 = 0, \text{ for } \tau = 0 \text{ and } \zeta \uparrow \infty. \tag{7.92b}$$

This problem (7.91), (7.92a, b) is analysed by Howarth (1951) and in particular, Howarth derives, for $p_1(\tau, \zeta)$, but with $Pr = 3/4$, the following third order equation:

$$\frac{4\gamma}{3}\frac{\partial^3 p_1}{\partial \zeta^2 \partial \tau} + \frac{\partial^2 p_1}{\partial \zeta^2} - \frac{\partial^2 p_1}{\partial \tau^2} = -\frac{\gamma(\gamma - 1)}{\pi \tau}\frac{\partial}{\partial \tau}\left[exp\left(-\frac{\zeta^2}{2\tau}\right)\right]. \tag{7.93}$$

This choice of $Pr = 3/4$ strongly simplifies the equation for p_1. Howarth (1951) and also Hanin (1960) consider he following condition for the temperature T_1:

$$\frac{\partial T_1}{\partial \zeta} = 0 \text{ on } \zeta = 0,$$

and in this case for the hyperbolic equation (7.93) we have the following homogeneous conditions:

$$p_1 = \frac{\partial p_1}{\partial \tau} = 0, \text{ for } \tau = 0\ (\zeta > 0) \text{ and } \zeta \uparrow \infty, \tag{7.94a}$$

$$\frac{\partial p_1}{\partial \zeta} = 0, \text{ on } \zeta = 0\ (\tau > 0). \tag{7.94b}$$

Via a Laplace transform it is possible to obtain a solution of the above problem (7.93), (7.94a, b) for $p_1(\tau, \zeta)$. In the paper by Hanin (1960, §5), the reader can find the asymptotic solution of the equations (7.91), with (7.92a, b) and $Pr = 3/4$, for large time. The outstanding property of the solution is the transition of the initial motion into the large-time flow. According to Hanin (1960, p. 197), in the initial stage the normal velocity and the variation of density are still small, and the dissipation of the main motion raises both the temperature and the pressure equally. The initial variations of the temperature and pressure resemble the temperature variation in the incompressible Rayleigh problem: they decrease from a constant value at the plate to zero in a narrow but expanding region. During the transition period the character of the flow is changed by the appearance of a *wave-like motion.* The pressure variation is transformed into a *progressing pulse,* and a similar normal-velocity pulse builds up. A density disturbance is developed, consisting of a *condensation wave moving forward and a rarefaction 'tail' ending at the plate.* The temperature variation also exhibits the formation of a pulse ahead of the expanding boundary region. The large-time flow can be regarded as a superposition of two distinct flows. The first flow is related closely to the incompressible solution of Rayleigh's problem, but the density variation and the vertival velocity do not vanish however, for the equation of state requires that the sum of density and temperature is equal to zero, from which the vertical velocity is then determined by continuity. The second flow describes, on the other hand, the propagation of an aperiodic, *isentropic,* acoustic wave in a compressible fluid, modified by effects of viscosity and heat conduction. Furthermore these terms asymptotically satisfy the equations of motion with the dissipation term omitted. Dissipation thus determines the wave indirectly by producing the generating disturbances at the plate and this wave produces variations of all the quantities. Finally, these variations form a single propagating pulse and at large times the peak of this pulse move with a speed approaching the speed of sound, the peak amplitude decrease like $1/\tau^{1/4}$ and the thickness of the wave region increases like $(1/\tau^{1/2})$. Below, in the schematic diagram (Fig.7.3) we have represented the various regions connected with the high-Reynolds number asymptotics of the unsteady NS-F equations.

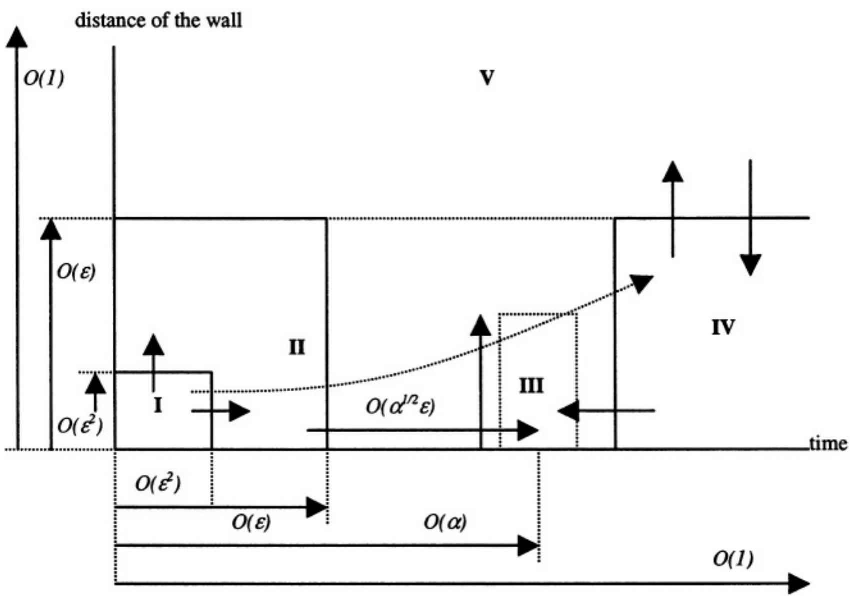


Fig. 7.3. Schematic diagram for the asymptotic structure at Re >> 1 of the unsteady NS-F equations.
I: Rayleigh-Howarth viscous region ($\theta = t/\varepsilon^2$ and $\eta = z/\varepsilon^2$);
II: Unsteady adjustment region ($t_a = t/\varepsilon$ and $Z = z/\varepsilon$);
III: Intermediate matching region ($t^* = t/\alpha$ and $z^* = z/\varepsilon\alpha^{1/2}$);
IV: Prandtl BL region ($t = O(1)$ and $Z = z/\varepsilon$);
V: Euler inviscid region ($t = O(1)$ and $z = O(1)$).

7.5. THE BLASIUS PROBLEM FOR A SLIGHTLY COMPRESSIBLE FLOW

A basic problem in the theory of fluids with a small viscosity (high Reynolds number) is that of steady 2D flow past a solid flat plate placed in a uniform stream. More precisely, the plate is understood to be a half-plane, say: $y > 0$, with $0 < x < \infty$, and the flow is taken to be uniform far upstream, with velocity parallel to the plate and normal to its edge, so that:

$$\boldsymbol{u} = U_0\, \boldsymbol{i}, \text{ for } x \to -\infty \text{ and all } x \text{ and } z, \tag{7.95a}$$

where $U_0 = constant > 0$ and $\boldsymbol{i}$ denotes the unit vector in the direction of increasing x. For $x \to -\infty$ we assume also that the pressure p, the density ρ and the temperature T have the constant prescribed values: p_0, ρ_0 and T_0.

In a compressible viscous and, thermally conducting flow, the temperature T has a prescribed value on the flat plate.

If this value is only a function of x, then the symmetry of the Blasius (1908) (steady) problem indicates that there should be a two dimensional flow, with $\boldsymbol{u} = (u, v, 0)$ such that u, v, p, ρ and T are only function of x and y, and attention will be restricted only to such a flow in this §7.5. The flat plate is understood to be impermeable, but with a variable temperature, so that we assume the following boundary conditions:

$$u = v = 0 \text{ and } T = T_0 + \Delta T_0\, \Theta(x), \text{ on } y = 0, 0 < x < \infty, \tag{7.95b}$$

where $\Delta T_0 > 0$ is a constant, reference wall temperature and $\Theta(x)$ a given function of x. In view of the symmetry of the flow, moreover there is no loss of generality in restricting attention to the half-space $y > 0$.

7.5.1. Basic dimensionless equations and conditions

We start from the NS-F equations for a compressible viscous and thermally conducting fluid and we suppose that the first, μ, and the second, λ, viscosity and heat conductivity, k, coefficients are functions of T only. First, the boundary conditions (7.95a) and (7.95b) make it possible to define a reference length; namely:

$$l_0 = \frac{\nu_0}{U_0}\left(\frac{\Delta T_0}{T_0}\right)^2, \tag{7.96a}$$

where

$$\nu_0 = \frac{\mu(1)}{\rho_0} = const, \text{ and } \mu = \mu\left(\frac{T}{T_0}\right).$$

In this case, with a proper choice of dimensionless quantities, the steady NS-F equations, for a thermally conducting perfect gas ($R = C_p - C_v$; $\gamma = C_p/C_v$), depend, in particular, of the Reynolds number (Re) and the upstream Mach number (M):

$$Re = \frac{U_0 l_0}{\nu_0},\ M = \frac{U_0}{[\gamma R T_0]^{1/2}}. \tag{7.96b}$$

The boundary condition for the temperature, according to (7.95b) in the dimensionless form includes the following parameter

$$\kappa = \frac{\Delta T_0}{T_0}, \tag{7.96c}$$

and from (7.96a, b, c) we have necessarily the following two similarity relations:

$$\kappa = M^2 \text{ and } M^2\, Re^{1/2} = 1. \tag{7.97a, b}$$

These two similarity relations (7.97) are consistent with the asymptotic analysis carry out below. With (7.97a, b) the dimensionless form of the NS-F steady equations for a two-dimensional flow is then:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0; \tag{7.98a}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{1}{\gamma M^2}\frac{\partial p}{\partial x}$$
$$= M^4\left\{\mu(T)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{d\mu}{dT}\right)\left[2\frac{dT}{dx}\frac{\partial u}{\partial x} + \frac{dT}{dy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\right.$$
$$\left. + \frac{\lambda_0}{\mu_0}\frac{d\lambda}{dT}\frac{\partial T}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left[\frac{\lambda_0}{\mu_0}\lambda(T) + \mu(T)\right]\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right\}, \tag{7.98b}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{1}{\gamma M^2}\frac{\partial p}{\partial y}$$

$$=M^4\left\{\mu(T)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\left(\frac{d\mu}{dT}\right)\left[2\frac{dT}{dy}\frac{\partial v}{\partial y}+\frac{dT}{dx}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right.$$

$$\left.+\frac{\lambda_0}{\mu_0}\frac{d\lambda}{dT}\frac{\partial T}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left[\frac{\lambda_0}{\mu_0}\lambda(T)+\mu(T)\right]\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}; \quad (7.98c)$$

$$\rho\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-\frac{\gamma-1}{\gamma}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)$$

$$=M^4\left\{k(T)\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\frac{dk}{dT}\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2\right]\right\}$$

$$+(\gamma-1)M^6\left\{\mu(T)\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2\right.\right.$$

$$\left.\left.+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]+\frac{\lambda_0}{\mu_0}\lambda(T)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2\right\}, \quad (7.98d)$$

with $p = \rho T$, where $Pr = \mu_0 C_p/k_0 \equiv 1$ and $\mu_0 = \mu(1)$, $\lambda_0=\lambda(1)$ and $k_0 = k(1)$.

All the variables are now understood to be in units of their respective scales (we do not change the notations). With the Stokes relation we have: $\lambda_0/\mu_0 = -2/3$, and with (7.97a) the nondimensional form of the boundary conditions (7.95b) is:

$$u = v = 0 \text{ and } T = 1 + M^2\,\Theta(x), \text{ on } y = 0,\ 0 < x < \infty. \quad (7.99a)$$

We have also the following conditions far upstream:

$$u \to 1,\ v \to 0,\ (p,\ \rho,\ T) \to 1, \text{ as } x \to -\infty \text{ for all } y. \quad (7.99b)$$

7.5.2. The limiting Euler equations for $M^2 \to 0$

We consider now the solution of the problem (7.98), (7.99) in the outer limit as:

$$M^2 \to 0, \text{ with } x \text{ and } y \text{ fixed}, \tag{7.100}$$

and we associate to the limit process (7.100) the following outer asymptotic representation:

$$u = 1 + M^2 u_1(x, y) + \ldots;\ v = M^2 v_1(x, y) + \ldots;$$

$$p = 1 + M^4 p_1(x, y) + \ldots; \tag{7.101}$$

$$(\rho, T) = (1, 1) + M^2 (\rho_1(x, y), T_1(x, y)) + \ldots..$$

The far upstream conditions (7.99b) show that: $\rho_1 = T_1$ and as a consequence, from the equations (7.98a, b, c), we derive the following Laplace equation:

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} = 0, \text{ where } u_1 = \frac{\partial \psi_1}{\partial y},\ v_1 = -\frac{\partial \psi_1}{\partial x}, \tag{7.102}$$

and ψ_1 is the associated "incompressible" stream function satisfying the far-upstream condition: $\psi_1(-\infty, y) = 0$. Next, we can determine the function $p_1(x, y)$ from the outer limit equation

$$\frac{\partial u_1}{\partial x} + \frac{1}{\gamma}\frac{\partial p_1}{\partial x} = 0, \text{ with } p_1(-\infty, y) = 0,$$

and as a consequence we obtain

$$p_1(x, y) = -\gamma u_1(x, y) = -\gamma \frac{\partial \psi_1}{\partial y}. \tag{7.103}$$

7.5.3. The limiting Prandtl equation for $M^2 \to 0$

Now we consider the solution of our problem (7.98a, b, c, d), (7.99a, b) in the inner limit as:

$$M^2 \to 0, \text{ with x and } \eta = \frac{y}{M^2} \text{ fixed.} \tag{7.104}$$

We associate to (7.104) the following asymptotic inner representation:

$$u = U_0(x,\ \eta) + M^2\ U_1(x,\ \eta) + ...;\ v = M^2\ [V_0(x,\ \eta) + M^2\ V_1(x,\ \eta) + ...];$$

$$p = 1 + M^4\ P_1(x,\ \eta) + ...; \tag{7.105}$$

$$(\rho, T) = (1,\ 1) + M^2\ (R_1(x,\ \eta),\ T_1(x,\ \eta)\ ) + ...$$

The substitution of (7.105) into the NS-F equations (7.98a, b, c) gives, for the functions $U_0(x,\ \eta)$ and $V_0(x,\ \eta)$, the classical Prandtl boundary layer equations for the associated incompressible Blasius problem:

$$\frac{\partial U_0}{\partial x} + \frac{\partial V_0}{\partial \eta} = 0\,,\ U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial \eta} = \frac{\partial^2 U_0}{\partial \eta^2}. \tag{7.106}$$

From (7.99a) we derive the following boundary conditions for (7.106):

$$U_0 = V_0 = 0, \text{ on } \eta = 0,\ 0 < x < \infty. \tag{7.107}$$

Matching between outer and inner representations, (7.101) and (7.105), gives also the conditions:

$$v_1(x,\ 0) = V_0(x,\ \infty),\ U_0\ (x,\ \infty) = 1. \tag{7.108a, b}$$

It is important to note that the matching condition (7.108a) is a direct consequence of similarity relation (7.97b) and gives, for the incompressible outer first order stream function $\psi_1(x,\ y)$, a solution different to zero (see below (7.111)).

7.5.4. Flow due to displacement thickness

According to (7.108a), it is necessary to have the behaviour of the Blasius problem (7.106), (7.107), when $\eta \to \infty$. For our purpose here the essential result of this incompressible Blasius problem is contained in the expansion of its solution for large η:

$$V_0(x,\infty)=\frac{\beta}{2}(x)^{-1/2}, \tag{7.109}$$

where $\beta = 1.7208$ (see, Meyer (1971, p.106)) and as a consequence we obtain in place of (7.108a), the following boundary condition for the outer stream function $\psi_1(x, y)$:

$$\psi_1(x, 0) = -\beta(x)^{1/2},\ 0 < x < \infty,$$

and for the outer, Euler, stream function $\psi_1(x, y)$ we derive the following classical problem for the flow due to displacement thickness:

$$\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial\psi_1}{\partial y^2}=0, \tag{7.110a}$$

$$\psi_1 = 0,\ x < 0 \text{ and } \psi_1(x, 0) = -\beta(x)^{1/2},\ 0 < x < \infty. \tag{7.110b}$$

The full solution of the outer problem (7.110a, b) is obvious from the viewpoint of complex-variable theory; namely we can write:

$$\psi_1(x, y) = -\beta\, Real\, [(x + iy)^{1/2}], \tag{7.111}$$

and because the displacement speed vanishes at $y = 0$, we have

$$\frac{\partial\psi_1}{\partial y}=0, \text{ on } y = 0. \tag{7.112}$$

7.5.5. Boundary-layer equations with weak compressibility

Substituting the inner expansions (7.105) into the full NS-F equations (7.98a, b, c, d) yields for the functions $U_1(x, \eta)$, $V_1(x, \eta)$, $P_1(x, \eta)$, $T_1(x, \eta)$ and $R_1(x, \eta)$, the following boundary-layer equations which take into account the weak compressibility effects:

$$\frac{\partial U_1}{\partial x}+\frac{\partial V_1}{\partial\eta}+U_0\frac{\partial R_1}{\partial x}+V_0\frac{\partial R_1}{\partial\eta}=0; \tag{7.113a}$$

$$U_0\frac{\partial U_1}{\partial x}+V_0\frac{\partial U_1}{\partial \eta}+U_1\frac{\partial U_0}{\partial x}+V_1\frac{\partial U_0}{\partial \eta}+R_1\left[U_0\frac{\partial U_0}{\partial x}+V_0\frac{\partial U_0}{\partial \eta}\right]$$
$$+\frac{1}{\gamma}\frac{\partial P_1}{\partial x}=\frac{\partial^2 U_1}{\partial \eta^2}+\left(\frac{d\mu}{dT}\right)_{T=1}\left[\frac{\partial^2 U_0}{\partial \eta^2}+\frac{\partial T_1}{\partial \eta}\frac{\partial U_0}{\partial \eta}\right] \qquad (7.113b)$$

$$\frac{\partial P_1}{\partial \eta}=0; \qquad (7.113c)$$

$$U_0\frac{\partial T_1}{\partial x}+V_0\frac{\partial T_1}{\partial \eta}=\frac{\partial^2 T_1}{\partial \eta^2}+(\gamma-1)\left(\frac{\partial U_0}{\partial \eta}\right)^2, \qquad (7.113d)$$

with $R_1 = -T_1$.

From (7.113c), we have $P_1 \equiv P_1(x)$ and matching with the outer flow gives

$$P_1(x)=p_1(x,0)=-\gamma\left(\frac{\partial \psi_1}{\partial y}\right)_{y=0}=0 \Rightarrow P_1(x)=0, \quad (7.114)$$

according to (7.103) and (7.112). For the boundary-layer equations (7.113a, b, d), for the inner functions U_1, V_1, and T_1, we have the following conditions, according to (7.99a) and by matching with the outer expansion:

$$U_1 = V_1 = 0,\ T_1 = \Theta(x), \text{ on } \eta = 0,\ 0 < x < \infty, \qquad (7.115a)$$

$$U_1 = 0 \text{ and } T_1 = 0, \text{ when } \eta \to \infty. \qquad (7.115b)$$

Now, the continuity equation (7.113a) makes it possible to introduce the following BL stream function $\Psi_1(x, \eta)$:

$$U_1-T_1U_0=\frac{\partial \Psi_1}{\partial \eta};\ V_1-T_1V_0=-\frac{\partial \Psi_1}{\partial x}, \qquad (7.116a)$$

since: with $\partial U_0/\partial x + \partial V_0/\partial \eta = 0$ we can also introduce an incompressible (Blasius) BL stream function $\Psi_0(x, \eta)$ with:

$$U_0 = \frac{\partial \Psi_0}{\partial \eta} \text{ and } V_0 = -\frac{\partial \Psi_0}{\partial x}. \qquad (7.116b)$$

7.5.6. Self-similar solution

If we assume that, in the boundary condition (7.115a),

$$\Theta(x) \equiv 1 \text{ for } 0 < x < \infty,$$

then a self-similar solution exists for the flat plate Blasius problem because the problem has a certain group property that permits its reduction to an ordinary differential equation. We choose the following form:

$$\Psi_0(x, \eta) = (x)^{1/2} f(Y), \text{ and } \Psi_1(x, \eta) = (x)^{1/2} F(Y) \text{ with } Y = \frac{\eta}{x^{1/2}}. \quad (7.117a, b)$$

Substituting (7.117a) into the boundary layer problem, (7.106), (7.107), (7.108b) gives:

$$2\frac{d^3 f}{dY^3} + f\frac{d^2 f}{dY^2} = 0, \; f(0) = \left(\frac{df}{dY}\right)_{Y=0} = 0, \; \left(\frac{df}{dY}\right)_{Y=\infty} = 1, \quad (7.118)$$

and this is the classical Blasius's problem.

Reintroducing the original dimensional variables at this point shows that the length l_0 (see (7.96a) disappears from f, F and Y in accordance with the fact that it is irrelevant for the boundary-layer on a flat plate with $\Theta(x) \equiv 1$, for $0 < x < \infty$, and this is an another way of motivating the group transformation (7.117).

For our purposes the essential results of the numerical solution are contained in the expansion for small Y:

$$f(Y) = \frac{1}{2}\alpha Y^2 + O(Y^5), \text{ where } \alpha = \left(\frac{d^2 f}{dY^2}\right)_{Y=0} = 0.33206. \quad (7.119)$$

If now we introduce the function: $G(Y) = -T_1(x, \eta)$, then we obtain for $G(y)$ the following classical thermal problem, from the equation (7.113d), the condition (7.115a) for T_1 with $\Theta(x) \equiv 1$, when $0 < x < \infty$, and (7.115b) for T_1, namely:

$$2\frac{d^2G}{dY^2}+f\frac{dG}{dY}=2(\gamma-1)\left[\frac{df}{dY}\right]^2,\ G(0)=-1,\ G(\infty)=0, \qquad (7.120)$$

and the full solution of (7.120) is obvious:

$$G(Y)=-1+\frac{1}{2}(3-\gamma)\frac{df}{dY}+\frac{1}{2}(\gamma-1)\left[\frac{df}{dY}\right]^2. \qquad (7.121)$$

Finally, substituting (7.116a, b), (7.117a, b) and the function $G(Y) = -T_1 = R_1$, into the equation (7.113b) and conditions (7.115a, b) for U_1 and V_1, we obtain for the function $F(Y)$ the following linear but inhomogeneous equation:

$$2\frac{d^3F}{dY^3}+f\frac{d^2F}{dY^2}+\left(\frac{d^2f}{dY^2}\right)F=2\left[\frac{df}{dY}\frac{d^2G}{dY^2}+2\frac{d^2f}{dY^2}\frac{dG}{dY}\right]$$
$$+f\frac{df}{dY}\frac{dG}{dY}-2\left(\frac{d\mu}{dT}\right)_{T=1}\left[\frac{d^3f}{dY^3}-\frac{dG}{dY}\frac{d^2f}{dY^2}\right] \qquad (7.122)$$

with the boundary conditions:

$$F(0)=0,\ \left(\frac{dF}{dY}\right)_{Y=0}=0 \text{ and } \left(\frac{dF}{dY}\right)_{Y=\infty}=0, \qquad (7.123)$$

where $f(Y)$ and $G(Y)$ are known functions from the classical Blasius problem (7.118) and the solution (7.121) for $G(Y)$.

In this stage the two-term inner expansion for the longitudinal velocity component u is:

$$u=\frac{df}{dY}+M^2\left[\frac{dF}{dY}+\frac{df}{dY}G\right]+O(M^4), \qquad (7.124)$$

and the skin friction coefficient is:

$$C_f = 0.6641(Re_x)^{-1/2}$$
$$+\frac{M^2}{(Re_x)^{1/2}}\left\{0.6641+2\left(\frac{d^2F}{dY_2}\right)_{Y=0}+0.6641\left(\frac{d\mu}{dT}\right)_{T=1}\right\}+\ldots, \quad (7.125)$$

where $Re_x = (U_0/\nu_0)x$, is the local Reynolds number, based on the dimensional distance x from leading edge of the flat plate. The second term in the skin friction coefficient (7.125) is a direct result of the slight effect of compressibility.

According to Godts and Zeytounian (1990) for: $\mu(T) = T^{\omega}$,when $0 < \omega < 1$, we obtain for $(d^2F/dY^2)_{Y=0}$ the value: -0.403 if $\omega = 0$. For the other values of $(d^2F/dY^2)_{Y=0}$ as function of ω, see Godts and Zeytounian (1990, p.69).

In conclusion, we note two fundamental points of our analysis.The first concerns the hyposonic viscous similarity relation:

$$\frac{1}{Re} = M^4$$

between the small Mach number and the large Reynolds number. The second is connected with the skin friction coefficient C_f in which we have a new term as a direct consequence of the slight effect of compressibility coupled with a small effect of viscosity. Naturally, it would be possible to generalize the above asymptotic analysis for an arbitrary body, but then a full numerical computation would be required to estimate the influence of a weak compressibility on the coefficient C_f. Such a generalization would be interesting for various low-speed, viscous fluid flow phenomena.

7.6. FLOW WITH VARIABLE VISCOSITY: A THREE-LAYER MODEL

We consider below a steady, two-dimensional flow past a realistic body of non-vanishing thickness. For definiteness we may envisage an aerofoil spanning the test section of a wind tunnel with plane walls, so that the flow would be uniform in the absence of the aerofoil. In any case the body is assumed to be solid, with an impermeable surface. Its shape will define a fixed reference length ($L°$), and hence a Reynolds number (Re), so that a non-dimensional formulation is possible from the start. Let $\boldsymbol{u}$ be the dimensionless velocity and p the dimensionless pressure. A rational approach is to assume that, for a fluid of small viscosity, the flow is one differing appreciably from that of an ideal (perfect) fluid only in the vicinity of the body surface. For that reason it is convenient to use curvilinear, orthogonal coordinates, usually denoted by s and n, such that the body surface is the line $n = 0$. A regular system of such coordinates certainly exists, in any case in a sufficiently small neighbourhood of the body surface, provided that the body surface has no corner. It is convenient to measure s along the body surface from the stagnation point that must be anticipated near the rounded nose of a realistic body, and to begin with it is desirable to exclude a neighbourhood of the origin point, $s = n = 0$, from consideration. It is similarly desirable to exclude a neighbourhood of the tail of the body. Since n measures normal distance from the body surface, we write for the velocity vector:

$$\boldsymbol{u} = u\,\boldsymbol{\tau} + v\,\boldsymbol{\nu},\ \nabla = h(s)\frac{\partial}{\partial s}\boldsymbol{\tau} + \frac{\partial}{\partial n}\boldsymbol{\nu}, \tag{7.126}$$

with: $h(s) = [1 + K(s)\,n]^{-1}$, where $K(s)$ is the curvature of the body surface.

We assume that $K(s)$ and its first derivative $dK(s)/ds$ are bounded or, in any case:

$$\frac{K(s)}{Re^{1/2}} \to 0 \text{ and } Re^{-1}\frac{dK(s)}{ds} \to 0 \text{ as } \frac{1}{Re} \equiv \varepsilon^2 \to 0.$$

In (7.126), obviously, $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are two unit vectors, tangent and normal (directed towards the fluid) to the wall of the body, respectively.

We start from the Navier equations (for a viscous and incompressible fluid flow) in dimensionless form: Here, these Navier equations have the following form:

$$\nabla \cdot \boldsymbol{u} = 0, \tag{7.127a}$$

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \varepsilon^2\left[\mu(n)\nabla^2\boldsymbol{u} + \left(\frac{d\mu}{dn}\right)\left(\frac{\partial \boldsymbol{u}}{\partial n} + \nabla v\right)\right], \tag{7.127b}$$

where (in dimensionless form) we have for the variable dynamic viscosity coefficient, $\mu(n)$, the following relationship:

$$\mu(n) = 1 + \mu^*\left(\frac{n}{\Delta(\varepsilon)}\right), \text{ with } \mu^*(\infty) = 0,$$

and $\Delta(\varepsilon)$ is a gauge such that: $\Delta(\varepsilon) << \varepsilon$, and 'in (7.127b)) $v = \boldsymbol{u} \cdot \boldsymbol{v}$.

We shall see that $\Delta(\varepsilon) = \varepsilon^2$ and the thickness of the lower-viscous layer is $O(1/Re)$. The case when: $\Delta(\varepsilon) >> \varepsilon$, is more subtle and requires the application of an homogenization technique for the microscopic description related to the variation of $\mu^*(n/\Delta(\varepsilon))$. We note that, with the chosen scaling:

$$\mu^\circ \mu(n) = \mu^\circ\left[1 + \mu^*\left(\frac{L^\circ n}{l^\circ}\right)\right],$$

where μ° is the characteristic constant value of the dynamic viscosity coefficient, l° is a characteristic length for the microscopic variation of the dynamic viscosity coefficient and $n' = L^\circ n$ is a dimensional coordinate normal to the body surface simulated by $n' = 0$. If L° is the reference length associated with the shape of the body, then:

$$\Delta(\varepsilon) = \frac{l^\circ}{L^\circ} << \varepsilon << 1.$$

7.6.1. The associated three limiting processes

7.6.1a. Outer Euler limit

The *first* (outer) limiting process is the usual Euler in viscid limit of the exact solution of (7.127a, b), with the above assumed structure for $\mu(n)$ and no-slip boundary condition $\boldsymbol{u} = 0$ on $n = 0$; namely we consider:

$$\varepsilon \to 0, \text{ for fixed } s \text{ and } n. \tag{7.128a}$$

In this case we search for an asymptotic representation of the flow (u, v, p):

$$u = u_0 + \varepsilon u_1 + \ldots, v = v_0 + \varepsilon v_1 + \ldots, p = p_0 + \varepsilon p_1 + \ldots. \tag{7.128b}$$

As usual, we get to leading-order the classical incompressible 2D Euler equations for (u_0, v_0, p_0), while to next order for (u_1, v_1, p_1) we have the linearized incompressible 2D Euler perturbation equations relative to the preceding Euler equations (see for instance §7.3.1).

7.6.1b. Intermediate Prandtl limit

The *second* (intermediate) limiting process is the Prandtl (BL) viscous limit of the exact solution as a function of s and $N = n/\varepsilon$, when:

$$\varepsilon \to 0, \text{ for fixed } s \text{ and } N. \tag{7.129a}$$

In this case, instead of (7.128b), we write

$$u = U_0 + \varepsilon U_1 + \ldots,\ v = \varepsilon V_1 + \varepsilon^2 V_2 + \ldots.,\ p = P_0 + \varepsilon P_1 + \ldots. \tag{7.129b}$$

For (U_0, V_1, P_0) we obtain the classical Prandtl boundary-layer equations:

$$\frac{\partial U_0}{\partial s} + \frac{\partial V_1}{\partial N} = 0;$$

$$U_0 \frac{\partial U_0}{\partial s} + V_1 \frac{\partial U_0}{\partial N} + \frac{\partial P_0}{\partial s} = \frac{\partial^2 U_0}{\partial N^2}; \tag{7.130}$$

$$\frac{\partial P_0}{\partial N} = 0,$$

and the following relations are derived:

$$v_0 \to 0 \text{ as } n \to 0,\ U_0 \to u_0(s, 0) = u_{e0}(s), \text{ as } N \to \infty, \tag{7.131a}$$

$$P_0 = p_0(s, 0) = p_{e0}(s), \text{ and } \frac{dp_{e0}(s)}{ds} + u_{e0}(s)\frac{du_{e0}(s)}{ds} = 0; \tag{7.131b}$$

$$v_1(s,0) = \int_0^\infty \left[\frac{du_{e0}(s)}{ds} - \frac{\partial U_0}{\partial s}\right] dN, \tag{7.131d}$$

after the matching of (7.128b) with (7.129b). The first relation of (7.131a), is directly related to the Euler inviscid/non-viscous equations and, in fact, makes it possible to write for these steady Euler equations the stationary slip condition on the wall $n = 0$.

For (U_1, V_2, P_1) we obtain the boundary-layer equations (7.132a, b) of the second approximation, where we take into account the curvature effects:

$$\frac{\partial U_1}{\partial s}+\frac{\partial V_2}{\partial N}=-K(s)\frac{\partial}{\partial N}(NV_1); \tag{7.132a}$$

$$\frac{\partial}{\partial s}(U_0U_1)+V_2\frac{\partial U_0}{\partial N}+V_1\frac{\partial U_1}{\partial N}+\frac{\partial P_1}{\partial s}-\frac{\partial^2 U_1}{\partial N^2}$$
$$=K(s)\left[NU_0\frac{\partial U_0}{\partial s}-U_0V_1+N\frac{dp_{e0}(s)}{ds}+\frac{\partial U_0}{\partial N}\right] \tag{7.132b}$$

and matching gives

$$U_1 \to u_1(s, 0) = u_{e1}(s), \text{ as } N \to \infty, \tag{7.133a}$$

and also

$$P_1 = p_1(s,0)+K(s)\int_N^\infty [U_0]^2\, dN\,, \tag{7.133b}$$

and, in fact, with (7.133b) we have two equations, (7.132a, b), for U_1, and V_2.

7.6.1c. Lower-viscous limit

For the time being we cannot take into account the influence of the microscopic coefficient $\mu^*(n/\Delta(\varepsilon))$ and to do so we envisage a tentative third (inner) lower-viscous layer (L-VL) limiting process; namely:

$$\varepsilon \to 0, \text{ for fixed } s \text{ and } \eta=\frac{n}{\Delta(\varepsilon)}, \tag{7.134a}$$

with the following local asymptotic representation of the flow:

$$u = \varepsilon^{\alpha} U^*_0(s, \eta) + \ldots,\ v = \varepsilon^{\beta} V^*_0(s, \eta) + \ldots, \tag{7.134b}$$

$$p = P^*_0(s, \eta) + \varepsilon P^*_1(s, \eta) + \ldots \tag{7.134c}$$

where the positive constants α and β are for the moment arbitrary. If we assume that:

$$\Delta(\varepsilon) = \varepsilon^{\lambda},\ \lambda > 1, \text{ then for } \lambda = \beta - \alpha,$$

we obtain for (U^*_0, V^*_0, P^*_0) the following system of equations in the lower-viscous layer:

$$\frac{\partial U^*_0}{\partial s} + \frac{\partial V^*_0}{\partial \eta} = 0\,,\ P^*_0 = p_{e0}(s); \tag{7.135a}$$

$$\frac{\partial}{\partial \eta}\left[(1+\mu(\eta))\frac{\partial U^*_0}{\partial \eta}\right] = 0\,, \tag{7.135b}$$

and now we can add the no-slip condition for the above equations (7.135a, b), namely:

$$U^*_0 = V^*_0 = 0 \text{ on } \eta = 0. \tag{7.136}$$

7.6.2. The interaction between the BL and the L-VL

The solution of the L-VL problem (7.135a, b), (7.136), is easy, and in particular we get

$$U^*_0(s,\eta) = A^\circ(s)\int_0^\eta [1+\mu(\tau)]^{-1} d\tau. \tag{7.137}$$

Now it is necessary to elucidate the behaviour of (7.137) as $\eta \to \infty$ and also to determine the arbitrary function $A^\circ(s)$. Through a straightforward argument we obtain

$$U^*_0(s,\ \eta) \sim A^\circ(s)\ [\eta + I^\circ + \ldots],\ \text{as } \eta \to \infty, \tag{7.138}$$

where

$$I^\circ = \int_0^\infty \left\{-1 + [1+\mu(\tau)]^{-1}\right\} d\tau < \infty\,, \tag{7.139}$$

and we see that this imposes a constraint on the function $\mu(\eta)$. But according to (7.129b), in the BL we have also the following behaviour for the tangential component of the velocity u:

$$u = U_0(s,0) + N\left(\frac{\partial U_0}{\partial N}\right)_{N=0} + \varepsilon U_1(s,0) + \dots, \text{ as } N \to 0.$$

Therefore, taking into account (7.134b) and the relation: $N = \varepsilon^{\lambda-1}\,\eta$, we verify that the matching between (7.134b, c) and (7.129b) is possible, according to (7.138) and (7.135a), if:

$$\alpha = 1,\ \lambda = \alpha + 1 = 2,\ \beta = 3, \tag{7.140}$$

and in this case we obtain the following three relations:

$$U_0(s,\,0) = 0,\ A^\circ(s) = \left(\frac{\partial U_0}{\partial N}\right)_{N=0},$$

$$U_1(s,\,0) = A^\circ(s)\,I^\circ, \tag{7.141}$$

$$V_1(s,\,0) = V_2(s,\,0) = 0.$$

From (7.141) we see that the classical laminar BL problem for $(U_0,\ V_1,\ P_0)$, is not affected by the appearance of the L-VL. This leading-order L-VL is active at the level of second-order BL equations (7.132a, b), with (7.133a, b), for $(U_1,\ V_2,\ P_1)$, in such a way that the boundary conditions in $N = 0$ are:

$$U_1(s,0) = I^\circ\left(\frac{\partial U_0}{\partial N}\right)_{N=0},\ V_2(s,\,0) = 0. \tag{7.142}$$

As $\Delta(\varepsilon) = \varepsilon^2 = (1/Re)$, the leading-order L-VL, governed by the equations (7.135a, b) with the boundary conditions (7.136), is a thin viscous layer (within the BL) with a thickness of $O(1/Re)$.

But for the time being we don't know if the second-order boundary-layer problem, (7.132a, b), (7.133a, b), (7.142) with (7.139) is well posed or not.

It seems that the rigorous proof of uniqueness of a solution in $C^\infty(0,\infty)$ is not easy! In the paper by Godts and Zeytounian (1991) the reader can find an application of the above three-layer asymptotic model to Blasius's basic problem of the theory of incompressible fluids of small viscosity. For this Blasius problem the expression obtained, in Godts and Zeytounian (1991), for the skin friction coefficient C_f shows that the classical Blasius value is multiplied by a positive term, directly linked to the variability of the dynamic viscosity coefficient. Finally, we note that, as a consequence of the resolution of the Blasius problem, it seems that the *uniqueness* of a solution of the linear second-order boundary-layer problem, (7.132a, b), (7.133a, b), (7.142), with (7.139), for (U_1, V_2, P_1), *is guaranteed* if we assume that the latter satisfies also the following condition at infinity:

$$lim_{N\to\infty}\frac{\partial U_1}{\partial N}=lim_{n\to 0}\frac{\partial u_0}{\partial n}, \text{ with } N=\frac{n}{\varepsilon}, \tag{7.143}$$

which follows from the matching between (7.129b) and (7.128b) for the vertical derivative $\partial u/\partial n$.

7.7. TAYLOR SHOCK LAYER

Usually, in the classical theory of the schock waves, it is assumed simply that all flow quantities are independent of time and depend only on the x-coordinate, and that the velocity vectors are parallel to the x direction. According to the theory of nonviscous compressible fluids, the only one - dimensional flow phenomena of this type are contact discontinuities and discontinuous shock waves. It is easily seen that viscosity and heat conductivity eventually completely obliterate a contact discontinuity, so that the corresponding stationary solution is that of uniform flow. On the other hand, the effect of viscosity and heat conductivity on a shock wave is to replace the shock discontinuity by a continuous transition between initial and final states.

The region of continuous transition is called a *shock layer*; it is in many respects analogous to a boundary layer. Note that the thickness of a boundary layer is of order $(\mu)^{1/2}$ (where μ is the dynamic viscosity) - but the thickness of a shock layer, on the other hand, is of the order $\mu^* = (4/3)\,\mu$.

For a first pertinent discussion of the shock thickness, the reader is invited to consult the Lagerstrom survey paper (1964; p. 177) and also the book by Meyer (1971; p. 170). Under normal conditions the greater part of the transition takes place in a *very narrow region*; however, strictly speaking, the shock layer is infinitely wide and the conditions ahead and behind the shock will have to be replaced by conditions at upstream and downstream infinity respectively.

With this reservation the same “jump conditions” should result whether viscosity and heat conduction are considered or not. In François (1981, §V.5) an asymptotic solution, for the one-dimensional NS-F equations, is derived when:

$$M_+ = \frac{U_+}{(\gamma R T_+)^{1/2}} \to 1 \qquad (7.144a)$$

and

$$Re_+ = \frac{L^\circ U_+}{\left(\frac{\mu^*}{\rho_+}\right)} \to +\infty, \qquad (7.144b)$$

where L° denotes here an arbitrary characteristic (macroscopic) length scale of the fluid flow problem. For a weak shock, when M_+ (upstream Mach number) is close to 1, we can introduce a small parameter, namely: $\alpha = (M_+)^2 - 1$, and in a such case the shock thickness is of the order of:

$$\delta = \frac{L^\circ}{\alpha Re_+} = \frac{\lambda^*}{M_+\left((M_+)^2 - 1\right)} \qquad (7.145)$$

where $\lambda^* = L^\circ M_+/Re_+$ is the mean free path, and in this case $\delta >> \lambda^*$.

Therefore this asymptotic theory furnishes an adequate basis for a theory of weak shocks (we note that δ/λ^* is independent of the arbitrary constant macroscopic length scale L°).

Concerning the systematic application of the asymptotic method to the structure of the shock layer, we mention the fundamental paper by Germain and Guiraud (1966) and in Germain (1972, Section IV) the reader can find a discussion of this asymptotic analysis. Here we give only some information concerning this Germain-Guiraud asymptotic theory.

For simplicity, the analysis is restricted to the case of classical gas dynamics for a flow of a viscous and heat conducting fluid. The limit of the flow when the dissipative effects, characterized by a small parameter ε - the inverse of a Reynolds, *Re,* number - tend toward zero is analysed. The first question to be answered is the following: assuming that a solution of NS-F equations has *very high* gradients inside a "layer", such that when $\varepsilon \to 0$, one gets a shock at a surface *G*, *G* being inside the layer, *how does one find the correct asymptotic expansion for small* ε *for such a solution?* As the case $\varepsilon \neq 0$ is a *singular* perturbation problem of the case $\varepsilon = 0$, it cannot be expected to have an asymptotic expansion uniformly valid everywhere. Accordingly, two different types of asymptotic expansions have to be introduced in order to represent what happens in the "layer" and in the neighbourhood of this layer. The first one - the outer expansion - will be valid far from *G*; far with respect to a distance of the order of some power of ε. The second - the inner expansion - will be valid inside the layer. Thus we have to obtain these two expansions and match them in such a way that they represent the same solution of the NS-F equations. Note that Germain and Guiraud (1966) obtain the complete expansions to any order. The outer expansion is obtained by starting with fixed geometrical coordinates and making $\varepsilon \to 0$, while the inner expansion requires the introduction of new (distorted) coordinates based on a normal coordinate counted from *G*, which is of the order of the dissipation length, say ε. Now, there is a second question to be answered, connected with a further use of the results in order to apply to the solution to some given global problem (in fact, the complete determination depends on the global problem and the authors consider only the local problem).

If we are interested only in the outer expansion, how must the expansions on both sides of *G* be related in order to be expansions of the same solution? These relations are, in fact, the shock conditions of the problem. Classical jump relations give the shock conditions to zeroth order, and classical shock structure also gives the inner expansion to zeroth order. The usual way of studying such a singular perturbation problem is to proceed by successive computation of the terms of the two expansions and to write at each step the relevant matching conditions. But Germain and Guiraud use a more direct method by assuming first the analytic properties of the different dependent variables in the neighbourhood of *G* and showing how to define these functions by using the NS-F equations. The validity of this method rest upon the check at each step of the consistency of the formulae which are found. Indeed, in the Germain and Guiraud theory (1966), *two* moments are important:

First, the introduction of the *functions NS,* and *then,* the concept of the "*star*" for these functions. In the shock conditions [up to order *1*; see, in Germain (1972), pp. 191 and 192], the "star terms" are important in a consistent theory (these terms take into account the influence of the thickness of the shock). The reader may find the details of this theory in Germain and Guiraud (1966).

It is necessary, however, to close with the remark that fluid dynamics does not furnish such an adequate basis for a theory of shocks as the foregoing might suggest! For fairly strong shocks (as in hypersonic flow), the shock thickness δ may be comparable to the mean free path λ^*; in this case the thickness of the shock layer depends very much on the assumed viscosity - temperature relation! Naturally, in this case the NS-F equations are inadequate for the description of a shock layer structure and it is necessary to work with the Boltzmann equation of the kinetic theory of gases. Concerning the use of the Boltzmann equation for fluid dynamics problems, see the recent pertinent review paper by Bellomo, LeTallec and Pertham (1995).

In the paper by Crighton (1986), the author notes that: "G. I. Taylor's solution in 1910 for the interior structure of a weak shock wave is, *with appropriate generalization,* an essential component of modern weak-shock theory".

In this paper the principal aim of Crighton is a study of the behaviour of a weakly nonlinear wave pattern, containing a weak-shock wave, after propagation over a very large distance - here the Taylor structure, valid for moderate range, loses its relevance in various ways (see the Introduction of the Crighton (1986, pp. 625-628) paper). Crighton (1986, p. 628) starts from the equation:

$$\frac{\partial u}{\partial t}+\left[a_0+\frac{1}{2}(\gamma+1)u\right]\frac{\partial u}{\partial x}+\frac{1}{2}a_0 u\frac{d}{dx}(ln\,A(x))=\frac{\delta}{2}\frac{\partial^2 u}{\partial x^2}, \qquad (7.146)$$

for the velocity fluctuation $u(t, x)$, where a_0 is a (source) sound speed, $A(x)$ the ray-tube or wave-front area. If $1/\omega$ is the typical time scale of the imposed motion, then $k_0 = a_0/\omega$ and U_0/a_0 is a (source) Mach number while: $\delta(k_0/U_0)$ is an inverse Reynolds number. The ratios: U_0/a_0, $\delta(k_0/U_0)$ and $1/k_0L$ are small, with the length $L = 1/|d/dx(lnA(x))|$. The equation (7.146) holds uniformly, as the small parameters vanish independently, to time $O(1/\eta,\ 1/\omega)$, where η is the smallest of the three above small parameters.

Concerning the Burgers equation, see in Chapter 8 the §8.1. For plane flow, when $A(x) \equiv constant$, (7.146) is the classical Burgers' equation which is linearized to the diffusion equation

$$\frac{\partial U}{\partial t}=\frac{\delta}{2}\frac{\partial^2 U}{\partial X^2}, \qquad (7.147a)$$

by the Hopf-Cole transformation (Bäcklund transformation)

$$u=-\frac{2\delta}{(\gamma+1)\dfrac{\partial\, lnU}{\partial X}}, \qquad (7.147b)$$

where $X = x - a_0\, t$.

But, unfortunately, when $A(x) \neq constant$ a such transformation do not exists and asymptotic and numerical methods are therefore the only tools currently available.

7.7.1. A simple description of the structure of the Taylor shock layer

In the viscous heat-conducting case the mass relation is the same but the other two conservation laws involve additional terms due to viscous stresses and heat flux. Within the framework of the one-dimensional steady NS-F equations these additional terms are proportional to the derivatives *du/dx* and *dT/dx* respectively. However, if we let x_0 tend to infinity these derivatives are "expected" to vanish. Hence the relations for change through a shock wave are the same in the viscous and in the nonviscous case. But in the former case the shock wave is considered to start at $-\infty$ and end at $+\infty$, in the latter case it is *infinitely thin.* In order to distinguish between the two cases, *the continuous shock transition is referred to as a shock layer.* Here, we consider this shock layer as a "*Taylor shock layer*" and give a simple description of its structure, and we start from the following stationary one-dimensional NS-F equations:

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = \frac{d}{dx}\left(\mu^* \frac{du}{dx}\right); \tag{7.148a}$$

$$\frac{d}{dx}(\rho u) = 0; \tag{7.148b}$$

$$\rho u \frac{dh}{dx} - u \frac{dp}{dx} = \frac{d}{dx}\left[\frac{\mu^*}{Pr^*} \frac{dh}{dx}\right] + \mu^* \left(\frac{du}{dx}\right)^2, \tag{7.148c}$$

where, according to Stokes relation: $\lambda + (2/3)\mu = 0$, we have: $\mu^* = (4/3)\,\mu$ and Pr^* is formed relative to μ^*. Now, if the term $u\, dp/dx$ in (7.148c) is eliminated with the aid of (7.148a), the shock layer equations may be integrated once with respect to x, giving:

$$mu + p - \mu^* \frac{du}{dx} = mU_+ + p_+ = mU_- + p_-; \tag{7.149a}$$

$$\rho u = m = \rho_+ U_+ = \rho_- U_-; \tag{7.149b}$$

$$m\left[\frac{1}{2}u^2+h\right]-\frac{\mu^*}{Pr^*}\frac{dh}{dx}-\mu^*\frac{d}{dx}\left[\frac{1}{2}u^2\right]$$
$$=m\left[\frac{1}{2}U_+{}^2+h_+\right]=m\left[\frac{1}{2}U_-{}^2+h_-\right] \qquad (7.149c)$$

where m denotes the constant mass flow and subscripts + and - denote the initial and the final state, respectively, i.e. conditions at upstream and downstream infinity. The constancy of the left-hand side of these above equations follows from the integration. If in addition it is assumed that u and T tend to finite values at $\pm\infty$, it follows that du/dx and dT/dx are zero there, and the constants may be evaluated as shown above. Equations (7.149a, b, c) also shows that as boundary conditions for (7.148a, b, c) one may prescribe the same quantities as in the theory of the discontinuous shock wave - as an example, one may prescribe U_+, p_+, and h_+ at $x = -\infty$. For a perfect gas with $\gamma = const$ and $Pr^* = 1$, the total enthalpy is constant:

$$H=\frac{1}{2}u^2+h=const=H_+=H_-.$$

In this case we can write in place of (7.149a, b, c)) the following two equations for the velocity u and temperature T:

$$\tau\equiv\mu^*\frac{du}{dx}=A\left(\frac{1}{u}\right)[(U_+-u)(u-U_-)], \qquad (7.150a)$$

$$q\equiv-k\frac{dT}{dx}=-\mu^*\frac{dh}{dx}=u\tau=A(U_+-u)(u-U_-), \qquad (7.150b)$$

where

$$A=-m\frac{(\gamma+1)}{2\gamma}. \qquad (7.151)$$

These above equations (7.150a, b) can be used to show that u is decreasing and T is increasing and both τ and q are everywhere negative. Since τ is negative it must reach its minimum value at the critical point where: $u = u_c = c^*$ this value is:

$$-\tau_m = m\frac{\gamma+1}{\gamma}\left[\frac{1}{2}(U_+ + U_-) - c^*\right]. \tag{7.152}$$

Similarly it follows that q has its minimum at: $u = (1/2)\,(U_+ + U_-)$ and that:

$$-q_m = m\frac{\gamma+1}{2\gamma}\left[\frac{1}{4}(U_+ + U_-)^2 - c^{*2}\right]. \tag{7.153}$$

In principle the shock layer structure may then be computed, i.e. u and h found as functions of x. One way to proceed is first to introduce a distorted coordinate:

$$\xi = \int_0^x \frac{dx}{\mu^*},\ \frac{d}{d\xi} = \mu^* \frac{d}{dx}, \tag{7.154}$$

and also the following notations:

$$U_a = \frac{1}{2}(U_+ + U_-),\ V = \frac{1}{2}(U_+ + U_-),\ \varepsilon = \frac{V}{U_a},$$

$$X = A\,\varepsilon\,\xi,\ u = V\left(W + \frac{1}{\varepsilon}\right). \tag{7.155}$$

In this case, for the function $W = W(X, \varepsilon)$, we obtain, in place of (7.150a), the following equation:

$$\frac{dW}{dx} = \left(1 - W^2\right)\left(1 + \varepsilon W\right), \tag{7.156a}$$

with the conditions:

$$-\ W \to \pm 1,\ \text{when } X \to \pm\infty. \tag{7.156b}$$

By integrating (7.156a, b) one obtains as solution:

$$X = tanh^{-1} W - \frac{\varepsilon}{2} log\left(1 - W^2\right). \tag{7.157}$$

This solution is normalized in such a way that: $W = 0$ $(u = V/\varepsilon = U_a)$, when $X = 0$ and thus when $x = 0$ (since in this case $\xi = 0$). For a weak shock (for ε *small*), we find immediately that:

$$W = W_W = tanh\, X. \tag{7.158}$$

We note that: *the thickness of a weak shock layer is inversely proportional to* ε.

Strictly speaking, a shock layer has infinite width since the limiting values U_+, U_-,...etc, are assumed to be reached only at $x = \pm\infty$.

However, one may define an "effective" *thickness of the shock layer* [as in classical Prandtl boundary-layer theory; see the review by Lagerstrom (1964, p.184)]. In the paper by Crocco (1970), shock waves in an inviscid gas are considered and the structure of a weak shock wave is determined through a solvability condition*).

7.7.2. The Taylor shock layer equations

For a consistent derivation of the so-called "Taylor shock layer" equations it is necessary to return to the formalism of the Section 7.4.1. Obviously, when $\varepsilon \to 0$ (for high Reynolds numbers), the outer, Euler, limit (7.2), is not valid in the vicinity of a shock wave.

As a consequence, it is necessary to define a new inner, Taylor, limit valid in the shock layer, and for this we write the full NS-F equations in the following matrix form (with the Strouhal number $S = 1$):

$$\frac{\partial U}{\partial t} + \frac{\partial \Phi_k(U)}{\partial x_k} = \varepsilon \frac{\partial}{\partial x_j}\left[L_{jk}(U)\frac{\partial U}{\partial x_k}\right], \tag{7.159}$$

where

---

* The so-called, "Fredholm alternative":
"Either the inhomogeneous boundary-value problem is solvable whatever the forcing terms may be, or the corresponding homogeneous problem has one or more eigenfunctions (non trivial solutions). In the first case the solution of the inhomogeneous problem is unique. In second case the inhomogeneous problem is solvable if and only if the forcing terms are orthogonal to all the eigenfunctions of the homogeneous problem".

$$U=\left(\rho,\rho u_1,\rho u_2,\rho u_3,\rho\left[e+\frac{1}{2}|\boldsymbol{u}|\right]^2\right)^T, \tag{7.160a}$$

$$\Phi_k(U)=\begin{bmatrix}\rho u_k\\ \rho u_1 u_k+\dfrac{1}{\gamma M^2}p\delta_{1k}\\ \rho u_2 u_k+\dfrac{1}{\gamma M^2}p\delta_{2k}\\ \rho u_3 u_k+\dfrac{1}{\gamma M^2}p\delta_{3k}\\ \rho\left[h+\dfrac{1}{2}|\boldsymbol{u}|^2\right]u_k\end{bmatrix} \tag{7.160b}$$

and

$$L_{jk}(U)\frac{\partial U}{\partial x_k}=\begin{bmatrix}0\\ \tau_{1j}\\ \tau_{2j}\\ \tau_{3j}\\ \left(\gamma M^2\right)\tau_{jk}u_k-q_j\end{bmatrix} \tag{7.160c}$$

with

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\lambda\frac{\partial u_s}{\partial x_s}\delta_{ij},$$

$$q_j=-\frac{\gamma}{Pr}k\frac{\partial e}{\partial x_j}, \tag{7.160d}$$

$$e=\frac{T}{\gamma-1},\ h=e+\frac{p}{\rho}.$$

7.7.2a. The outer-inviscid Elder expansion

In fact, we can introduce in the NS-F equations, written as a matrix equation (7.159), new time-space coordinates, τ, ξ_1, ξ_2, ξ_3, such that the surface:

$\xi_3 \equiv z = 0$, is the shock wave in the *inviscid* (Eulerian) fluid.

Namely, we write:

$$\tau = t,\ \xi_j = \xi_j(t, x_1, x_2, x_3),\ d\tau = dt,\ d\xi_j = F_{j0}\, dt + F_{jk}\, dx_k \qquad (7.161a)$$

and

$$U(t, x_i) = W(\tau, \xi_k(t, x_i)),\ F_{jp} = F_{jp}(\tau, \xi_1, \xi_2, \xi_3),\ p = 0, k. \qquad (7.161b)$$

In (7.159) the small parameter ε is a singular perturbation parameter and, first, we consider an *outer* asymptotic expansion, namely:

$$W = W^{(0)} + \varepsilon W^{(1)} + \ldots,\ \Phi_k(W) = \Phi_k^{(0)} + \varepsilon \Phi_k^{(1)} + \ldots . \qquad (7.162)$$

With (7.161a, b) and (7.162), in place of (7.159), we derive the following equation for $W^{(0)}$:

$$\frac{\partial W^{(0)}}{\partial \tau} + \sum_{p=1}^{2}\left[F_{p0}\frac{\partial W^{(0)}}{\partial \xi_p} + \sum_{k=1}^{3} F_{pk}\frac{\partial W^{(0)}}{\partial \xi_p}\right] + F_{30}\frac{\partial W^{(0)}}{\partial z} + \sum_{k=1}^{3} F_{3k}\frac{\partial \Phi_k^{(0)}}{\partial z} = 0 . \qquad (7.163)$$

We observe that: $d\Phi_k = A_k(W)\, dW$, where $A_k(W)$ is a matrix 5 X 5 of W, and we consider the following expression, according to (7.163),

$$\left[F_{30} I + \sum_{k=1}^{3} F_{3k} A_k(V)\right]\frac{\partial V}{\partial z} = Y , \qquad (7.164)$$

on the shock wave $z = 0$, in the Eulerian inviscid fluid, we have identically:

$$F_{30} dt + \sum_{k=1}^{3} F_{3k} dx_k = 0 . \qquad (7.165)$$

But if N_k is the unit normal to shock and V_N the normal velocity propagation of the shock, then on $z = 0$, identically: $F_{30} + V_N N_k F_{3k} = 0$, and as a consequence, from (7.164), we obtain:

$$\text{on } z = 0\text{: } \sum_{k=1}^{3} F_{3k}[A_k(V) - V_N N_k I]\frac{\partial V}{\partial z} = Y. \qquad (7.166)$$

On the one hand, in the inviscid fluid on the both side of the shoch wave:

$$det\left\{\sum_{k=1}^{3} F_{3k}[A_k(V) - V_N N_k I]\right\} \neq 0, \qquad (7.167)$$

and as a consequence

$$\text{on } z = 0\text{: } \frac{\partial V}{\partial z} = \left\{\sum_{k=1}^{3} F_{3k}[A_k(V) - V_N N_k I]\right\}^{-1} Y. \qquad (7.168)$$

On the other hand, in the *inviscid fluid*, from the equation (7.159) we can associate the following zeroth-order *inviscid fluid shock wave relation:*

$$\left[\left[\sum_{k=1}^{3} F_{3k}\left\{\Phi_k\left(W^{(0)}\right) - V_N N_k W^{(0)}\right\}\right]\right] = 0. \qquad (7.169)$$

The above relations relative to shock wave, located in inviscid fluid on $z = 0$, do not give any information relative to Taylor internal structure of the Shock wave!

Such a internal structure is obtained via an "inner-internal structure-Taylor" expansion.

7.7.2b. Inner-internal structure-Taylor expansion

Now it is necessary to consider an *inner* asymptotic expansion. For this we introduce an inner coordinate: $\zeta = z/\varepsilon$, and the inner expansions:

$$W = W^*(\tau, \xi_1, \xi_2\ \zeta;\ \varepsilon) = W^{*(0)} + \varepsilon W^{*(1)} + \ldots, \qquad (7.170a)$$

$$\Phi_k(W^*) = \Phi^{*(0)}_k + \varepsilon\, \Phi^{*(1)}_k + \ldots, \qquad (7.170b)$$

$$F_{jp} = F_{jp}^{*}(\tau,\xi_1,\xi_2,\varepsilon\zeta) = F_{jp}^{*}(\tau,\xi_1,\xi_2,0) + O(\varepsilon), \qquad (7.170c)$$

with $p = 0, k$. In this case, with (7.170) and $z = \varepsilon\,\zeta$, in place of (7.159), we derive the following equation for $W^{*(0)}$:

$$F_{30}^{*(0)}\frac{\partial W^{*(0)}}{\partial\zeta} + \sum_{k=1}^{3} F_{3k}^{*(0)}\frac{\partial \Phi^{*}{}_{k}^{(0)}}{\partial\zeta}$$
$$-\sum_{j=1}^{3}\sum_{k=1}^{3} F_{3j}^{*(0)}\frac{\partial}{\partial\zeta}\left[L_{jk}\left(W^{*(0)}\right)F_{3k}^{*(0)}\frac{\partial W^{*(0)}}{\partial\zeta}\right] = 0 \qquad (7.171)$$

where $F_{jp}^{*}(\tau,\ \xi_1,\ \xi_2\ 0) \equiv F_{jp}^{*(0)}$.

Since the $F_{jp}^{*(0)}$ are independent of the inner coordinate ζ, in place of (7.171), we obtain the following equation for the *Taylor internal structure of the shock wave* at the zeroth-order, namely:

$$\frac{\partial}{\partial\zeta}\left\{\sum_{j=1}^{3}\sum_{k=1}^{3} F_{3j}^{*(0)}F_{3k}^{*(0)}L_{jk}\left(W^{*(0)}\right)\frac{\partial W^{*(0)}}{\partial\zeta}\right\}$$
$$+\frac{\partial}{\partial\zeta}\left\{-F_{30}^{*(0)}W^{*(0)} - \sum_{k=1}^{3} F_{3k}^{*(0)}\Phi^{*}{}_{k}^{(0)}\right\} = 0. \qquad (7.172)$$

The unique solution of (7.172) is such that:

$$lim_{\zeta\to-\infty} W^{*(0)} = W^{*(0),-} \text{ and } lim_{\zeta\to+\infty} W^{*(0)} = W^{*(0),+}, \qquad (7.173)$$

but, according to $F_{30} + V_N N_k F_{3k} = 0$, we can write: $F_{30}^{*(0)} = -\ V_N N_k F_{3k}^{*(0)}$, and as a consequence, in place of (7.172) we derive the following *zeroth-order shock wave condition*, namely:

$$\left[\left[\sum_{k=1}^{3} F_{3k}^{*(0)}\left\{\Phi_k\left(W^{*(0)}\right) - V_N N_k W^{*(0)}\right\}\right]\right] = 0, \qquad (7.174)$$

which is equivalent to:

$$\sum_{k=1}^{3} F_{3k}{}^{*(0)}\left[\Phi_k\left(W^{*(0),+}\right) - V_N N_k W^{*(0),+}\right]$$

$$-\sum_{k=1}^{3} F_{3k}{}^{*(0)}\left[\Phi_k\left(W^{*(0),-}\right) - V_N N_k W^{*(0),-}\right] = 0\,. \qquad (7.174a)$$

Obviously, we can, via this above method, to derive also the corresponding shock relation and Taylor internal structure of the shock wave at the first-order, but here we shall not consider this tedious derivation. Indeed, from the above formal approach we can recover the main results of the Germain and Guiraud (1966) paper.

CHAPTER 8

SOME MODELS OF NONLINEAR ACOUSTICS

A good introduction to acoustics from the point of view of the fluid dynamics is the book by Marvin E. Goldstein (1976), where the reader can find information concerning: the acoustics of moving media, aerodynamic sound, and the effects of solid boundaries for uniform and nonuniform (mean) flows. On the other hand, the reader can consult the review papers by Ffowcs Williams (1977) and Crighton (1979). Among various interesting papers we mention, first, the paper by Lesser and Crighton (1975) which is devoted to 'physical acoustics and the method of matched asymptotic expansion', then the papers by Blackstok (1972) and BjØrnØ (1976) devoted to nonlinear acoustics. We mention also the books by Beyer (1974), Rudenko and Soluyan (1977) and Pierce (1991), the last of which being devoted its physical principles and applications. The reader can find in the review paper by Coulouvrat (1992) various more recent references. Finally, a very pertinent research and review paper, in the context of low-Mach number far-field effects, is the recent paper by Ting and Miksis (1990). In this Chapter 8, we consider only some particular models which are related to nonlinear acoustics and are derived consistently in the spirit of asymptotic modelling (see Fig.8.1).

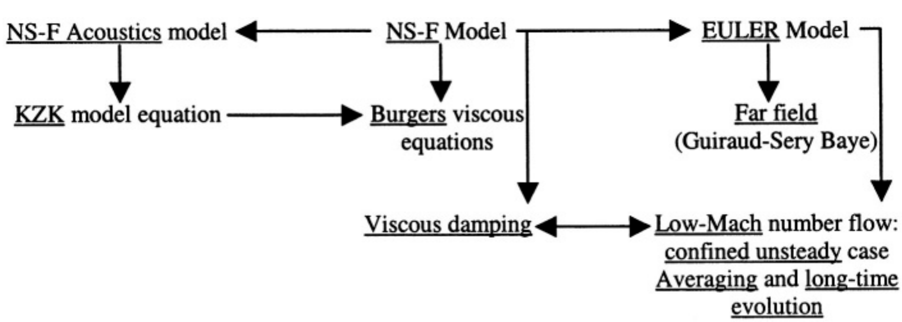


Fig. 8.1 The various model closely linked with the acoustics

8.1. BURGERS EQUATIONS FOR LARGE-TIME IN THE DISSIPATIVE CASE

An important and difficult mathematical question asks for the large-time behaviour of small-amplitude acoustic waves experiencing weak dissipation.

In fact, the small-amplitude condition allows one to begin with a linear (acoustic) mathematical problem and granted the observed persistence of acoustic waves, the large-time and weak-dissipation restrictions are quite natural.

The equations of gas dynamics (in fact, the compressible unsteady Euler equations) constitute a quasi-linear hyperbolic system. These unsteady equations of inviscid fluid dynamics (Euler equations) are "properly embedded" in full viscous and thermally conducting equations (NS-F equations) and one of the remarkable equations of fluid dynamics, the *Burgers equation,* illustrates this embedding. The Burgers equation describes the motion of weak nonlinear (acoustic) waves in gases when a first accounting of dissipative effects is needed. In the limit of vanishing dissipation, this Burgers equation provides the proper interpretation for the inviscid solution. The history of the Burgers equation is too rich to detail here.

It was proposed by Burgers (1948) as a model equation for one-dimensional turbulence. Note, however, that the essential ideas concerning the asymptotic consistent derivation of the Burgers equation, from the full NS-F equations, may be found in Guiraud (1967; pp. 18-20) short paper. In Germain (1971) the reader can find a review of the main studies on the progressive wave concept with an emphasis on the general ideas and techniques which allow to present this topic in a unified way to underline the close relations of many applications which have been published in various fields.

Here Burgers equation is derived from the unsteady one-dimensional NS-F equations, when the characteristic Mach number is a small parameter. But for a consistent asymptotic derivation of the Burgers equation it is necessary to consider also the case of high Strouhal and Reynolds numbers and as a consequence we have one small parameter and two large parameters and it is necessary to impose two similarity relations (see the relations: (8.1) and (8.2) below).

One of the most characteristic features of acoustic waves is their persistence. These waves survive for long periods and can transmit disturbances over very long distances. In fact, acoustic waves assume their most distinctive forms after travelling a "large" distance from the region in which they are generated. As a consequence, the Burgers equation is a very significant model equation.

8.1.1. The one-dimensional dimensionless "acoustic form" of the NS-F equations

The appropriate non-dimensionalization of the independent variables (below, x^* and t^* are the dimensional physical variables), is one that measures distances in units of a typical wavelength l_0 and time in units of the time it takes the wave to propagate the distance l_0.

Namely: $x^* = l_0 x$, $t^* = t_0 t$, and if (our first similarity relation):

$$S\,M = 1, \tag{8.1}$$

then $t_0 = l_0/a_0$, where: $S = l_0/U_0 t_0$ and $M = U_0/a_0$ are the high Strouhal and the small Mach numbers, while $a_0 = [\gamma R T_0]^{1/2}$ the characteristic sound speed (at the reference constant temperature T_0) for a perfect gas with constant specific heats C_P and C_V.

The choice of the characteristic velocity U° is such that (our second similarity relation):

$$M\,Re = Re^* = O(1), \tag{8.2}$$

and in this case, $U_0 \approx [(\mu_0/\rho_0)\, a_0/l_0]^{1/2}$, where $Re = l_0 U_0/(\mu_0/\rho_0)$ is the high Reynolds number, with μ_0 the constant value of dynamic viscosity at T_0 and ρ_0, where ρ_0 is the reference constant density and $p_0 = RT_0\rho_0$.

The velocity u^* (speed in the x^*-direction) and the thermodynamic functions, p^*, ρ^* and T^*, are made dimensionless as follows:

$$u^* = U_0 M\, u,\ p^* = p_0[1 + M^2\pi],\ \rho^* = \rho_0[1 + M^2\,\omega], \tag{8.3a}$$

$$T^* = T_0[1 + M^2\,\theta]. \tag{8.3b}$$

With the above notations the dimensionless (exact) one-dimensional NS-F equations for the dimensionless functions u, π, ω and θ, are written in the following form:

$$\frac{\partial \omega}{\partial t} + \frac{\partial u}{\partial x} = -M^2 \frac{\partial}{\partial x}(u\omega), \tag{8.4a}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\gamma}\frac{\partial \pi}{\partial x} = M^2\left[\frac{1}{Re^*}\left(\frac{4}{3} + \frac{\mu_V{}^\circ}{\mu^\circ}\right)\frac{\partial^2 u}{\partial x^2} - \omega\frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x}\right] + O(M^4), \tag{8.4b}$$

$$\frac{\partial\theta}{\partial t}+(\gamma-1)\frac{\partial u}{\partial x}$$

$$=M^2\left[\frac{1}{Pr'Re^*}\frac{\partial^2\theta}{\partial x^2}-\omega\frac{\partial\theta}{\partial t}-u\frac{\partial\theta}{\partial x}-(\gamma-1)\pi\frac{\partial u}{\partial x}\right]+O(M^4) \quad (8.4c)$$

$$\pi-(\theta+\omega)=M^2\,\theta\,\omega, \quad (8.4d)$$

where $Pr'=C_V\mu_0/k_0$, is a (modified) Prandtl number, with k_0 the constant thermal conductivity at T_0 and ρ_0.

8.1.2. Large-time behaviour: the viscous Burgers equation

For the large-time behavior it is necessary to introduce (with the "fast" time *t*) a second "slow" time

$$\tau=M^2\,t \text{ and }, \ \frac{\partial}{\partial t}=\frac{\partial}{\partial t}+M^2\frac{\partial}{\partial\tau}, \quad (8.5)$$

and in this case the uniformly valid expansion for the solution of equations (8.4a, b, c, d) take the form:

$$u=u^*(t,x,\tau;M^2)=u_0+M^2\,u_2+\ldots, \quad (8.6a)$$

$$\omega=\omega^*(t,x,\tau;M^2)=\omega_0+M^2\,\omega_2+\ldots,$$

$$\pi=\pi^*(t,x,\tau;M^2)=\pi_0+M^2\,\pi_2+\ldots, \quad (8.6b)$$

$$\theta=\theta^*(t,x,\tau;M^2)=\theta_0+M^2\,\theta_2+\ldots.$$

The $(M)^0$ leading-order limiting equations obtained are:

$$\frac{\partial\omega_0}{\partial t}+\frac{\partial u_0}{\partial x}=0, \quad (8.7a)$$

$$\frac{\partial u_0}{\partial t}+\frac{1}{\gamma}\frac{\partial\pi_0}{\partial x}=0, \quad (8.7b)$$

$$\frac{\partial \theta_0}{\partial t} + (\gamma - 1)\frac{\partial u_0}{\partial x} = 0, \tag{8.7c}$$

$$\pi_0 - (\theta_0 + \omega_0) = 0, \tag{8.7d}$$

and an obvious solution of the system (8.7) is (sound waves moving to the right):

$$u_0 = F^\circ(\sigma, \tau),\ \omega_0 = F^\circ(\sigma, \tau),\ \pi_0 = \gamma F^\circ(\sigma, \tau),$$

$$\theta_0 = (\gamma - 1) F^\circ(\sigma, \tau), \tag{8.8}$$

where

$$\sigma = x - t \text{ and } \frac{\partial}{\partial x} = \frac{\partial}{\partial \sigma},\ \frac{\partial}{\partial t} = -\frac{\partial}{\partial \sigma}. \tag{8.9}$$

Next, at the order M^2 limiting equations, with (8.8) and (8.9), can be written in the following form

$$\frac{\partial \omega_2}{\partial t} + \frac{\partial u_2}{\partial x} = A(F^\circ(\sigma,\tau)),$$

$$\frac{\partial u_2}{\partial t} + \frac{1}{\gamma}\frac{\partial \theta_2}{\partial x} + \frac{1}{\gamma}\frac{\partial \omega_2}{\partial x} = B(F^\circ(\sigma,\tau)), \tag{8.10}$$

$$\frac{\partial \theta_2}{\partial t} + (\gamma - 1)\frac{\partial u_2}{\partial x} = C(F^\circ(\sigma,\tau)),$$

since $\pi_2 - (\theta_2 + \omega_2) = (\gamma - 1)[F^\circ(\sigma, \tau)]^2$, where:

$$A(F^\circ(\sigma,\tau)) = -\left[\frac{\partial F^\circ}{\partial \tau} + \frac{\partial (F^\circ)^2}{\partial \sigma}\right]; \tag{8.11a}$$

$$B(F^\circ(\sigma,\tau)) = \frac{1}{Re^*}\left[\frac{4}{3} + \frac{\mu_v{}^\circ}{\mu^\circ}\right]\frac{\partial^2 F^\circ}{\partial \sigma^2} - \frac{\partial F^\circ}{\partial \tau} - \frac{\gamma - 1}{\gamma}\frac{\partial (F^\circ)^2}{\partial \sigma}; \tag{8.11b}$$

$$C(F^\circ(\sigma,\tau)) = \frac{\gamma(\gamma-1)}{\gamma Pr' Re^*}\frac{\partial^2 F^\circ}{\partial\sigma^2} - (\gamma-1)\left[\frac{\partial F^\circ}{\partial\tau} + \gamma F^\circ \frac{\partial F^\circ}{\partial\sigma}\right]. \tag{8.11c}$$

From (8.10) we derive by a straightforward combination the following equation:

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = -\frac{\partial}{\partial\sigma}\left[B + \frac{1}{\gamma}(A+C)\right]. \tag{8.12}$$

As a consequence of (8.12), with (8.11a, b, c), the only way for that our low-Mach number expansions (8.6) can be non-secular is to make the source term:

$$B + \frac{1}{\gamma}(A+C) = 0,$$

and this is the compatibility condition, which gives the sought-after classical viscous Burgers equation:

$$\frac{\partial F^\circ}{\partial\tau} + \frac{1}{2}(\gamma+1)F^\circ\frac{\partial F^\circ}{\partial\sigma} = \nu^\circ\frac{\partial^2 F^\circ}{\partial\sigma^2}, \tag{8.13}$$

with the following dissipative coefficient (also called the 'Stokes number' or 'diffusivity of sound')

$$\nu^\circ = \frac{1}{2Re^*}\left[\frac{4}{3} + \frac{\mu_V{}^\circ}{\mu^\circ} + \frac{\gamma-1}{Pr}\right], \tag{8.14}$$

since $\gamma Pr' = Pr$. This Stokes number measures the combined influence of all dissipative effects due to viscosity or thermal conduction. We note, again, that the Burgers equation (8.13) is valid in the framework of low-Mach number asymptotics, only if the above two similarity relations (8.1), (8.2), between *M*, *Re* and *S* are satisfied. Although the Burgers equation is derived in the context of the low Mach number flows, it should be emphasized that this equation arise in a wide variety of different physical context and so the Burgers equation is, in a sense, typical of nonlinear dissipative systems.

Burgers equation played a fundamental role in the understanding of nonlinear wave propagation. It was first derived by Burgers (1948) as a simple model for turbulent motions. Mendousse (1953) was the first to use it in nonlinear acoustics for a viscous, perfect gas. Afterwards, it was generalized by different authors to thermoviscous fluids (see Blackstock (1964) for a brief history). The direct derivation of that equation in the frame of secular perturbation problems is presented by Leibovich and Seebass (1974). One main point of interest is that the exact solution of Burgers equation is known, which can be obtained by means of the transformation achieved independantly by Hopf (1950) and Cole (1951). This solution enable us in particular to test the validity and the precision of asymptotic approximations. In the framework of Burgers equation, the fundamental role played by shock waves and shock structures occuring along the wave propagation is now clearly understood.

8.2. THE MODEL EQUATIONS OF NONLINEAR ACOUSTICS

8.2.1. The acoustic wiewpoint

Note that in acoustics, in place of Reynolds number $Re = l_0U_0/(\mu_0/\rho_0)$, where U_0 can be interpreted as a 'source velocity', it is customary to introduce an 'acoustic Reynolds number', which is denoted here by Re_a, see, for instance, Coulouvrat (1992, p. 326). The definition of this Re_a in our notation is:

$$Re_a = \frac{\rho_0 a_0}{\omega\mu_0},$$

where ω is a reference frequency, determined by the main frequency $f^o = \omega/2\pi$ of the source-signal: $U_0 f(\omega t)$, and the associated wave number is $k = \omega/a_0$, the reference length scale being $1/k = l_0 = a_0/\omega$. As a consequence, we can write the following relation between Re, M and Re_a:

$$Re_a = \frac{l_0 a_0^{\ 2}}{a_0\left(\dfrac{\mu_0}{\rho_0}\right)} = \frac{Re}{M}. \tag{8.15}$$

In classical nonlinear acoustics a fundamental model equation is the so-called "parabolic KZK equation" was first derived by Zabolotskaya and Khokhlov (1969) in the inviscid case and then generalized by Kuznetsov (1970) to the dissipative as an approximation of the Kuznetsov equation. Concerning this Kuznetsov equation the reader can find in Coulouvrat

(1992) a derivation of this equation (see, in Coulouvrat (1992), pages 333-334). Using a multiple scale asymptotic method the KZK equation was derived for the first time by TjØtta and TjØtta (1981). The same method was used (though in a slightly different way) in Coulouvrat (1992, §3.2).

If we take into account, again, the similarity relation (8.1) but, in place of (8.2), we introduce the acoustic Reynolds number Re_a, such that: $1/Re = 1/MRe_a$, and in place of (8.3a), (8.3b), the following relations:

$$u^* = U_0 u,\ p^* = p_0[1 + M\pi],\ \rho^* = \rho_0[1 + M\,\omega], \tag{8.16a}$$

$$T^* = T_0[1 + M\,\theta], \tag{8.16b}$$

then, when the dissipative coefficients are assumed constant, in place of equations (8.4a, b, c) and (8.4d), we obtain the following 3D system of unsteady dimensionless (acoustics) NS-F equations:

$$\frac{\partial \omega}{\partial t} + \nabla \cdot \boldsymbol{u} = -M\nabla \cdot (\omega \boldsymbol{u}), \tag{8.17a}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{1}{\gamma}\nabla \pi = \frac{1}{Re_a}\left\{\nabla^2 \boldsymbol{u} + \left(\frac{1}{3} + \frac{\mu_V{}^\circ}{\mu^\circ}\right)\nabla(\nabla \cdot \boldsymbol{u})\right\}$$
$$- M\left[\omega \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] + O(M^2), \tag{8.17b}$$

$$\frac{\partial \theta}{\partial t} + (\gamma - 1)\nabla \cdot \boldsymbol{u} = \frac{1}{Re_a}\frac{\gamma}{Pr}\nabla^2 \theta$$
$$+ M\left[\frac{\gamma(\gamma-1)}{Re_a}\left[2Tr(\boldsymbol{D}^2) + \left(\frac{1}{3} + \frac{\mu_V{}^\circ}{\mu^\circ}\right)(\nabla \cdot \boldsymbol{u})^2\right]\right.$$
$$\left. - \left(\omega \frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta + (\gamma - 1)\pi \nabla \cdot \boldsymbol{u}\right)\right] + O(M^2) \tag{8.17c}$$

$$\pi - (\theta + \omega) = M\,\theta\,\omega, \tag{8.17d}$$

with $Pr = C_P \mu_0 / k_0$. These 'dominant' equations (8.17a, b, c) with (8.17d) are the main starting point for the derivation of various model equations in nonlinear acoustics.

For the frequencies and the media commonly used in nonlinear acoustics, the acoustic Reynolds number Re_a is always very large compared to unity and this means that the (continuous) medium is weakly dissipative at the chosen frequency - in the most common experimental situations the equations (8.17a, b, c) with (8.17d), are accurate enough.

As a consequence, the above three equations (8.17a, b, c) turn out to be perturbation equations with two small parameters M and $1/Re_a$. Obviously, since we neglect the terms proportional to $O(M^2)$, the above equations (8.17b, c) make sense only if:

$$M^2 << \frac{1}{Re_a}. \tag{8.18}$$

From these above set of 'dominant' equations (8.17a, b, c, d), various model equations of acoustics can be derived, but, unfortunately, the derivation of these equations is not fully consistent with the spirit of rigorous and logical asymptotic modelling. I *do not claim* that these model equations, derived with a 'pseudo-asymptotic' method, are not interesting for various applications related to nonlinear acoustics, namely: sound attenuation in seawater, thermoviscous and nonlinear attenuation in near and far-field, analysis of the dispersion relation - more examples of various applications are provided in Proceedings of the International Symposium on Nonlinear Acoustics (Hamilton and Blackstock Eds. (1990)). Nevertheless, it appears necessary that these acoustic models should be seriously 'revisited' to examine the consistency of the results and conclusions obtained and this, obviously, is a somewhat difficult but still challenging problem! Below I give (as an example) a tentative derivation of the KZK equation for the far-field.

8.2.2. A tentative derivation of the KZK equation

More particularly, concerning the derivation of the KZK equation, the main idea is that, basically it is assumed that the 3D acoustic field is locally plane, such that the nonlinear wave propagates in the same way as linear plane wave over a few wavelengths, the wave profile or amplitude being significantly altered only over large distances away from the source and obviously the parabolic approximation, which leads the KZK model equation, may not be valid close to the source (near field). The condition for that approximation to be valid is that the acoustic source width '*d*' should be much larger than the wavelength $1/k$, so that transverse field variations are slow compared to longitudinal variations along the acoustic axis. Close to the origin of the acoustic axis near the acoustic source we have a "near field" and at large a distance from this acoustic source a far-field - the KZK equation is significant (asymptotically) just for this acoustic far-field. In this case, the purely linear acoustic homogeneous field, which is derived when the Mach number tends to zero ($1/Re_a$ being, in fact, of the order of M), depends mostly on the retarted time $\chi = t - x$, the dependence on two other (transverse) space variables being slow.

Thus, if the dimensionless 3D position variable is $\mathbf{x} = (x, y, z)$, then two new dimensionless variables η and ζ are defined as:

$$\eta = \alpha y \text{ and } \zeta = \alpha z, \tag{8.19a}$$

where $\alpha = 1/kd << 1$ and tends to zero as $M \downarrow 0$. But, the transverse components of the fluid velocity should be changed also. Namely, if the dimensionless velocity vector is $\mathbf{u} = (u, v, w)$, then we set:

$$V = \frac{v}{\alpha} \text{ and } W = \frac{w}{\alpha}. \tag{8.19b}$$

In this case, we can write:

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \alpha\frac{\partial}{\partial \eta}\boldsymbol{j} + \alpha\frac{\partial}{\partial \zeta}\boldsymbol{k} \equiv \frac{\partial}{\partial x}\boldsymbol{i} + \alpha\nabla_{\perp}. \tag{8.19c}$$

and it is necessary to assume that the functions u, V, W, π, ω, θ are dependent on ξ, χ, η, ζ and M where:

$$\xi = M\,x, \text{ and } \frac{\partial}{\partial x} = -\frac{\partial}{\partial \chi} + M\frac{\partial}{\partial \xi},\ \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \chi^2} + O(M), \tag{8.19d}$$

since with $\chi = t - x$, $\partial/\partial t = \partial/\partial\chi$ and $\partial/\partial x = -\partial/\partial\chi$. But, for a consistent asymptotic derivation it is necessary also to assume the existence of two similarity relations between the three small parameters, $1/Re_a$, M and α, namely:

$$\frac{1}{Re_a} = \kappa M \text{ and } \frac{\alpha^2}{M} = \beta \text{ with } \kappa = O(1) \text{ and } \beta = O(1). \tag{8.20}$$

Now, for low Mach numbers we consider the following asymptotic expansions:

$$U = (u, V, W, \pi, \theta, \omega) = U_0 + M\, U_1 + \ldots, \tag{8.21}$$

and for the leading order (with subscript "$_0$") we derive again the system of acoustic equations (8.7) and we can write the following solution:

$$U_0 \equiv U_0(\chi, \xi, \eta, \zeta),$$

such that

$$u_0 = H^\circ(\chi, \xi, \eta, \zeta),\ \omega_0 = H^\circ(\chi, \xi, \eta, \zeta),\ \pi_0 = \gamma H^\circ(\chi, \xi, \eta, \zeta),$$
$$\theta_0 = (\gamma - 1) H^\circ(\chi, \xi, \eta, \zeta), \tag{8.22}$$

and also

$$\frac{\partial V_0}{\partial t} + \frac{1}{\gamma}\frac{\partial \pi_0}{\partial \eta} = 0,\quad \frac{\partial W_0}{\partial t} + \frac{1}{\gamma}\frac{\partial \pi_0}{\partial \zeta} = 0, \tag{8.23a}$$

or

$$\frac{\partial}{\partial t}\left[\frac{\partial V_0}{\partial \zeta} - \frac{\partial W_0}{\partial \eta}\right] = 0, \tag{8.23b}$$

as a consequence of the vectorial equation (8.17b), written for the transverse components of the velocity vector, and the similarity relations (8.20).

In this case, at the next order M, thanks to the similarity relations (8.20), we obtain the following system of equations for the functions: u_1, ω_1, and θ_1,

$$\frac{\partial \omega_1}{\partial t} + \frac{\partial u_1}{\partial x} = \frac{\partial H^{\circ 2}}{\partial \chi} - \frac{\partial H^\circ}{\partial \xi} - \beta\left(\frac{\partial V_0}{\partial \eta} + \frac{\partial W_0}{\partial \zeta}\right); \tag{8.24a}$$

$$\frac{\partial u_1}{\partial t} + \frac{1}{\gamma}\frac{\partial \theta_1}{\partial x} + \frac{1}{\gamma}\frac{\partial \omega_1}{\partial x} = \kappa\left(\frac{4}{3} + \frac{\mu_v{}^\circ}{\mu^\circ}\right)\frac{\partial^2 H^\circ}{\partial \chi^2} - \frac{\partial H^\circ}{\partial \xi} + \frac{\gamma - 1}{\gamma}\frac{\partial H^{\circ 2}}{\partial \chi}; \tag{8.24b}$$

$$\frac{\partial \theta_1}{\partial t}+(\gamma-1)\frac{\partial u_1}{\partial x}=\kappa(\gamma-1)\frac{\gamma}{Pr}\frac{\partial^2 H^\circ}{\partial \chi^2}-(\gamma-1)\frac{\partial H^\circ}{\partial \xi}$$
$$+\gamma(\gamma-1)H^\circ\frac{\partial H^\circ}{\partial \chi}. \tag{8.24c}$$

We do not write the two inhomogeneous equations for V_1 and W_1, since these equations are related to a derivation of a (linear) second-order KZK equation!

Now from the above system (8.24a, b, c) we derive again (by analogy with (8.12)) a single inhomogeneous equation for u_1, namely:

$$\frac{\partial^2 u_1}{\partial t^2}-\frac{\partial^2 u_1}{\partial x^2}=\frac{\partial}{\partial \chi}\left[B+\frac{1}{\gamma}(A+C)\right]+\frac{\beta}{\gamma}\left(\frac{\partial^2 H^\circ}{\partial \eta^2}+\frac{\partial^2 H^\circ}{\partial \zeta^2}\right), \tag{8.25}$$

where:

$$A=\frac{\partial H^{\circ 2}}{\partial \chi}-\frac{\partial H^\circ}{\partial \xi};$$

$$B=\kappa\left(\frac{4}{3}+\frac{\mu_v{}^\circ}{\mu^\circ}\right)\frac{\partial^2 H^\circ}{\partial \chi^2}-\frac{\partial H^\circ}{\partial \xi}+\frac{\gamma-1}{\gamma}\frac{\partial H^{\circ 2}}{\partial \chi}; \tag{8.26}$$

$$C=\kappa(\gamma-1)\frac{\gamma}{Pr}\frac{\partial^2 H^\circ}{\partial \chi^2}-(\gamma-1)\frac{\partial H^\circ}{\partial \xi}+\gamma(\gamma-1)H^\circ\frac{\partial H^\circ}{\partial \chi}.$$

Finally, since the right-hand-side source term in (8.25) gives, in the solution for u_1, a term which will be ultimately greather than u_0, whatever the smallness of the Mach number, the only way for that expansion to be non-secular is the:

$$\frac{\partial}{\partial \chi}\left[B+\frac{1}{\gamma}(A+C)\right]+\frac{\beta}{\gamma}\left(\frac{\partial^2 H^\circ}{\partial \eta^2}+\frac{\partial^2 H^\circ}{\partial \zeta^2}\right)=0,$$

and as a consequence we derive the following KZK equation:

$$\frac{\partial}{\partial \chi}\left(\frac{\partial H^\circ}{\partial \xi}\right) - \frac{1}{2}(\gamma+1)\frac{\partial}{\partial \chi}\left[H^\circ \frac{\partial H^\circ}{\partial \chi}\right]$$

$$-\frac{1}{2}\frac{\beta}{\gamma}\left[\frac{\partial^2 H^\circ}{\partial \eta^2} + \frac{\partial^2 H^\circ}{\partial \zeta^2}\right] = S\frac{\partial^3 H^\circ}{\partial \chi^3} \tag{8.27}$$

where

$$S = \frac{\kappa}{2}\left[\frac{4}{3} + \frac{\mu_V{}^\circ}{\mu^\circ} + \frac{\gamma - 1}{Pr}\right]. \tag{8.28}$$

The parameter β, in the KZK equation (8.27), measures the relative magnitude orders of magnitude of diffraction and nonlinearity and for $\beta = 0$, we derive again a Burgers equation valid for the far-field (at a large distances from source) in the following form:

$$\frac{\partial H^\circ}{\partial \xi} - \frac{1}{2}(\gamma+1)H^\circ \frac{\partial H^\circ}{\partial \chi} = S\frac{\partial^2 H^\circ}{\partial \chi^2}, \tag{8.29}$$

after an integration relative to χ. Since $\chi = -\sigma$, then we recover, in fact, the equation (8.13) with ξ in place of τ.

The above KZK equation (8.27) is the simplest equation taking into account, simultaneously in an asymptotic consistent way, nonlinearity, dissipation and diffraction.

8.3. THE EMERGENCE OF THE FLOW ACOUSTICS AT LARGE DISTANCE AND AERODYNAMIC SOUND GENERATION

In a sense defined by Lighthill (1962), aerodynamic sound generation is a process whereby an unsteady flow of limited extent produces noise as a by-product and such noise can be recorded outside the real flow region, for instance, as pressure fluctuations propagating with the speed of sound. As long as feedback mechanisms can be neglected, the assumption that the noise is only a by-product of unsteady flows is confirmed by experimental observations.

From the point of view of nonlinear acoustics, where flow and sound fields interact, a substantial advance was made in a well-known paper by Lighthill in 1952 at about the same time that jet-powered aircraft entered the commercial air traffic scene. Lighthill (1952) rearranged the equations of fluid motion into an inhomogeneous wave equation for the flow density with a formal source distribution on the right hand side, i.e., he obtained:

$$\Delta p - \frac{1}{a^2}\frac{\partial \rho}{\partial t}$$
$$= \frac{\partial^2}{\partial x_i \partial x_j}\left[\rho u_i u_j + \left(p - a^2 p\right)\delta_{ij} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right] \quad (8.30)$$

where δ_{ij} is the Kronecker-Symbol. However, it is necessary to put in place of a^2, the square of the speed of sound in the medium at rest, a_0^2, and assume in addition that the right hand side of equation (8.30) decreases sufficiently fast outside the actual flow region D. The effects on the sound generation caused by solid bodies inside the flow region D were discussed by Curle (1955) in a generalization of the Lighthill theory.

It is our aim here to discuss singular perturbation methods and we mention, first, Van Dyke (1964) book. It was recognized shortly thereafter that asymptotic expansions methods would be applied to aeroacoustics as well. The essential step was made independently by Crow (1966, 1970), Lauvstad (1968), Obermeier (1967) and Viviand (1970). The more recent paper by Obermeier (1977), where the inner flow and outer flow (where the sound is observed) regions are analyzed and the matching via an intermediate region is discussed, gives a pertinent overview of the low-Mach number asymptotics applied to aerodynamic sound generation. We will not go into details of these above mentioned papers, the reader is refered to the literature.

Here, we note only that if we assume that the Strouhal number $S = 1$, one must deal, in the general case, with the smallness of both Mach number, M, and inverse Reynolds number, $1/Re$, and a way of dealing with such a situation is to enforce a relation such that $1/Re = O(M^{\alpha})$ with a properly chosen exponent α.

But, in the inviscid case, from the above mentioned works of Lauvstad, Crow and Viviand one may infer that the low-Mach number expansion should proceed as follows, when we do not take into account the influence of the Reynolds number in the isentropic case:

$$p = 1 + M^2 p_2 + M^3 p_3(t) + M^4 p_4 + \ldots,\ \boldsymbol{u} = \boldsymbol{u}_0 + M^2 \boldsymbol{u}_2 + M^3 \boldsymbol{u}_3 + \ldots, \quad (8.31)$$

and these formulae exhibit some peculiar features of the low-Mach number expansion which show that it is not as straightforward as might be expected.

The first point is that $p_3(t)$ is a function of time only. This was shown by Crow (1970) and Viviand (1970) who discovered that the expansion (8.31) is indeed a "proximal" (valid only when $|x| = O(1)$) one, which is not valid at large distance from the wall bounding the unsteady external fluid motion. In fact, according to Viviand (1970), we have the following Poisson equation for the term p_{S+2},

$$\Delta p_{S+2} = \frac{\partial^2 p_s}{\partial t^2} - \gamma \nabla \cdot [\nabla \cdot (u_0 u_S + u_S u_0 + \rho_S u_0 u_0)], \qquad (8.32)$$

and it is clear that it is not possible to find a solution of (8.32) which is vanishes at a large distance from the wall! We note that for $s = 2$, the behaviour at infinity of p_2 is $O(1/|x|)$ and as a consequence: p_4 is $O(|x|)$ for $|x| \to +\infty$.

Following Lauvstad (1968), Crow and Viviand were able to match the proximal expansion (8.31), with t and $|x|$ fixed, with a "distal" expansion

$$p = 1 + M^{c+1} p^*_{c+1}(t, \xi) + \ldots, \; u = M^c u^*_c(t, \xi) + \ldots, \qquad (8.33)$$

where $\xi = M x$, valid when $\xi = O(1)$. With an error of $O(M^{c+1})$ we find in this case that p^* and u^* with $\rho^* = (1/\gamma)p^*$ are solutions of the equations of acoustics, namely for p^* we derive a classical wave equaton

$$\frac{\partial^2 p^*}{\partial t^2} - \Delta^* p^* = 0, \qquad (8.34)$$

where $\Delta^* = M^2 \Delta$.

The solution of the equation (8.34) must satisfy the (Sommerfeld) radiation condition at infinity, the matching conditions when $|\xi|$ tends to zero and this solution must be singular at $|\xi| = 0$ (or else this solution is identically zero!). In Viviand (1970) the author utilizes a composite expansion for the pressure, namely:

$$C(p) = 1 + M^2 P_2(t, x, \xi) + M^3 P_3(t, x, \xi)) + M^4 P_4(t, x, \xi) + \ldots, \qquad (8.35a)$$

and for each P_S we have

$$P_S = p_S(t, x) - \Pi_S(t, x) + p^*_S(t, \xi) - \Phi^*_S(t, \xi), \qquad (8.35b)$$

where $\Pi_S(t, \boldsymbol{x})$ is the part of the proximal expansion of p_S which does not tend to zero when $|\boldsymbol{x}| \to +\infty$, and $\Phi^*_S(t, \boldsymbol{\xi})$ is an unknown function of $\boldsymbol{\xi}$ which must be determined from the condition of the uniformly validity of (8.35a).

In Viviand (1970, p. 590) the reader can find this composite expansion up to the term proportional to M^5. In Viviand (1970, §6) the results are compared with Lighthill's theory in the case of small Mach numbers, and are found to agree with it up to terms of order M^5.

We observe that (8.33) matches either with a source (if $c = 2$ and $p_3(t)$ in (8.31) is proportional to the second derivative of the mass flow) or with a dipole (if $c = 3$ and $p_3(t) = 0$) or with a quadrupole (if $c = 4$ and also $p_3(t) = 0$).

The quadrupole situation is the one appropriate for sound generation and for a pertinent account, see Ffowcs Williams (1984).

If one comes back to (8.31) one finds that the pair $[(1/\gamma)p_2, \boldsymbol{u}_0]$ is governed by the equations of incompressible (inviscid) aerodynamics while the pair $[(1/\gamma)p_4, \boldsymbol{u}_2]$ and so on... exhibits some slight compressibility.

Finally, we observe that it is well known that the whole field of acoustics is related to the concept of flow at low Mach number but one must stress that this results from two different settings, which are formally equivalent but lead to different interpretations.

Both correspond to

$$p = 1 + M\,\pi,\ \rho = 1 + M\,\omega,$$

and the equations of acoustics govern π, ω and $\boldsymbol{u}$ with an error $O(M)$, where we assume $M\,S = O(1)$ in one case, and $t = O(M)$ in the other one which is suited for studying the transient behaviour of a flow set into motion from rest by the displacement of a body (see, for instance, the §5.2 in Chapter 5).

The reader should consider this, apparently curious phenomenon (but known long ago, probably since Rayleigh), that sligthly compressible (inviscid) external aerodynamics becomes compressible flow at large distances from the sources which create the motion, and furhermore, is in fact acoustic dominated.

We note that this conclusion is also true for a viscous compressible and heat conducting flow, when in place of the Euler equations we consider the full NS-F equations and the above conclusion is true even at moderate Reynolds numbers. It is in this apparently simple remark that the theory of sound propagation is profoundly rooted. A fairly comprehensive investigation of this subject appears in the early work of Lighthill (1952) and for an application of singular perturbation methods to aerodynamic sound generation, see the paper by Obermeier (1977).

8.3.1. The Guiraud - Sery Baye investigation of the far field for the Steichen equation

Following the Guiraud unpublished Notes, Sery Baye (1994) built up an asymptotic algorithm (in the inviscid potential case) for the scalar velocity potential Φ, which satisfies the unsteady Steichen equation written in the diumensionless form (see the Section 6.7.2 in §6.7, Chapter 6):

$$\Delta\Phi - SM^2\frac{\partial^2\Phi}{\partial t^2} = M^2\left\{S\frac{\partial}{\partial t}\left[\frac{1}{2}|\nabla\Phi|\right]^2 + \left[(\gamma-1)\Delta\Phi + (\nabla\Phi\cdot\nabla)\right]\left[S\frac{\partial\Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2\right]\right\} \quad (8.36)$$

where S is the Strouhal number and $M = U^\circ/a^\circ$ the characteristic Mach number. For the above second-order in time equation (8.36) it is necessary to impose two conditions for $t = 0$, namely: at $t = 0$: $\Phi = \Phi^0$ and $S\partial\Phi/\partial t = \Phi^1$.

This asymptotic algorithm is expressed in terms of the integer powers of M, including the product of such powers with $LogM$, and the calculation is truncated at M^6. The terms M^p, with p an odd integer and $M^q\, LogM$, when $q \geq 4$, appear in the asymptotic expansions for the far-field because of the behavior of the classical Janzen-Rayleigh expansion, in the unsteady case, when $\|x\|$ goes to infinity. In fact, one of the difficulties is that the expansion cannot be generated by iterative application of the limit $M \downarrow 0$, keeping the time t and position x fixed. As a consequence two asymptotic expansions are needed, one (proximate) for $\|x\| = O(1)$, the other (distant) for $\|x\| = O(1/M)$. Relative to the position x, Sery Baye brought great care to bear upon the matching conditions. Indeed, for the matching the proximate expansion is written in the following form, for the solution of the unsteady Steichen equation,

$$\Phi = \sum_{p=0}^{6} M^p\Phi_p(t,x) + \sum_{q=4}^{6} M^q LogM\,\Phi_{q,1}(t,x). \quad (8.37)$$

The associated distant asymptotic expansion, for the far-field, is then written as:

$$\Phi = M^S \Phi^*(t, x^*) = \sum_{p=0}^{6-s} M^p \Phi^*_p(t, x^*) + \sum_{q=4-s}^{6-s} M^q LogM \Phi^*_{q,1}(t, x^*), \quad (8.38)$$

where $x^* = M x$ and we introduce $r^* = |x^*|$.

In the distant expansion (8.38) the scalar $s > 0$ is determined by the behaviour at large distance, when $r = |x| \to +\infty$, of the first term of (8.37), $\Phi_0(t, x)$, which is a solution of the Laplace equation, $\Delta\Phi_0 = 0$.

In fact, at large distance: $\Phi_0(t, x) = O(1/r^s)$; more precisely, if we write only the three first terms which are necessary for the discussion below:

$$\Phi_0 = \frac{m^\circ}{r} - D^\circ \cdot \nabla\left(\frac{1}{r}\right) - Q^\circ : \nabla\nabla\left(\frac{1}{r}\right) + O\left(\frac{1}{r^4}\right), \text{ when } r \to +\infty, \quad (8.39)$$

but, in Viviand (1970), the reader can find a more precise formula written with an error of $O(1/r^7)$. In (8.39), m°, D° and Q° are arbitrary functions (a scalar, a vector and a second-order tensor) depending on time t. As is noticed in Sery Baye (1994, p.37), we have:

if $m^\circ \neq 0$, then $s = 1$ (monopole-like behaviour of Φ_0, when $r \to +\infty$),
if $m^\circ = 0$ but $D^\circ \neq 0$, then $s = 2$ (dipole-like behaviour of Φ_0, when $r \to +\infty$),
if $m^\circ = D^\circ = 0$, but $Q^\circ \neq 0$, then $s = 3$ (quadrupole-like behaviour of Φ_0, when $r \to +\infty$).

When the behaviour of Φ_0 is quadrupole-like, for $r \to +\infty$, then the terms proportional to $M^q\, LogM$ are zero in (8.37) and (8.38).

The results obtained by Sery Baye (1994) extend the results of Viviand (1970) and also of Leppington and Levine (1987).

Finally, we note, again, that low-Mach numbers asymptotics is doubly singular, respectively: *near* $t = 0$ (where the exact initial conditions are imposed) and at *infinity* when $|x| \to +\infty$, because of the singular behavior of the Janzen-Rayleigh proximate expansion. Below, in Fig. 8.2, the reader can find a sketch of the various regions related to the limiting case $M \to 0$.

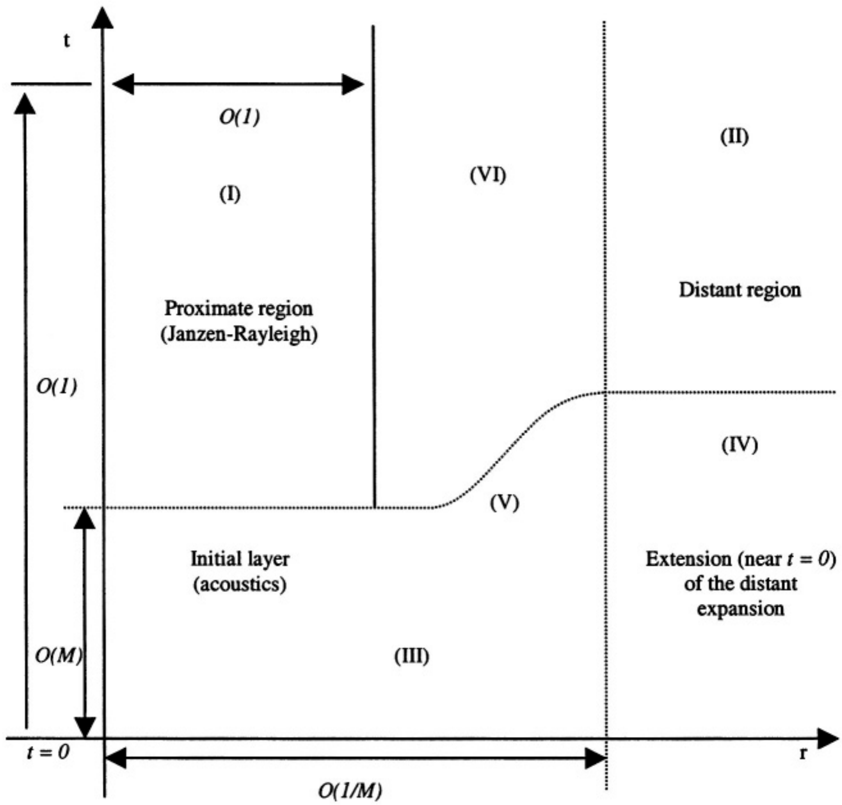


Fig. 8.2. Sketch of the various regions related to the limiting case $M \to 0$.

According to the above sketch, we have a three-region singular problem when the Mach number $M \to 0$.

The proximate (Janzen-Rayleigh) region (I) matches the acoustic (initial-layer) region (III) and also the distant (acoustic) region (II).

Regions (V) and (VI) are, in fact, the matching (intermediate) regions.

Finally, region (IV) is the extension of region (II), for small time, near $t = 0$.

The main region (I) is an "incompressible" region, while regions (III) and (II), with region (IV), are "acoustic" regions.

It is true that there exists also a "large time" region where the limit equation is the Burgers model equation (8.13).

8.4. LOW-MACH NUMBER FLOW AFFECTED BY ACOUSTIC EFFECTS IN A CONFINED GAS OVER A LONG TIME

8.4.1. Some introductory remarks

In the unpublished working Notes of Zeytounian and Guiraud (1980), a first approach to the inviscid perfect gas flow which occurs within a bounded domain $D(t)$, the walls, $\partial D(t) = \Sigma(t)$, of which deform with motion very slowly in comparison with the speed of sound, has been investigated.

The result of a 'naïve' asymptotic single-time and space scale analysis, is that the appropriate (quasi-incompressible Euler) model is one for which the velocity is divergence-free while the density is not a constant but a function of time. Unfortunately, this 'naïve' main model is *only valid* if the component normal to $\Sigma(t)$, $W_\Sigma = \boldsymbol{V}_\Sigma \cdot \boldsymbol{n}$, of the wall displacement velocity $\boldsymbol{V}_\Sigma$, at $t = 0$ *is zero*! Obviously, this is not the case when the motion of $\Sigma(t)$ is started impulsively from rest and the same holds if the motion of $\Sigma(t)$ is accelerated from rest to a finite velocity in a short (dimensionless) time $O(M)$.

As a consequence, it is necessary to elucidate how the 'naïve' main (quasi-incompressible Euler) model fits with the initial conditions (at $t = 0$) associated with the exact unsteady compressible Euler equations.

For this it is necessary to consider an initial layer in the vicinity of $t = 0$ and to introduce the short time: $\tau = t/M$. But, when $\tau \to \infty$, we see again that only the case when the motion of the wall from the rest is progressive, during a time $O(1)$, ensures the matching of the local acoustic model with the 'naïve' main (quasi-incompressible Euler) model when $t \to \infty$.

As a consequence, if the wall $\Sigma(t)$ is started impulsively from rest, then at the end of the stage $t = O(M)$ the acoustic oscillations remain undamped and come in the stage $t = O(1)$. A multiple-time scale technique is necessary for the investigation of the long-time evolution of these rapid oscillations.

Unfortunately, a *double-time scale* (t and $\tau = t/M$) technique, *is not sufficient* to eliminate the all secular terms. Indeed, with the unit of speed chosen, the speed of sound is $O(1/M)$ and consequently the periods of natural (free) vibrations of the bounded domain $D(t)$ are $O(M)$.

Therefore, we expect that the solution oscillates on a time scale $O(M)$, while, on the other hand, it evolves on a time scale $O(1)$ due to the slow evolution of $\Sigma(t)$. As a consequence we must introduce into the structure of the solution a multiplicity (an 'enumerable' infinity!) of fast time scales. First we should use the time t, a slow time, and then we should bring into the solution an infinity of fast times designed to cope with the infinity of periods of free vibrations of $D(t)$.

With

$$U = (\boldsymbol{u}, p, \rho, S)^T,$$

where $\boldsymbol{u}$ is the velocity, p, ρ, S, the pressure, density, and specific entropy, we set U^* for U expressed through this variety of time scales and we write:

$$\frac{\partial U}{\partial t} = \frac{\partial U^*}{\partial t} + \frac{1}{M}\boldsymbol{D}U^*, \tag{8.40}$$

where $\partial U^*/\partial t$ stands for the time derivative computed when all fast times are maintened constant, while $\boldsymbol{D}U^*$ is the time derivative operator (in a non-explicit form) occuring through fast times. Then, it is necessary to introduce the following decomposition:

$$U^* = \langle U^* \rangle + U^{*\prime}, \tag{8.41}$$

such that the averaged function $\langle U^* \rangle$ is only dependent on the slow time t, while the fluctuation $U^{*\prime}$ is a function of all fast time. Obviously $\langle U^{*\prime} \rangle = 0$.

In this case, via the asymptotic expansion:

$$U^* = \sum_{n \geq 0} M^n U^*_n, \tag{8.42}$$

and in the framework of the Euler equations, first, we can study the rapid acoustic oscillations and derive the ordinary differential equation which governs the long- time evolution of their amplitudes.

Then, for the slow variation, we can derive a system of averaged quasi-incompressible Euler equations, with a fictitious pressure which includes the energy of the acoustic oscillations.

When the Reynolds number, Re, is not infinite but very large, for slightly viscous flow, we must start from the full equations for a compressible, viscous, and heat conducting fluid. In this case, we must expect that the oscillations are damped out, but a precise analysis of this damping phenomenon appears to be a difficult problem and raises many questions.

8.4.2. Formulation of the inviscid problem

The inviscid Euler equations for a perfect gas with constant specific heats, C_P and C_V ($\gamma = C_P/C_V$), read in the *dimensionless* form as follows:

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\boldsymbol{u} = 0\,, \tag{8.43a}$$

$$\frac{D\boldsymbol{u}}{Dt} + \frac{1}{\gamma M^2}\nabla p = 0\,, \tag{8.43b}$$

$$\frac{DS}{Dt} = 0\,, \tag{8.43c}$$

$$p = \rho^{\gamma}\exp(S), \tag{8.43d}$$

where $D/Dt = \partial/\partial t + \boldsymbol{u}\cdot\nabla$, with standard notation.

In equations (8.43a, b, c), the Strouhal number, $S = L^\circ/U^\circ t^\circ \equiv 1$, where L° is a typical length scale of the bounded domain $D(t)$, U° *is* a characteristic constant velocity related to the motion of the wall $\partial D(t) \equiv \Sigma(t)$ and t° is a characteristic time, such that: $t^\circ = L^\circ/U^\circ$.

When $t = 0^-$, we have, as dimensionless initial conditions:

$$\boldsymbol{u} = 0,\ S = 0,\ p = \rho = 1. \tag{8.44}$$

On the wall $\Sigma(t)$, for the Eulerain inviscid fluid flow, we write the slip condition:

$$\boldsymbol{u}\cdot\boldsymbol{n} = W_{\Sigma}(t, \boldsymbol{P}), \tag{8.45}$$

where the velocity $W_{\Sigma}(t, \boldsymbol{P})$, characterizes the normal displacement of the wall $\Sigma(t)$. In condition (8.45), $\boldsymbol{n}$ is the unit vector normal to the wall $\Sigma(t)$, directed inside $D(t)$ and $\boldsymbol{P}$ is the position vector on $\Sigma(t)$.

The low compressibility is characterized by a small reference (constant), Mach number: $M = U^\circ/a^\circ << 1$, where a° is the associated reference sound speed, and for a perfect gas with constant specific heats: $a^\circ = (\gamma p^\circ/\rho^\circ)^{1/2}$, where p° and ρ° are the corresponding reference constant pressure and density (when $\boldsymbol{u} = 0$).

But, it is also necessary to take into account the conservation of the global mass $\boldsymbol{M}^\circ$ of the bounded domain. In the dimensionless reduced form we write:

$$\rho = \frac{1}{V(t)}, \tag{8.46}$$

with

$$V(t) = \frac{\rho^\circ |D(t)|}{M^\circ},\ V(0) = 1, \tag{8.47}$$

where $|D(t)|$ is the volume of the bounded domain $D(t)$ - a known function of t.

Below we consider the case when the motion of the wall $\Sigma(t)$ is started impulsively from rest and in this case:

$$\boldsymbol{u} \cdot \boldsymbol{n} = H(t)\, w_\Sigma(P), \text{ all along } \Sigma(t), \tag{8.48}$$

where $H(t)$ is the so-called Heaviside (or unit) function, such that

$$\lim_{t \to 0^+} H(t) \equiv 1 \text{ and } H(t = 0^-) \equiv 0. \tag{8.49}$$

8.4.3. The persistence of acoustic oscillations

Now we consider the asymptotic expansion (8.42), and from the Euler equations (8.43a, b, c) we find, first, that to zeroth-order, p_0^* and ρ_0^* cannot depend on any fast time but may depend on the slow time t; they do not depend on position either, furthemore:

$$p_0^*(t) = \rho_0^*(t)^\gamma \text{ and } \rho_0^* = \frac{1}{V(t)}. \tag{8.50a, b}$$

To order $O(M)$, we derive the following system of equations from (8.43a, b, c, d):

$$\boldsymbol{D}\rho^*{}_1 + \rho^*{}_0(\nabla \cdot \boldsymbol{u}^*{}_0) + \frac{d\rho^*{}_0}{dt} = 0, \tag{8.51a}$$

$$\frac{1}{\gamma}\nabla p^*{}_1 + \rho^*{}_0 \boldsymbol{D}\boldsymbol{u}^*{}_0 = 0, \tag{8.51b}$$

$$\boldsymbol{D} S^*{}_1 = 0, \tag{8.51c}$$

with

$$p^*{}_1 = \gamma \frac{p^*{}_0}{\rho^*{}_0} \rho^*{}_1, \tag{8.51d}$$

and

$$\boldsymbol{u}^*{}_0 \cdot \boldsymbol{n} = W_\Sigma(t, \boldsymbol{P}), \text{ all along } \Sigma(t). \tag{8.51e}$$

From the above system (8.51a, b, c, d, e) we derive, first, the following averaged equation for $<\boldsymbol{u}^*{}_0>$:

$$\nabla \cdot <\boldsymbol{u}^*{}_0> = \frac{d \log(\boldsymbol{V}(t))}{dt}, \tag{8.52a}$$

with

$$<\boldsymbol{u}^*{}_0> \cdot \boldsymbol{n} = W_\Sigma(t, \boldsymbol{P}), \text{ all along } \Sigma(t). \tag{8.52b}$$

Then, from (8.51c), if we take into account the initial condition (8.44) for S, and (8.50b), we have:

$$S^*{}_1(t, \boldsymbol{x}) \equiv 0. \tag{8.52c}$$

Next, from (8.51d), obviously

$$p^{*\prime}{}_1 = \gamma \frac{p^*{}_0}{\rho^*{}_0} \rho^{*\prime}{}_1. \tag{8.52d}$$

We note that, as a consequence of the conservation of the global mass of $D(t)$, which is expressed by (8.46), the averaged value of $\rho^*{}_1$ is obviously zero and this is also true for $p^*{}_1$ (see (8.51d)). Indeed, the fluctuations $\rho^{*\prime}{}_1$ and $\boldsymbol{u}^{*\prime}{}_0$, according to the equations (8.51a, b), are solution of the following system of two equations:

$$\boldsymbol{D}\rho^{*\prime}{}_1 + \rho^*{}_0[\nabla \cdot \boldsymbol{u}^{*\prime}{}_0] = 0, \tag{8.53a}$$

$$\frac{1}{\gamma} \nabla p^{*\prime}{}_1 + \rho^*{}_0 \boldsymbol{D} \boldsymbol{u}^{*\prime}{}_0 = 0, \tag{8.53b}$$

with

$$\boldsymbol{u}^{*\prime}{}_0 \cdot \boldsymbol{n} = 0, \text{ all along } \Sigma(t), \tag{8.53c}$$

where $p^{*\prime}{}_1$ is given, as a function of $\rho^{*\prime}{}_1$, by (8.52d).

We can easily write a solution of (8.53a, b), for $\rho^{*\prime}{}_1$ and $\boldsymbol{u}^{*\prime}{}_0$, if we take into account (8.52d) and introduce the normal modes of D(t), $[\boldsymbol{U}_n(t, \boldsymbol{x}), \rho_n(t, \boldsymbol{x})]$, with eigen-frequencies $\omega_n(t)$; namely:

$$\boldsymbol{u}^{*\prime}{}_0 = \sum_{n\geq 1}\left\{A_n(t)\cos\left[\frac{\phi_n(t)}{Ma}\right] - B_n(t)\sin\left[\frac{\phi_n(t)}{Ma}\right]\right\}\boldsymbol{U}_n(t,\boldsymbol{x}), \tag{8.54a}$$

$$\rho^{*\prime}{}_1 = \rho_0{}^*\left[\frac{\rho_0{}^*}{p_0{}^*}\right]^{1/2}\sum_{n\geq 1}\left\{A_n(t)\sin\left[\frac{\phi_n(t)}{Ma}\right] - B_n(t)\cos\left[\frac{\phi_n(t)}{Ma}\right]\right\}\rho_n(t,\boldsymbol{x}), \tag{8.54b}$$

where

$$\frac{d\phi_n(t)}{dt} = \left[\frac{p_0{}^*(t)}{\rho_0{}^*(t)}\right]^{\frac{1}{2}}\omega_n(t),\ \phi_n(0) = 0. \tag{8.55a}$$

More precisely, $\omega_n(t)$ is one of the acoustic frequencies corresponding to the shape of the bounded domain $D(t)$ at time t, while the pair: $[\boldsymbol{U}_n(t,\boldsymbol{x}), \rho_n(t,\boldsymbol{x})]$ serves to define the normal modes of oscillation at frequency $\omega_n(t)$, normalised according to the relation:

$$\int_{D(t)}\left[|\boldsymbol{U}_n|^2 + \rho_n{}^2\right]dv = 1. \tag{8.55b}$$

For $[\boldsymbol{U}_n(t, \boldsymbol{x}), \rho_n(t, \boldsymbol{x})]$ we have the following differential system:

$$\omega_n\rho_n + \nabla\cdot\boldsymbol{U}_n = 0,\ -\omega_n\boldsymbol{U}_n + \nabla\rho_n = 0, \tag{8.55c}$$

$$\boldsymbol{u}_n \cdot \boldsymbol{n} = 0, \text{ all along } \Sigma(t). \tag{8.55d}$$

We note that the above relation (8.55a) gives, in fact, the definition of the dimensionless fast time in relation to the sound speed within $D(t)$, at time t, and $\omega_n(t)$. From (8.55c), each $\boldsymbol{U}_n(t, \boldsymbol{x})$ is irrotational, namely $\boldsymbol{U}_n = \nabla(\rho_n/\omega_n)$.

8.4.4. Averaged equations for the slow variation

First, to order $O(M^2)$, we obtain again that:

$$DS^*_2 = 0 \Rightarrow S^*_2 = S^*_2(t, x) \equiv 0. \tag{8.56}$$

Then from the equation of state (8.43d) we derive for p^*_2 the following relation:

$$p^*_2 = \frac{p^*_0}{\rho^*_0}\left[\rho^*_2 + \frac{1}{2\rho^*_0}(\gamma - 1)(\rho^*_1)^2\right]. \tag{8.57}$$

Now, to order $O(M^2)$, we derive a system of two equations, from (8.43a, b), for ρ^*_2/ρ^*_0 and u^*_1, if we take into account (8.57); namely

$$D\left(\frac{\rho^*_2}{\rho^*_0}\right) + \nabla \cdot u^*_1 = G, \tag{8.58a}$$

$$\frac{p^*_0}{\rho^*_0}\nabla\left(\frac{\rho^*_2}{\rho^*_0}\right) + Du^*_1 = F, \tag{8.58b}$$

where

$$G = -\left\{\frac{\partial}{\partial t}\left(\frac{\rho^*_1}{\rho^*_0}\right) + \nabla \cdot \left[\frac{\rho^*_1}{\rho^*_0}u^*_0\right]\right\} + \frac{\rho^*_1}{\rho^*_0}\frac{d\log(V(t))}{dt}, \tag{8.59a}$$

$$F = -\left\{\frac{\partial u^*_0}{\partial t} + (u^*_0 \cdot \nabla)u^*_0 + (\gamma - 2)\frac{p^*_0}{\rho^*_0}\frac{\rho^*_1}{\rho^*_0}\nabla\left(\frac{\rho^*_1}{\rho^*_0}\right) + \frac{\rho^*_1}{\rho^*_0}Du^*_0\right\}. \tag{8.59b}$$

The equations (8.58a, b) are inhomogeneous (the boundary condition being homogeneous) and their homogeneous version has the free oscillations of the bounded domain *D(t)* as eigen-functions. Examining the structure of the right hand side (according to (8.59a, b)) we find that they contain terms which oscillate with precisely these frequencies and this leads to resonances that we must eliminate, otherwise they would contribute to secular terms in ρ^*_2 and u^*_1. Indeed, in place of (8.58b), with (8.59b), if we do not take account of (8.57), we obtain as a consequence of (8.43b) the following equation:

$$\frac{\partial \boldsymbol{u}^*_0}{\partial t} + (\boldsymbol{u}^*_0 \cdot \nabla)\boldsymbol{u}^*_0 + \nabla\left(\frac{p^*_2}{\gamma\rho^*_0}\right) + \boldsymbol{Du}^*_1 + \frac{\rho^*_1}{\rho^*_0}\boldsymbol{Du}^*_0 = 0,$$

and when we apply the averaging process (over all rapid oscillations), we obtain:

$$\frac{\partial < \boldsymbol{u}^*_0 >}{\partial t} + < (\boldsymbol{u}^*_0 \cdot \nabla)\boldsymbol{u}^*_0 > + \nabla\left(\frac{< p^*_2 >}{\gamma\rho^*_0}\right) + < \frac{\rho^*_1}{\rho^*_0}\boldsymbol{Du}^*_0 >= 0.$$

If now we utilize the solution (8.54b) and (8.54a) for the fluctuations $\rho^{*\prime}_1$ and $\boldsymbol{u}^{*\prime}_0$, and also (8.55c), then we can write in place of this above averaged equation, the following *averaged equation of motion* for $<\boldsymbol{u}^*_0>$:

$$\frac{\partial < \boldsymbol{u}^*_0 >}{\partial t} + (< \boldsymbol{u}^*_0 > \cdot \nabla) < \boldsymbol{u}^*_0 > + \nabla \Pi_0 = 0, \tag{8.60}$$

with

$$\Pi_0 = \frac{< p^*_2 >}{\gamma\rho^*_0} + \frac{1}{4}\sum_{n\geq 1}(A_n^2 + B_n^2)\left[|\mathcal{U}_n|^2 - |\rho_n|^2\right]. \tag{8.61}$$

Thus, at this stage we have derived an averaged system of two equations for the averaged velocity $<\boldsymbol{u}^*_0>$ and fictitious pressure Π_0; namely equations (8.52a) and (8.60).

As a initial condition (at $t = 0$), we can write:

$$t = 0: < \boldsymbol{u}^*_0 >= -\sum_{n\geq 1} A_n(0)\mathcal{U}_n(0, \boldsymbol{x}), \tag{8.62}$$

if we take into account the solution (8.54a) for $\boldsymbol{u}^*_0$. As slip condition on $\Sigma(t)$ we have [according to (8.52b)]:

$$<\boldsymbol{u}^*_0> \cdot \boldsymbol{n} = w_\Sigma(P), \text{ all along } \Sigma(t),\ t > 0, \tag{8.63}$$

if we take into account (8.49). From equation (8.60) we find that the averaged velocity (over all rapid oscillations) behaves as if the perfect gas were incompressible with density a function of the slow time *t*, as seen from (8.52a).

8.4.5. The long time evolution of the rapid oscillations

Indeed, in above Section 8.4.4, we have only eliminated part of the secular terms in ρ^*_2/ρ^*_0 and $\boldsymbol{u}^*_1$. Here, we shall eliminate all others secular terms. The elimination of these secular terms is based on the classical Fredholm alternative. But, the system (8.58a, b), for ρ^*_2/ρ^*_0 and $\boldsymbol{u}^*_1$, is linear, and as a consequence we can write its solution as a superposition of particular solutions corresponding to various terms in the right hand sides of the equations (8.58a) and (8.58b) of this system according to (8.59a, b).

First, the right hand sides (8.59a, b) can be written in the form:

$$\boldsymbol{G} = <\boldsymbol{G}> + \left(\frac{p^*_0}{\rho^*_0}\right)^{1/2} \sum_{n\geq 1}\left[G_{nc}\cos\left[\frac{\phi_n}{Ma}\right] - G_{ns}\sin\left[\frac{\phi_n}{Ma}\right]\right] + G^*, \quad (8.64a)$$

$$\boldsymbol{F} = <\boldsymbol{F}> + \left(\frac{p^*_0}{\rho^*_0}\right) \sum_{n\geq 1}\left[\boldsymbol{F}_{ns}\sin\left[\frac{\phi_n}{Ma}\right] - \boldsymbol{F}_{nc}\cos\left[\frac{\phi_n}{Ma}\right]\right] + \boldsymbol{F}^*, \quad (8.64b)$$

In the above relations (8.64a) and (8.64b), the G^* and $\boldsymbol{F}^*$, include all terms proportional to $\cos[(\phi_p \pm \phi_q)/Ma]$ or $\sin[(\phi_p \pm \phi_q)/Ma]$, and below we assume that there are no resonant triads satisfying the relation:

$$|\phi_p \pm \phi_q| = \phi_r,$$

thus none of the terms in G^* and $\boldsymbol{F}^*$ can interfere with any of the terms in $\Sigma_{n\geq 1}$.

Then, we can write the solution of the two equations (8.58a, b), corresponding to the terms in $\Sigma_{n\geq 1}$, in the following form:

$$\frac{\rho^*_2}{\rho^*_0} = \sum_{n\geq 1}\left[R_{nc}\cos\left[\frac{\phi_n}{Ma}\right] + R_{ns}\sin\left[\frac{\phi_n}{Ma}\right]\right], \quad (8.65a)$$

$$u^*_1 = \left(\frac{p^*_0}{\rho^*_0}\right)^{1/2} \sum_{n\geq 1}\left[U_{nc}\cos\left[\frac{\phi_n}{Ma}\right] - U_{ns}\sin\left[\frac{\phi_n}{Ma}\right]\right], \tag{8.65b}$$

and, for example, the amplitudes R_{ns} and U_{nc} satisfy the system:

$$\omega_n R_{ns} + \nabla\cdot U_{nc} + G_{nc} = 0, \tag{8.66a}$$

$$-\omega_n U_{nc} + \nabla R_{ns} + F_{ns} = 0, \tag{8.66b}$$

$$\mathbf{U}_{nc}\cdot\mathbf{n} = 0 \text{ on } \Sigma(t). \tag{8.66c}$$

Obviously, for R_{nc} and U_{ns} we obtain a similar system, but in place of G_{nc} we have G_{ns} and, in place of F_{ns}, F_{nc}.

Finally, if we utilize (8.55c, d) and also (8.66a, b, c) we derive a compatibility relation (as a consequence of the Fredholm alternative), which ensures the existence of a solution of the above system (8.66a, b, c); namely:

$$\iint_{D(t)} [G_{nc}\rho_n - F_{ns}\cdot \mathcal{U}_n] dv = 0, \tag{8.67}$$

and a similar relation exists if we write G_{ns} and F_{nc} in place of G_{nc} and F_{ns} respectively.

Now, it is necessary to calculate the terms G_{nc}, F_{ns}, G_{ns} and F_{nc} explicitly according to (8.64a, b), with (8.59a, b), (8.54a, b), (8.52a, b) and also (8.55b).

A straightforward calculation gives the following ordinary differential equation for the amplitudes $A_n(t)$ and $B_n(t)$:

$$\frac{dY_n}{dt} + \beta_n(t) Y_n = 0, \tag{8.68a}$$

where

$$\beta_n(t) = \frac{1}{2\rho_0^*}\frac{d\rho_0^*}{dt} + \iint_{D(t)} [\mathcal{U}_n \cdot [\nabla < u^*_0 >]\cdot \mathcal{U}_n] dv. \tag{8.68b}$$

In (8.68a), $Y_n(t)$ stands for either $A_n(t)$ or $B_n(t)$.

We derive initial conditions for $A_n(t)$ and $B_n(t)$ by applying the initial conditions (8.44), and this gives (8.62) and $\rho_1^*(0, \boldsymbol{x}) = 0$. As a consequence $B_n(0) = 0$ and thanks to equation (8.68a) we obtain:

$$B_n(t) \equiv 0 \text{ (is zero for all } t). \tag{8.69}$$

Concerning $A_n(0)$ its value must be derived from (8.62) and depends on the value of $<\boldsymbol{u}^*_0>$ at $t = 0$. On the other hand, *obviously*, if:

$$W_\Sigma(0, \boldsymbol{P}) = 0, \text{ then } <\boldsymbol{u}^*_0> \text{ is also zero at } t = 0, \text{ then } A_n(0) = 0,$$

which implies that also:

$$A_n(t) \equiv 0 \text{ (is zero for all } t),$$

and then the *oscillations are absent*!

But, if the motion of the deformable wall of the bounded domain is started impulsively from rest (or accelered from rest to a finite velocity in a time O(M)), then according to (8.48) we have:

$$W_\Sigma(0^+, \boldsymbol{P}) \equiv w_\Sigma(\boldsymbol{P}) \neq 0, \tag{8.70}$$

and the same holds for the averaged velocity, $<\boldsymbol{u}^*_0>$. *In this case:*

$$A_n(0) \neq 0 \text{ and } \textit{as a consequence } A_n(t) \textit{ is also non zero.}$$

As a conclusion:

If the motion of the deformable wall of the bounded domain, where the inviscid gas is confined, is started impulsively from rest, then the acoustic oscillations remain present and have an effect on the pressure which would be felt by a gauge, and would not be related to the mean (averaged) motion. The same holds if the motion of the wall is accelerated from rest to a finite velocity in a time O(M).

In this both case, for the averaged functions we obtain the following system:

$$\nabla\cdot <\boldsymbol{u}^*_0> = \frac{d\,log(\boldsymbol{V}(t))}{dt},$$

$$\frac{\partial < u^*_0 >}{\partial t} + \left(< u^*_0 > \cdot \nabla\right) < u^*_0 > + \nabla \Pi_0 = 0,$$

with

$$\Pi_0 = \frac{< p^*_2 >}{\gamma \rho^*_0} + \frac{1}{4} \sum_{n \geq 1} A_n^2 \left[|u_n|^2 - |\rho_n|^2 \right],$$

and the pressure is strongly affected by the acoustic oscillations via the second term in the relation for Π_0.

8.4.6. On viscous damping of the oscillations

All the preceding asymptotic analysis was completed on the assumption that the fluid is inviscid. If we deal with a slightly viscous flow, then we must start from the full NS-F equations in place of the Euler equations (8.43a, b, c, d) and bring into the asymptotic analysis a second small parameter $1/Re = \varepsilon^2$, the inverse of a characteristic Reynolds number. Then we must expect that the oscillations are damped out with time?

A precise analysis of this damping phenomenon appears to be a difficult problem and raises many questions. Here we shall be content with an order of magnitude estimate of the damping time, which is obtained when we assume that:

$$M\,Re >> 1.$$

First it is necessary to change the relation (8.40); namely we write:

$$\frac{\partial U}{\partial t} = \frac{\partial U^*}{\partial t} + \frac{1}{M} \mathbf{D} U^* + \delta \mathbf{T} U^*, \tag{8.71}$$

with $\delta << 1$, where the derivative operator $\mathbf{T}$ is related with the damping short times. Then we may carry out an analysis analogous to the one of Section 8.4.4 to order $O(\delta M)$. It appears that the secular terms related to the derivative operator $\mathbf{T}U^*_0$ can be eliminated only if the condition, on the wall Σ, for the velocity is inhomogeneous!

Such an inhomogeneous condition may be provided by a boundary-layer analysis of the NS-F equations for $Re >> 1$ $(\varepsilon^2 << 1)$ near Σ.

For our problem with $M << 1$ and $\varepsilon << 1$, the BL is in fact a Stokes layer of thickness $O(\varepsilon M^{1/2}) = O\left(\sqrt{M/Re}\right)$.

Next, it is found that the flux away from the Stokes layer is also of order $O\left(\sqrt{M/Re}\right)$ and must be equilibrated with the TU^*_0 terms of order δM.

Finally, we find the required estimate,

$$t \sim \sqrt{MRe}\,, \tag{8.72}$$

since

$$\delta M \sim \sqrt{\frac{M}{Re}} \Rightarrow \delta = \frac{1}{\sqrt{MRe}} \text{ and } \delta t = O(1).$$

8.5. LOW-MACH NUMBER FLOWS AND COMPUTATIONAL AEROACOUSTICS

Computational aeroacoustics has been emerging as a new branch of computational fluid dynamics to predict sound generation and propagation in fluids. Besides noise prediction, the computation of compressible low-Mach number flow has important applications in heat and mass transfer and in combustion. In the framework of the present book, obviously, it is not our role to discuss the various aspects of this new branch of CFD. We give only some information concerning recent references and first we mention the review paper (Lecture) by Müller (1999), where the first part of Lecture gives insight into the low Mach number limit of the NS-F equations by means of asymptotic methods. Unfortunately, again, the derived models are questionable!For example, Müller (1998) considers (via a double time scale analysis) the compressible Navier-Stokes equations at low Mach number, when the slow flow is affected by acoustic effects in a bounded domain *D* over a long time, and as an application, Müller (1998), mentions a closed piston-cylinder system, in which the isentropic compression due to a slow piston motion is modified by acoustic waves. According to Schneider's book (1978, pp.235-240), Müller (1998) claims that secular terms appearing in the double time scale (!) analysis of the closed piston-cylinder system can be eliminated by perturbing the coordinates. That 'technique' is referred by Schneider (1978) as the analytic method of characteristics and mentioned by Müller (1998, p.98). Schnieder (1978) provided an enlightening description of the non-reacting, inviscid piston-cylinder problem and suggested a first asymptotic solution ansatz, which, however, *did not lead to a uniformly valid long time solution at the time.* Nevertheless, Schneider's approach has greatly inspired the work by Klein and Peters (1988). In Muller (1996 and 1999) the reader can find various interesting references related to the low-Mach number asymptotics of NS-F equations and its applications to aeroacoustics.

It seems that a fundamental paper in this area is the paper by Crighton (1993). Concerning reacting flows we mention the paper by Dwyer (1990). In the paper by Gustafsson and Stoor (1991), the NS model for an 'almost' incompressible flow is considered and Zank and Matthaeus (1991) discuss the equations of 'nearly' incompressible fluids. Finally, we mention the thesis by Viozat (1998) concerning the numerical computations of steady and unsteady flows at low Mach number. The above cited papers will be of interest to numericians concerned with the proposed 'new strategy for the construction of uniformly applicable discretization'. These papers give, in fact, some "ad hoc" approximate equations and methods for the resolution of various fluid dynamical problems.

The approach requires a close interplay between application-oriented asymptotics (!) and numerical analysis and is a highly intuitive process that is based, to a large extent, on physical reasoning (the 'craft', by which this approach is accomplished is called "Practical Asymptotics"!).

But, I think that the major problem with this 'sort' of 'pseudo-asymptotics oriented papers', is that they are not devoted to asymptotic modelling in the sense of being assisted by the spirit of asymptotic techniques, in a rational, coherent and consistent framework.

Finally, concerning the recently introduced notion of "asymptotically adaptive numerical method" (see, for instance, the recent paper by Klein *et al.* (1999)), which is possibly a good idea and gives a fruitful approach for the numericians, obviously, at least from my point of view, this approach does not have any relation to asymptotic modelling.

CHAPTER 9

LOW-REYNOLDS NUMBERS ASYMPTOTICS

In this Chapter we consider mainly, as exact governing equations, the Navier equations for an incompressible and viscous fluid. But, first, in §9.1, we discuss briefly the compressible case when the exact governing equations are the full NS-F equations.

In the book by Happel and Brenner (1965), devoted to Low Reynolds Number Hydrodynamics, the reader can find a classical approach - the treatment developed in this book being based almost entirely on the linearized form of the equations of motion which results from omitting the inertial terms from the Navier-Stokes equations, giving the so-called creeping motion or Stokes equations. According to the authors of the above mentioned book:

This is tantamount to assuming that the particle Reynolds numbers are very small. Many systems which involve bulk flow relative to external boundaries at high Reynolds numbers are still characterized by low Reynolds numbers as regards the movement of particles relative to the fluid. Also, inertial effects are less important for systems consisting of a number of particles in a bounded fluid medium than they are for the motion of a single particle in an unbounded fluid.

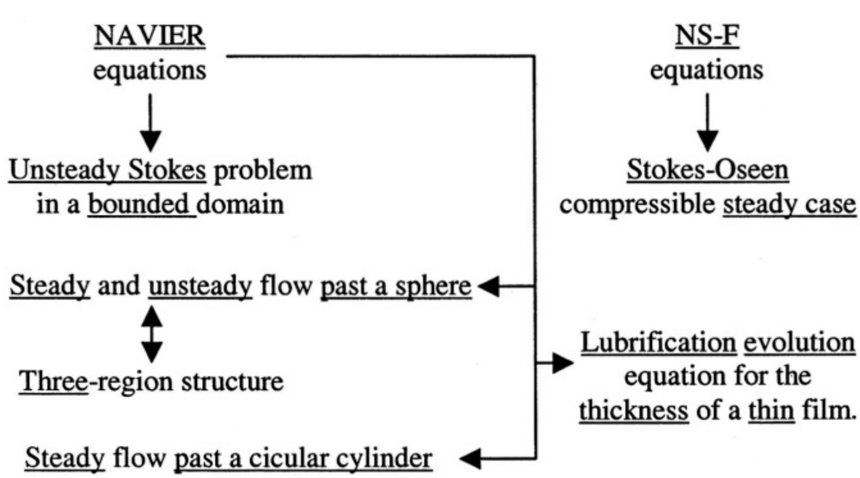


Fig. 9.1 Low-Reynolds numbers asymptotics

9.1. COMPRESSIBLE FLOW AT LOW REYNOLDS NUMBER

Two limit processes play a fundamental role (see the discussion of P.A. Lagerstrom (1964)) in the asymptotic study of low Reynolds number flow, namely the Stokes limit (inner limit) and the Oseen limit (outer limit).

But for a compressible flow, when $Re \to 0$, it is necessary to specify also the role of the Mach number M! In fact, it is necessary to pose the problem concerning the behaviour of solutions of the full dimensionless Navier-Stokes-Fourier equations, (2.56a, b, c), with (2.56d), when simultaneously:

$$Re \to 0 \text{ and } M \to 0. \tag{9.1}$$

Naturally, for the validity of the NS-F equations, it is obvious that it is assumed that the limiting compressible flow at low Reynolds and Mach numbers remains a continuous medium and this implies that the Knudsen (dimensionless) number is a small parameter:

$$Kn = \frac{M}{Re} << 1. \tag{9.2}$$

As a consequence, the double limit process (9.1) must be made with the following similarity relation:

$$M = R^*(Re)^{1+a}, \text{ with } R^* = O(1) \text{ and } a > 0 \text{ when } Re \to 0. \tag{9.3}$$

First the Stokes (compressible - proximal) limit is denoted by Lim^{St} and defined by (with $Bo = 0$):

$$Lim^{St} = [Re \to 0, M \to 0;\ \|\boldsymbol{x}\|, t, R^*, S, Pr,\ \gamma \text{ fixed} = O(1)]. \tag{9.4}$$

Similarly, the Oseen (compressible - distal) limit is denoted by Lim^{Os} and defined by (again for $Bo = 0$):

$$Lim^{Os} = [Re \to 0, M \to 0;\ \|\boldsymbol{\xi}\|, t, R^*, S, Pr, \gamma \text{ fixed} = O(1)], \tag{9.5}$$

where $\boldsymbol{\xi}$ is the Oseen space variable, namely:

$$\boldsymbol{\xi} = Re\, \boldsymbol{x}. \tag{9.6}$$

In general, the distal, Oseen limit is applied only at points which are not on the boundary of the body - in fact, for the points far away from boundary of the body. The Stokes, proximal limit may be thought of as the limit where the "viscosity tends to infinity" while the other parameters (except the Mach number, *M,* which satisfies the similarity relation (9.3)) are kept fixed.

From this it is evident that: in the Stokes limit the influence of the boundary is strong and for this reason the Stokes limit is called a proximal limit.

Obviously, the value of the velocity vector $\boldsymbol{u}$ at the wall, i.e., zero, will have an increasingly large influence on any fixed point, but not on the boundary, as the viscosity tends to infinity.

In fact, for any given viscosity one can find points sufficiently far away from the boundary of the body such that $\boldsymbol{u}$ there is arbitrary close to $U_\infty \boldsymbol{i}$, when we assume ($\boldsymbol{u}$ is a dimensionless velocity vector) that: $\boldsymbol{u} \to U_\infty \boldsymbol{i}$, at infinity, where the free stream direction is taken to coincide with the positive horizontal x_1 direction so that $\boldsymbol{u}(\infty)$ may be written as: $U_\infty \boldsymbol{i}$, with $U_\infty =$ *const.*

The Oseen limit may be thought of as the limit where the characteristic length L_C (a measure of a finite body) tends to zero while the other parameters (except *M*) and the point (in three-dimensional space) are fixed. The body then tends to a conical body, i.e. a body for which no length can be defined! The limiting conical body may in particular be a point, a line, or a semi-infinite line. Such objects have no arresting power, i.e. they cannot cause a finite disturbance in the fluid and as a consequence:

In the Oseen limit, conditions at infinity determine the value of the limit.

For this reason the Oseen limit is called a distal limit, and the discussion below is mainly concerned with finite two-dimensional (2D) or three-dimensional (3D) bodies. In the former case the limiting cone is a 2D point, i.e. an infinite line in 3D space, and in the later case the limiting cone is a point.

Note, also, that the Oseen limit is non-uniform at the body. No matter how small L_C is, points sufficiently close to the body may be found where $\boldsymbol{u}$ is arbitrarily small. Following Lagerstrom (1964, p. 164), we now let $\varepsilon_j(Re)$ be the sequence of functions of *Re* only, such that:

$$\varepsilon_0(Re) = 1, \text{ and } lim[\frac{\varepsilon_{j+1}(Re)}{\varepsilon_j(Re)}] = 0, Re \to 0. \tag{9.7}$$

The ε_j are thus of successively smaller order of magnitude.

For a 2D flow, the following form of the composite asymptotic expansion has been proposed by S.Kaplun (1957) - but, indeed, only for the incompressible case

$$\boldsymbol{u} \approx \sum \varepsilon_j(Re)\left[\boldsymbol{F}_j(t,\xi)+\boldsymbol{G}_j(t,\boldsymbol{x})\right], j = 0,\ldots,\infty. \quad (9.8)$$

The terms $\boldsymbol{F}_j(t, \xi)$ are obtained from $\boldsymbol{u}$ by a repeated application of the distal (outer-Oseen) limit and the $\boldsymbol{G}_j(t, \boldsymbol{x})$ are then transcendentally small except at the body. They are corrections terms needed to make the expansion unifomly valid in the neighbourhood of the body. Formally the terms $\boldsymbol{G}_j(t, \boldsymbol{x})$ are obtained by a repeated application of the Stokes, proximal, limit to:

$$\frac{\left[\boldsymbol{u}-\sum \varepsilon_j(Re)\boldsymbol{F}_j(t,\xi)\right]}{\varepsilon_j(Re)}, j = 0,\ldots,n$$

where n is a sufficiently large number.

A 'pure' Oseen expansion: $\boldsymbol{F}_0 + \varepsilon_1(Re)\,\boldsymbol{F}_1 + \ldots$, is not uniformly valid near the body. For example, the term $\varepsilon_1(Re)\boldsymbol{G}_1$, which is transcendentally small off the boundary, is of order unity at the boundary.

The equations for the first terms of the Stokes and Oseen expansions may be found by inserting the expansion (9.8) into the NS-F equations (2.56a)-(2.56c). Certain conditions for matching the inner flow and the outer flow have to be used as boundary conditions. These matching conditions are somewhat more complicated than in the high Reynolds number case (see the Chapter 7). An example of the matching procedure (for the incompressible case) is given in §9.3.

9.1.1. The Stokes limiting case and the steady compressible Stokes equations

We consider the Stokes limit of the steady NS-F equations. If we take into account the similarity relation (9.3), then, when $Re \to 0$, it is necessary to study the approximate solutions of the NS-F equations in the following asymptotic form (when $Bo = 0$):

$$\boldsymbol{u} = \boldsymbol{u}_S + \ldots;\ p = 1 + (Re)^{1+2a}\,[p_S + \ldots];\ \rho = \rho_S + \ldots;\ T = T_S + \ldots, \quad (9.9)$$

where the "Stokes" limiting functions (with "$_S$" as subscript) depends only on the position variable $\boldsymbol{x} = (x_i)$, $i = 1, 2, 3$. In this case for $\boldsymbol{u}_S$, p_S, ρ_S and T_S

we derive an "à la Stokes" steady compressible system which is written in three parts:

$$\nabla\cdot[k(T_S)\nabla T_S]=0, \tag{9.10a$_1$}$$

$$T_S = T_W(P,\ \tau) \equiv 1 + \tau\ \Xi(P), P \in \partial\Omega, \text{ on } \partial\Omega; \tag{9.10a$_2$}$$

$$\rho_S = \frac{1}{T_S}; \tag{9.10b}$$

$$\nabla\cdot\boldsymbol{u}_S = \boldsymbol{u}_S\cdot\nabla \log T_S, \tag{9.10c$_1$}$$

$$\nabla p_S = \gamma(R^*)^2\nabla\cdot[2\mu(T_S)\boldsymbol{D}(\boldsymbol{u}_S)+\lambda(T_S)(\nabla\boldsymbol{u}_S)\boldsymbol{I}], \tag{9.10c$_2$}$$

$$\boldsymbol{D}(\boldsymbol{u}_S)=\frac{1}{2}\left[\nabla\boldsymbol{u}_S+(\nabla\boldsymbol{u}_S)^T\right], \tag{9.10c$_3$}$$

$$\boldsymbol{u}_S = 0 \text{ on } \partial\Omega. \tag{9.10c$_4$}$$

The equation (9.10a$_1$) with the associated boundary condition on $\partial\Omega$ (9.10a$_2$) determines T_S, as soon as the temperature behaviour at infinity is specified. The equation (9.10b) is a relation (a limiting form of the equation of state (2.56d)) between T_S and ρ_S and determines ρ_S when T_S is known.

Finally, the two equations (9.10c$_1$, c$_2$) give a closed system for the determination of the velocity vector $\boldsymbol{u}_S$ and the perturbation pressure p_S.

When the rate of temperature fluctuation τ tends to zero as $Re \to 0$, then we can obtain a particularly simple solution for (9.10a, b), namely:

$$T_S = 1,\ \rho_S = 1, \tag{9.11}$$

and equations (9.10c$_1$, c$_2$) with (9.10c$_3$, c$_4$) reduce to the classical Stokes equations for an incompressible fluid:

$$\nabla\cdot\boldsymbol{u}_S = 0,\ \nabla p_S = \gamma(R^*)^2\mu(1)\nabla^2\boldsymbol{u}_S. \tag{9.12}$$

We note that the Stokes equations (9.12) for an incompressible flow may be obtained either by linearization or by letting Re tend to zero.

That these two procedures give the same result in the incompressible case is fortuitous; for compressible fluids the low Reynolds number Stokes equations are nonlinear as shown by the above system of equations (9.10).

In fact, in solving the equations (9.10), one first finds T_S from the energy equation and then ρ_S from the equation of state. The continuity and momentum equations then become linear equations whose variables coefficients involve the known function T_S.

9.1.2. The Oseen limiting case and the steady compressible Oseen equations

An alternative form of the steady dimensionless NS-F equations, using the Oseen (outer) space variable: $\xi = Re\, x$, is (when $Bo = 0$):

$$\boldsymbol{u} \cdot \nabla^{Os} \rho + \rho \nabla^{Os} \cdot \boldsymbol{u} = 0; \tag{9.13a}$$

$$\rho(\boldsymbol{u} \cdot \nabla^{Os})\boldsymbol{u} + \frac{1}{\gamma M^2} \nabla^{Os} p = \nabla^{Os} \cdot \left[2\mu \boldsymbol{D}^{Os}(\boldsymbol{u}) + \lambda(\nabla^{Os} \cdot \boldsymbol{u})\boldsymbol{I}\right]; \tag{9.13b}$$

$$\rho \boldsymbol{u} \cdot \nabla^{Os} T + (\gamma - 1) p \nabla^{Os} \cdot \boldsymbol{u} = \frac{\gamma}{Pr} \nabla^{Os} \cdot \left[k \nabla^{Os} T\right]$$
$$+ M^2 \gamma(\gamma - 1)\left(2\mu Tr\left(\boldsymbol{D}^{Os}(\boldsymbol{u})\right)^2 + \lambda\left(\nabla^{Os} \cdot \boldsymbol{u}\right)^2\right) \tag{9.13c}$$

with $p = \rho T$, where the gradient vector ∇^{Os} is formed with respect to the Oseen space variable $\boldsymbol{\xi} = (\xi_i)$, $i = 1, 2, 3$, and we have the following relations for the gradient operator and the rate of strain tensor: $\nabla = Re\ \nabla^{Os}$ and $\boldsymbol{D} = Re\, \boldsymbol{D}^{Os}$.

The unknown functions, $\boldsymbol{u}$, p, ρ, T and the rate of strain tensor $\boldsymbol{D}^{Os}(\boldsymbol{u})$ depend on (we consider a steady flow) all of the Oseen space variables ξ_i.

In the steady NS-F equations (9.13a, b, c), the Reynolds number, Re, has been eliminated! However, Re will reappear in the boundary conditions.

For example, if the finite solid body is a sphere of diameter L_C, then the boundary of the body is given in Oseen coordinates ξ_i by

$$(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 = \frac{Re}{2},$$

and when $Re \to 0$, *with* ξ_i ($i = 1, 2$ and 3) *fixed,* the body *shrinks to a point*! For a steady flow past a solid finite body at rest and when the domain is infinite, the following uniform conditions have to be added

$$u = U_\infty i,\ p = \rho = T = 1, \text{ at infinity.} \tag{9.14}$$

Now, we assume for: u, p, ρ and T the following asymptotic expansions for small Re:

$$u = U_\infty i + \mu_1(Re)\, u^{Os}(\xi) + \ldots,\ p = 1 + (Re)^{2+2a}[\mu_1(Re)\, p^{Os}(\xi) + ..], \tag{9.15a}$$

$$\rho = 1 + \mu_1(Re)\, \rho^{Os}(\xi) + \ldots,\ T = 1 + \mu_1(Re)\, T^{Os}(\xi) + \ldots, \tag{9.15b}$$

where $\mu_1(Re)$ is some suitable gauge function of Re which tends to zero as $Re \downarrow 0$. In the Oseen, distal-outer region a finite 3D body *shrinks to a point which cannot cause a finite disturbance in the Oseen fluid flow* and hence the value of u, p, ρ and T, at any distant fixed point, in the Oseen region, will tend to the free-stream values (9.14). The expansions (9.15) take into account this property.

If one then inserts these expansions (9.15) into the NS-F equations as written in Oseen variable [Eqs. (9.13a, b, c), with $p = \rho T$] and retains only terms of order $\mu_1(Re)$ it is found that $u^{Os}(\xi)$, $p^{Os}(\xi)$, $\rho^{Os}(\xi)$ and $T^{Os}(\xi)$ satisfies the following Oseen (linear) limiting equations:

$$\nabla^{Os} \cdot u^{Os} = U_\infty \frac{\partial T^{Os}}{\partial \xi_1}, \tag{9.16a}$$

$$U_\infty \frac{\partial u^{Os}}{\partial \xi_1} + \frac{1}{\gamma (R^*)^2} \nabla^{Os} p^{Os} = \mu(1)\left[\nabla^{Os2} u^{Os} + \frac{U_\infty}{3} \nabla^{Os} \frac{\partial T^{Os}}{\partial \xi_1}\right], \tag{9.16b}$$

$$U_\infty \frac{\partial T^{Os}}{\partial \xi_1} = \frac{k(1)}{Pr} \nabla^{Os2} T^{Os}, \tag{9.16c}$$

with

$$\rho^{Os} = -\, T^{Os}, \tag{9.16d}$$

if we assume that, $\lambda(1) = -(2/3)\mu(1)$ and we take into account the similarity relation (9.3)).

For the above Oseen equations (9.16a, b, c), with (9.16d), we can write the following boundary conditions:

$$U^{Os}, p^{Os}, T^{Os} \text{ and } \rho^{Os} \to 0 \text{ at infinity.} \qquad (9.17)$$

When T^{Os} is identically zero we recover the classical steady incompressible Oseen equations for $\boldsymbol{u}^{Os}$ and p^{Os}:

$$\nabla^{Os} \cdot \boldsymbol{u}^{Os} = 0\,;\; U_\infty \frac{\partial \boldsymbol{u}^{Os}}{\partial \xi_1} + \frac{1}{\gamma (R^*)^2} \nabla^{Os} p^{Os} = \mu(1) \nabla^{Os2} \boldsymbol{u}^{Os}. \qquad (9.18)$$

Therefore, the fundamental problem for the compressible Oseen equations (9.16a, b, c), with (9.16d), is the study of the behavior of the temperature fluctuation $T^{Os}(\xi)$, which is a solution of the linear equation

$$\left[\frac{\partial^2}{\partial \xi^2{}_1} + \frac{\partial^2}{\partial \xi^2{}_2} + \frac{\partial^2}{\partial \xi^2{}_3}\right] T^{Os} - \beta \frac{\partial T^{Os}}{\partial \xi_1} = 0, \qquad (9.19a)$$

when

$$(\xi_1)^2 + (\xi_2)^2 + (\xi_3)^2 \to 0, \qquad (9.19b)$$

according to matching with the Stokes equations (9.10a$_1$). In (9.19a),

$$\beta = \frac{U_\infty Pr}{k(1)}.$$

A consequence of the incompressible Oseen equations (9.18) is that p^{Os} is a harmonic function of ξ:

$$\nabla^{Os2} p^{Os} = 0. \qquad (9.20)$$

Surprisingly, this property is also true for the compressible Oseen equations (9.16a, b, c), with (9.16d), if we assume that: $Pr = (3/4)\,((k(1)/\mu(1))$.

In incompressible flow the Oseen equations (9.18) are actually uniformly valid, since the inner (Stokes) limit of these equations gives the classical incompressible Stokes equations (9.12).

This is a fortuitous coincidence closely linked with the fact that the Stokes (proximal) equations for incompressible flow are linear. The Oseen equations for compressible flow are not uniformly valid near the body. In solving a problem of low Reynolds number compressible flow past a solid finite body, it is not sufficient to use the Oseen solution only; the outer (distal-Oseen) solution must be matched with an inner solution, i.e., a solution of the compressible Stokes equations.

The method of matching is in principle the same as in the case of incompressible flow (see §9.3) although the computational difficulties are of course considerably greater in the case of compressible flow. In order to obtain a " crude estimate" of the solution one may solve the compressible Oseen equations as if they were uniformly valid!

Actually, the matching in the compressible case is an open problem!

9.2. THE STOKES PROBLEM IN A BOUNDED DOMAIN AND THE UNSTEADY ADJUSTMENT PROCESS TO INITIAL CONDITION

We consider now the undeady Navier system for an incompressible and highly viscous fluid motion in a *bounded domain* $\Omega(t)$. More precisely we start from the equations (see equations (5.23a, b)):

$$\nabla \cdot \boldsymbol{u}_N = 0, \tag{9.21a}$$

$$\frac{\partial \boldsymbol{u}_N}{\partial t} + \boldsymbol{u}_N \cdot \nabla \boldsymbol{u}_N + \frac{1}{\gamma} \nabla \pi_N = \frac{1}{Re} \nabla^2 \boldsymbol{u}_N . \tag{9.21b}$$

For the unsteady Navier equation (9.21b) it is necessary to write an initial condition for the velocity vector $\boldsymbol{u}$ at $t = 0$. Namely:

$$t = 0: \boldsymbol{u} = \boldsymbol{u}^0(\boldsymbol{x}). \tag{9.22a}$$

On the other hand, if $\partial\Omega(t) = \Sigma(t)$, then:

$$\boldsymbol{u} = \boldsymbol{U}(t, \boldsymbol{P}) \text{ on } \Sigma(t). \tag{9.22b}$$

The two vectors $\boldsymbol{u}^0(\boldsymbol{x})$ and $\boldsymbol{U}(t, \boldsymbol{P})$ are given functions, with $\boldsymbol{P} \in \Sigma(t)$ and

$$\boldsymbol{u}^0(\boldsymbol{P}) = \boldsymbol{U}(0, \boldsymbol{P}) \text{ on } \Sigma(0).$$

When the Reynolds number *Re* tends to zero, it is necessary, first, to introduce a reduced pressure and for this we write:

$$\boldsymbol{u}_N = \boldsymbol{u}_S + O(Re),\ \frac{\pi_N}{\gamma} = \frac{1}{Re}\left[\pi_S + O(Re)\right], \tag{9.23}$$

then, under the Stokes limiting process:

$$Lim^{St} = [\ Re \text{ tends to zero with } t, \text{ and } \boldsymbol{x} \text{ fixed}], \tag{9.24a}$$

we derive the classical (steady) Stokes system, namely:

$$\nabla \cdot \boldsymbol{u}_S = 0,\ \nabla^2 \boldsymbol{u}_S = \nabla \pi_S\,. \tag{9.24b}$$

With (9.23) we can only satisfy the no-slip condition (9.22b):

$$\boldsymbol{u}_S = \boldsymbol{U}(t, \boldsymbol{P}) \text{ on } \Sigma(t), \tag{9.24c}$$

and in this case we obtain a family of Stokes solutions where the time t plays the role of a parameter. But, in this case, *it is not possible* to satisfy the initial condition (9.22a).

9.2.1. The adjustment process to initial condition

Obviously, the steady Stokes equations, (9.24b), for $\boldsymbol{u}_S$ and π_S, are not valid close to the initial time $t = 0$. As a consequence, in the Navier system (9.21) we introduce, in place of t, a new 'short' time:

$$\tau = \frac{1}{Re} t\,, \tag{9.25a}$$

and we write:

$$\boldsymbol{u}(Re\ \tau, \boldsymbol{x}) = \boldsymbol{u}^*(\tau, \boldsymbol{x}, Re) \text{ and } \frac{Re}{\gamma}\pi_N(Re\,\tau, \boldsymbol{x}) = \pi^*(\tau, \boldsymbol{x}, Re)\,. \tag{9.25b}$$

If, with (9.25a, b), we consider now the following local-time limiting process,

$$Lim^l = [Re \downarrow 0, \text{ with } \tau, \boldsymbol{x} \text{ fixed}], \tag{9.26a}$$

then for the limit functions:

$$[\boldsymbol{u}^*_0(\tau, \boldsymbol{x}),\ \pi^*_0(\tau, \boldsymbol{x})] = Lim^l\,[\boldsymbol{u}^*,\ \pi^*], \tag{9.26b}$$

we derive the following *unsteady Stokes* system, which is consistent close to initial time $\tau = 0$:

$$\nabla\cdot\boldsymbol{u}^*_0 = 0,\ \frac{\partial \boldsymbol{u}^*_0}{\partial \tau} = \nabla^2\boldsymbol{u}^*_0 - \nabla\pi^*_0, \tag{9.27a}$$

and for the unsteady system (9.27a) we can write the original conditions (9.22a, b), namely:

$$\tau = 0:\ \boldsymbol{u}^*_0 = \boldsymbol{u}^0(\boldsymbol{x}),\ \boldsymbol{u}^*_0 = \boldsymbol{U}(0, \boldsymbol{P}) \text{ on } \Sigma(0), \tag{9.27b}$$

when we assume that:

$$\boldsymbol{U}(t, \boldsymbol{P}) = \boldsymbol{U}(Re\ \tau, \boldsymbol{P}) = \boldsymbol{U}(0, \boldsymbol{P}) + Re\ \tau\boldsymbol{U}'(0, \boldsymbol{P}) + \ldots.$$

If now we write the solution of the unsteady linear adjustment problem (9.27a, b) in the following form:

$$\boldsymbol{u}^*_0 = \boldsymbol{u}_S|_{t=0} + \boldsymbol{v}^*_0,\ \pi^*_0 = \pi_S|_{t=0} + \Pi^*_0, \tag{9.28}$$

where $\boldsymbol{u}_S|_{t=0}$ and $\pi_S|_{t=0} \equiv \nabla^{-1}(\nabla^2\boldsymbol{u}_S|_{t=0})$ are the values of the Stokes solution of the problem (9.24b, c) at $t = 0$.

For $\boldsymbol{v}^*_0(\tau, \boldsymbol{x})$ and $\Pi^*_0(\tau, \boldsymbol{x})$, we obtain the following linear problem:

$$\nabla\cdot\boldsymbol{v}^*_0 = 0, \tag{9.29a}$$

$$\frac{\partial \boldsymbol{v}^*_0}{\partial \tau} = \nabla^2\boldsymbol{v}^*_0 - \nabla\Pi^*_0, \tag{9.29b}$$

$$\tau = 0:\ \boldsymbol{v}^*_0 = \boldsymbol{u}^0(\boldsymbol{x}) - \boldsymbol{u}_S\big|_{t=0}, \tag{9.29c}$$

$$\boldsymbol{v}^*_0 = 0, \text{ on } \Sigma(0). \tag{9.29d}$$

In this case for $(\boldsymbol{v}^*_0,\ \Pi^*_0)$ we can write the following eigen-value problem:

$$\nabla^2\Phi_k - \nabla\Pi_k = \lambda_k\Phi_k,\ \nabla\cdot\Phi_k = 0,\ \Phi_k = 0 \text{ on } \Sigma(0), \tag{9.30a}$$

where

$$\left(v^*{}_0, \Pi^*{}_0\right) = \sum_{k=1}^{\infty}\alpha_k \exp[-\lambda_k\tau](\Phi_k(x), \Pi_k(x)). \tag{9.30b}$$

From the classical spectral theory we find that the eigen-values $[\lambda_k]$ are real and positive and also: $\lambda_k \to \infty$ when $k \to \infty$. Then the following relation is satisfied:

$$\alpha_k \int_\Omega |\Phi_k|^2 dv = \int_\Omega \Phi_k \cdot \left(u^0(x) - u_S|_{t=0}\right)dv. \tag{9.30c}$$

From (9.30b), when $\tau \to \infty$, $(v^*_0,\ \Pi^*_0) \to 0$ exponentially and the adjustment is assured! Finally, we obtain by matching:

$$lim_{\tau\to\infty}\left(u^*{}_0, \pi^*{}_0\right) = \left(u_S|_{t=0}, \pi_S|_{t=0}\right). \tag{9.31}$$

Since in general: $u^0(x) - u_S|_{t=0} \neq 0$, then from (9.30c) we can determine the coefficients α_k, when the solution of the eingen-value problem (9.30a) is known. We note that the solution of the Stokes system (9.24b) is unique in the bounded domain $\Omega(t)$, such that the condition is well satisfied on $\Sigma(t)$.

9.3. UNSTEADY NAVIER FLOW PAST A SPHERE

9.3.1. Formulation of the problem

We start from the classical unsteady Navier equations:

$$\nabla\cdot u_N = 0,\ \left[S\frac{\partial}{\partial t} + u_N\cdot\nabla\right]u_N + \nabla\pi = \frac{1}{Re}\Delta u_N, \tag{9.32}$$

for the velocity vector u_N and pressure fluctuation π. Alternative forms may be obtained from the relations

$$S\frac{Du_N}{Dt} = \left[S\frac{\partial}{\partial t} + u_N\cdot\nabla\right]u_N = S\frac{\partial u_N}{\partial t} + \nabla\left[\frac{1}{2}|u_N|^2\right] + \omega_N \wedge u_N,$$

and

$$\Delta \boldsymbol{u}_N = \nabla^2 \boldsymbol{u}_N = -\nabla \wedge \omega_N \text{, where } \omega_N = \nabla \wedge \boldsymbol{u}_N .$$

By applying the operator $\boldsymbol{curl}* = (\nabla \wedge *)$ to the Navier equation and using the above relations one obtains the law for the propagation of vorticity ω_N:

$$S\frac{\partial \omega_N}{\partial t} + (\omega_N \cdot \nabla)\boldsymbol{u}_N - (\boldsymbol{u}_N \cdot \nabla)\omega_N = \frac{1}{Re}\Delta \omega_N . \tag{9.33}$$

Now, a spherical coordinate system (r, θ, ϕ), with r = 0 at the centre of the sphere and $\theta = 0$ in the direction of the undisturbed stream is chosen and in this case: $\boldsymbol{u}_N = (u_r, u_\theta, u_\phi)$. The motion is assumed to be axially symmetric and hence all quantities are independent of the azimuthal coordinate ϕ. Moreover it is assumed that no swirling motion occurs and hence: $u_\phi = 0$.

The motion may be described by radial and polar components of velocity (u_r, u_θ) in a plane through the axis of symmetry. There is then only one component of vorticity ζ, in the ϕ direction, given by:

$$\zeta = \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right].$$

In this case the divergence operator $\nabla \cdot \boldsymbol{u}_N$ is of the following form:

$$\nabla \cdot \boldsymbol{u}_N = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, u_\theta) . \tag{9.34}$$

and, for the two-dimensional flow parallel to the (r, θ) plane, the vorticity vector reduces to $\omega_N = \zeta \boldsymbol{k}$. Finally, for $\zeta(t, r, \theta)$ we obtain the following equation, in place of (9.33):

$$S\frac{\partial \zeta}{\partial t} + \nabla \cdot (\zeta \boldsymbol{u}) = \frac{1}{Re}\nabla \cdot (\nabla \zeta), \tag{9.35}$$

where, $\nabla = (\partial/\partial r, (1/r)\partial/\partial \theta)$.

As as consequence of $\nabla \cdot \boldsymbol{u}_N = 0$, with (9.34), we can introduce the stream function $\psi(t, r, \theta)$ related to the velocity components (u_r, u_θ) in the (r, θ) directions by:

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \; u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}. \tag{9.36}$$

As a consequence of the above relations, when $S\,Re = 1$, we can write in place of (9.35), the following single equation for $\psi(t, r, \theta)$:

$$\left[\frac{\partial}{\partial t} - D^2\right] D^2 \psi = Re\left\{\frac{1}{r^2 \sin\theta} \frac{\partial(\psi, D^2\psi)}{\partial(r,\theta)} - \frac{2D^2\psi}{r^3 \sin^2\theta} \frac{\partial(\psi, r\sin\theta)}{\partial(r,\theta)}\right\} \tag{9.37}$$

where, by definition:

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right]. \tag{9.38a}$$

and for two functions $f(r, \theta)$ and $g(r, \theta)$ we have:

$$\frac{\partial(f, g)}{\partial(r,\theta)} = \frac{\partial f}{\partial r} \frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta} \frac{\partial g}{\partial r}. \tag{9.38b}$$

For the equation (9.37) the boundary conditions are:

$$\psi = \frac{\partial \psi}{\partial r}, \text{ for } r = 1, 0 \le \theta \le \pi, \tag{9.39a}$$

$$\psi \to \frac{r^2}{2} \sin^2\theta \, H(t), \text{ as } r \to \infty, \tag{9.39b}$$

and the initial condition is

$$\psi = 0, \text{ for } t = 0, 1 < r < \infty, \tag{9.39c}$$

where $H(t)$ is the Heaviside step function ($H(t) \equiv 1$ for $t > 0$ and zero otherwise). We note that the condition (9.39b) implies that the entire expanse of incompressible fluid undergoes acceleration.

9.3.2. Steady motion

In the case of steady motion, in place of equation (9.37) and conditions (9.39a, b), we obtain the following steady problem for $\psi_S(r,\ \theta)$:

$$D^4\psi_S = Re\frac{1}{r^2\sin\theta}\left[\frac{\partial\psi_S}{\partial\theta}\frac{\partial}{\partial r} - \frac{\partial\psi_S}{\partial r}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi_S}{\partial r} - \frac{2}{r}\frac{\partial\psi_S}{\partial\theta}\right]D^2\psi_S, \tag{9.40a}$$

$$\psi_S = \frac{\partial\psi_S}{\partial r} = 0,\ \text{for } r = 1,\ 0 \le \theta \le \pi, \tag{9.40b}$$

$$\psi_S \to \frac{r^2}{2}\sin^2\theta,\ \text{as } r \to \infty. \tag{9.40c}$$

9.3.2a. Inner, near-Stokes field

Near the surface of the sphere $(r = 1)$, the 2-term Stokes expansion has the form:

$$\psi_S = f_0(r,\ \theta) + Re\, f_1(r,\ \theta) + \ldots, \tag{9.41}$$

with

$$f_0(r,\theta) = \frac{1}{4}\left[2r^2 - 3r + \frac{1}{r}\right]\sin^2\theta, \tag{9.42a}$$

$$f_1(r,\theta) = -\frac{3}{32}\left[2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right]\sin^2\theta\cos\theta. \tag{9.42b}$$

But, if the solution $f_0(r,\ \theta)$ is a "good" solution of the full steady problem (9.40a, b, c), for $Re = 0$, on the contrary the solution $f_1(r,\ \theta)$ is only a particular integral of the inhomogeneous Stokes equation:

$$D^4 f_1 = -\frac{9}{4}\left[\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^5}\right]\sin^2\theta\cos\theta, \tag{9.42c}$$

that satisfies the surface conditions (9.40b).

As a consequence, (Whitehead's paradox) it is necessary to consider an outer, far-Oseen field. In fact, the corresponding correct 2-term steady Stokes expansion of (9.40a, b, c), in the vicinity of the sphere for low Reynolds number, given by Proudman and Pearson (1957) is (when $Re \to 0$ with r fixed):

$$\psi_S = \frac{1}{4}\left[2r^2 - 3r + \frac{1}{r}\right]\sin^2\theta$$

$$-\frac{3}{32}Re\left\{\left[2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right]\sin^2\theta\cos\theta + \left[2r^2 - 3r + \frac{1}{r}\right]\sin^2\theta\right\} + \dots \tag{9.43}$$

9.3.2b. Outer, far-Oseen field

In the outer, far-Oseen field, we introduce the following outer variables:

$$R = Re\, r, \text{ and } \psi^{Os}(R,\theta;Re) = Re^2\psi_S\left(\frac{R}{Re},\theta\right), \tag{9.44}$$

in terms of which the governing equation (9.40a) can be written as

$$D^{*4}\psi^{Os} = \frac{1}{R^2\sin\theta}\left[\frac{\partial\psi^{Os}}{\partial\theta}\frac{\partial}{\partial R} - \frac{\partial\psi^{Os}}{\partial R}\frac{\partial}{\partial\theta} + 2\cot\theta\frac{\partial\psi^{Os}}{\partial R} - \frac{2}{R}\frac{\partial\psi^{Os}}{\partial\theta}\right]D^{*2}\psi^{Os} \tag{9.45}$$

where D^{*2} is the same operator (9.38), D^2, but with r replaced by R, according to the first relation of (9.44): $D^{*2} = (1/Re)^2 D^2$. The appropriate Oseen expansion for $\psi^{Os}(R, \theta; Re)$ (when $Re << 1$, but R fixed) is:

$$\psi^{Os} = \frac{R^2}{2}\sin^2\theta + Re\, g_1(R,\theta) + \dots, \tag{9.46}$$

the first term of which represents a uniform flow according to (9.40c). Substituting (9.46) into the full equation (9.45) yields for $g_1(R, \theta)$ the classical linearized Oseen equation:

$$\left[D^{*2} - \cos\theta \frac{\partial}{\partial R} + \frac{\sin\theta}{R}\frac{\partial}{\partial \theta}\right] D^{*2} g_1 = 0. \qquad (9.47)$$

The solution of (9.47) is the Oseen fundamental solution:

$$g_1(R,\theta) = -2C(1+\cos\theta)\left\{1 - \exp\left[-\frac{1}{2}R(1-\cos\theta)\right]\right\}, \qquad (9.48)$$

which describes the disturbance field produced at great distance by any finite three-dimensional non-lifting body. The constant C depends upon certain details of the flow near the body.

Now we find this constant, C, by applying the asymptotic matching principle, according to Van Dyke (1964, p. 90):

"the m-term inner expansion of (the n-term outer expansion)
= the n-term outer expansion of (the m-term inner expansion)".

Writing the Oseen expansion (9.46) in the Stokes variable r and expanding for small Reynolds number yields:

$$\text{1-Stokes (2-Oseen) } \psi_S = \frac{r^2}{2}\sin^2\theta - Cr\sin^2\theta, \qquad (9.49a)$$

and writing the Stokes solution (9.42a) in terms of the Oseen variable $R = Re\ r$ and expanding for small Re gives as its 2-term Oseen expansion:

$$\text{2-Oseen (1-Stokes) } \psi_S = \frac{1}{Re}\left[\frac{R^2}{2Re} - \frac{3}{4}R\right]\sin^2\theta. \qquad (9.49b)$$

As a consequence, (9.49a) matches (9.49b) if $C = 3/4$.

9.3.2c. Uniform approximation

With (9.48) and $C = 3/4$ we have found two terms of the Oseen expansion (9.46). Now, we can find two terms of the Stokes expansion, and for this we rewrite (9.46) in the Stokes variables:

$$\psi_S \sim \frac{r^2}{2}\sin^2\theta - \frac{3}{2}\frac{1}{Re}(1+\cos\theta)\left\{1-\exp\left[-\frac{1}{2}rRe(1-\cos\theta)\right]\right\},$$

as $Re \to 0$, with $R = Re\,r$ fixed. (9.50)

Now, the 2-term Stokes expansion of the Oseen expansion (9.46) is found to be:

$$\text{2-Stokes (2-Oseen) } \psi_S = \frac{1}{4}\left(2r^2-3r\right)\sin^2\theta + \frac{3}{16}r^2 Re(1-\cos\theta)\sin^2\theta. \tag{9.51}$$

In order to match this, the Stokes expansion must have the following form:

$$\psi_S \sim \frac{1}{4}\left[2r^2-3r+\frac{1}{r}\right]\sin^2\theta + ReF_1(r,\theta)+\ldots. \tag{9.52}$$

The equation for $F_1(r, \theta)$ is (9.42c) and its particular integral (9.42b) remains valid. The original Stokes approximation (9.42a) provides the only complementary function with the proper symmetry that no more singular at infinity. Thus we get the following solution:

$$F_1(r,\theta) = A\left[2r^2-3r+\frac{1}{r}\right]\sin^2\theta - \frac{3}{32}\left[2r^2-3r+1-\frac{1}{r}+\frac{1}{r^2}\right]\sin^2\theta\cos\theta \tag{9.53}$$

The constant A is found by matching. Carrying out the Oseen expansion of (9.53) yields

$$\text{2-Oseen (2-Stokes) } \psi_S = \frac{1}{2}R^2\frac{\sin^2\theta}{Re^2} + \frac{1}{Re}\left[2AR^2 - \frac{3}{16}R^2\cos\theta - \frac{3}{4}R\right]\sin^2\theta \tag{9.54}$$

and this matches (9.51) if <u>$A = 3/32$.</u> Thus we have found two terms of the Stokes expansion for the stream function in the vicinity of the sphere (see (9.43)).

Finally, we can construct a uniformly valid composite expansion using the rule:

" the sum of the inner and outer expansions is corrected by substracting the part they have in common, so that it is not counted twice",

for a so-called "additive" composition (Van Dyke (1964; p.94)).

The result gives a uniform approximation to the perturbation field. It is found to be exactly the solution of the Oseen equation given by Oseen himself:

$$\psi_S = \frac{1}{4}\left[2r^2 + \frac{1}{r}\right]\sin^2\theta - \frac{3}{2Re}(1+\cos\theta)\left\{1 - exp\left[-\frac{1}{2}rRe(1-\cos\theta)\right]\right\} \tag{9.55}$$

This confirms the statement that the linearized equation yields a uniform first approximation. Near the body the last term in (9.55) reduces to the "stokeslet" of the Stokes approximation ($(3/4)\ r\ sin^2\theta$); it may by analogy be called an "oseenlet". Higher approximations can be found by continuing the preceding asymptotic analysis, and Proudman and Pearson (1957) have carried it far enough to show that the next Stokes approximation contains a term in $Re^2\ log\ Re$ as well as Re^2, and that logarithms are thereby introduced also into the Oseen expansion beginning with $Re^3\ log\ Re$. They have calculated only the term in $Re^2\ log\ Re$ in the Stokes expansion.

9.3.3. Unsteady flow

Bentwich and Miloh (1978) considered the unsteady matched Stokes-Oseen solution for the flow past a sphere when a constant rectilinear velocity is suddenly imparted to the sphere. In fact, the solution obtained by these authors represents the entire process of transition from stagnation to the steady state envisaged by Proudman and Pearson (1957).

But in the paper by Sano (1981) it is shown that the matching procedure proposed by Bentwich and Miloh in 1978, is incomplete.They divided the (r, t) plane into two regions; one is the *L*- shaped region adjacent to the r and t axes (*inner domain = small-time + large-time inner domains*), and the other is the rectangular region far from the axes (outer domain + large - time outer domain). In fact, the *L*-shaped region suggested by Bentwich and Miloh (1978) must be subdivided into two domains as shown in figure 9.2 below:

One is a small-time domain where $t = O(1)$ and the other a large-time inner domain where: $t = O(1/(Re)^2)$ and $r = O(1)$, including a large-time outer domain where: $t = O(1/(Re)^2)$ and $r = O(1/Re)$.

As a consequence, the unsteady problem has a three-region structure.

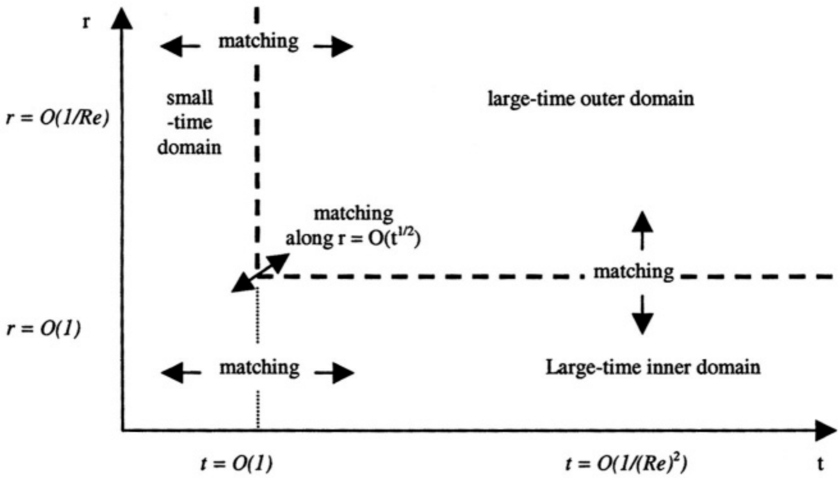


Fig. 9.2. Schematic sketch demonstrating the matching procedure

In the small-time domain, the vorticity layer is confined to the inner region near the surface ($r = O(1)$) and hence the assumption that the nonlinear inertia terms in the unsteady equation (9.37) are neligible, when $Re << 1$, is valid throughout the flow field. The solution in this domain is given by the following small-time expansion:

$$\psi = h_0(t, r, \theta) + Re\, h_1(t, r, \theta) + ..., \tag{9.56}$$

which is required to satisfy the boundary conditions, (9.39a, b), both on the surface and at infinity as well as the initial condition (9.39c).

In the large-time inner domain, the $\partial/\partial t$ term in the full equation (9.37) is also small for $Re << 1$, along with the nonlinear term; thus, the motion in this (T, r) domain, where $T = (Re)^2\, t$, is quasi-steady.

The requirement of the outer domain in r in the large-time region is due to the fact that, as the time t increases, the vorticity layer diffuses into the outer region where all the terms in the momentum equation are of the same order of magnitude and the (large-time) inner solution fails. In the large-time outer domain we introduce the following outer variables:

$$T = Re^2\, t \text{ and } R = Re\, r. \tag{9.57}$$

The construction of the two asymptotic expansions in the large-time region is made in such a way that:
(i) the inner expansion satisfies the boundary condition on the surface;
(ii) the outer expansion satisfies the boundary condition at infinity;
(iii) the two expansions match identically in the overlapping domain in space where both expansions are valid and also match the small-time expansion (9.56) at small values of T (when t tends to infinity).

The matching along $r = O(t^{1/2})$, proposed by Bentwich and Miloh (1978), is incomplete to obtain the higher-order terms and Bentwich and Miloh's way of dividing the $(r,\ t)$ plane is correct only if the expansion in the L-shaped region contains only one term. In the large-time inner domain $(T,\ r)$, the analysis for the steady flow suggests that the appropriate inner expansion is:

$$\psi^*(T,\ r,\ \theta;\ Re) = H_0(T,\ r,\ \theta) + Re\, H_1(T,\ r,\ \theta) + Re^2\,(\log Re)\, H_2 + \ldots, \quad (9.58)$$

where

$$\psi\left(\frac{T}{Re^2}, r, \theta\right) \equiv \psi^*(T, r, \theta; Re).$$

Finally, in the large-time outer domain $(T,\ R)$, the appropriate outer expansion is

$$Re^2\psi = \psi^{**}(T,R,\theta;Re) = \frac{1}{2}R^2 \sin^2\theta + Re\,G_1(T,R,\theta) + \ldots, \quad (9.59)$$

the first term of which represents a uniform flow, where:

$$Re^2\psi\left(\frac{T}{Re^2}, \frac{R}{Re}, \theta\right) \equiv \psi^{**}(T,R,\theta;Re).$$

9.3.3a. Analysis of the large-time inner domain

In the large-time inner domain $(r,\ T)$, in the limit $Re \to 0$ the unsteady term, in the governing unsteady equation (9.37) rewritten with the variables r and large time $T = Re^2\, t$, can be neglected. It follows that long after motion has begun the flow field close to the sphere is quasi-steady. In other words, time plays no role in the equation governing the motion in that field and its time dependence is determined by the outer field via matching.

The time dependence so obtained is verified later by the fact that the large-time inner expansion matches that prevailling when motion is just beginning.

Substituting the expansion (9.58) into the "inner" unsteady equation:

$$\left[Re^2 \frac{\partial}{\partial T} - D^2\right] D^2\psi^*$$

$$= Re\left\{\frac{1}{r^2 \sin\theta}\frac{\partial(\psi^*, D^2\psi^*)}{\partial(r,\theta)} - \frac{2}{r^3 \sin^2\theta} D^2\psi^* \frac{\partial(\psi^*, r\sin\theta)}{\partial(r,\theta)}\right\} \quad (9.60)$$

we derive the following equations for the three terms:

$$D^4 H_0 = 0; \quad (9.61a)$$

$$D^4 H_1 = \frac{1}{r^2 \sin\theta}\frac{\partial(H_0, D^2 H_0)}{\partial(r,\theta)} - \frac{2}{r^3 \sin^2\theta} D^2 H_0 \frac{\partial(H_0, r\sin\theta)}{\partial(r,\theta)}; \quad (9.61b)$$

$$D^4 H_2 = 0. \quad (9.61c)$$

The solution of (9.61a) which matches with (9.59) is clearly the steady Stokes solution

$$H_0 = \left[\frac{1}{2}r^2 - \frac{3}{4}r + \frac{1}{4r}\right]\sin^2\theta, \quad (9.62)$$

and hence (9.61b) becomes (see (9.42c)):

$$D^4 H_1 = -\frac{9}{4}\left[\frac{2}{r^2} - \frac{3}{r^3} + \frac{1}{r^5}\right]\sin^2\theta\cos\theta. \quad (9.63)$$

The solution of (9.63) satisfying the boundary condition on the surface is:

$$H_1 = -\frac{3}{32}\left[2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right]\sin^2\theta\cos\theta + A(T)\left[2r^2 - 3r + \frac{1}{r}\right]\sin^2\theta \tag{9.64}$$

where $A(T)$ is an integration constant depending on the large time T and is to be determined through matching with the outer solution.

9.3.3b. Analysis of the large-time outer domain

In the large-time outer domain (R, T), we have the following "outer" equation for the function ψ^{**},

$$\left[\frac{\partial}{\partial T} - D^{*2}\right]D^2\psi^{**} = \frac{1}{R^2\sin\theta}\frac{\partial\left(\psi^{**}, D^{*2}\psi^{**}\right)}{\partial(R,\theta)} - \frac{2}{R^3\sin^2\theta}D^{*2}\psi^{**}\frac{\partial\left(\psi^{**}, R\sin\theta\right)}{\partial(R,\theta)} \tag{9.65}$$

such that:

$$Re^2\psi\left(\frac{T}{Re^2}, \frac{R}{Re}, \theta\right) \equiv \psi^{**}(T, R, \theta : Re),$$

where D^{*2} is, again, the same operator (9.38), D^2, but with r replaced by R and

$$D^{*2} = \left(\frac{1}{Re}\right)^2 D^2 .$$

From the equation (9.65), with the expansion (9.59), we see that the limiting equation for the second term $G_1(T, R, \theta)$, in this outer expansion (9.59), is the unsteady linearized Oseen equation:

$$\left[\frac{\partial}{\partial T} - D^{*2} + \cos\theta\frac{\partial}{\partial R} - \frac{\sin\theta}{R}\frac{\partial}{\partial\theta}\right]D^2G_1 = 0 . \tag{9.66}$$

A solution which matches to both the small-time and Stokes solutions has been obtained by Bentwich and Miloh (1978, p. 21-23). According to their solution, the asymptotic behaviour of G_1 for small R is given by:

$$G_1 \sim -\frac{3}{4} R \sin^2\theta + \frac{3}{16}\sin^2\theta \left\{ \left[1 + \frac{4}{T^2} erf\left[\frac{1}{2}T^{1/2}\right]\right] + \left(\frac{4}{\pi T}\right)^{1/2}\left[1-\frac{2}{T}\right] exp\left[-\frac{1}{4}T\right] - \cos\theta \right\} R^2 + \ldots \qquad (9.67)$$

Therefore, in order to satisfy the matching condition between the inner (9.58) and outer (9.59) expansions, the asymptotic behavior of H_1 for large r should be of the form

$$H_1 \sim \frac{3}{16}\sin^2\theta \left\{ \left[1 + \frac{4}{T^2} erf\left[\frac{1}{2}T^{1/2}\right]\right] + \left(\frac{4}{\pi T}\right)^{1/2}\left[1-\frac{2}{T}\right] exp\left[-\frac{1}{4}T\right] - \cos\theta \right\} r^2 + \ldots \qquad (9.68)$$

From (9.64) and (9 68), the function $A(T)$ can be determined as:

$$A(T) = \frac{3}{32}\left\{ \left[1 + \frac{4}{T^2}\right] erf\left[\frac{1}{2}T^{1/2}\right] + \left(\frac{4}{\pi T}\right)^{1/2}\left[1-\frac{2}{T}\right] exp\left[-\frac{1}{4}T\right] \right\}. \qquad (9.69)$$

In (9.67)-(9.69) we note that:

$$erf[X] = 1 - \left(\frac{4}{\pi}\right)^{1/2} \int_X exp(-a^2)da. \qquad (9.70)$$

Thus, H_1 has been determined as follows:

$$H_1 = -\frac{3}{32}\left[2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right]\sin^2\theta\cos\theta + \frac{3}{32}\left\{\left[1 + \frac{4}{T^2}\right]erf\left[\frac{T^{1/2}}{2}\right]\right.$$
$$\left. + \left(\frac{4}{\pi T}\right)^{1/2}\left[1 - \frac{2}{T}\right]exp\left[-\frac{T}{4}\right]\right\}\left[2r^2 - 3r + \frac{1}{r}\right]\sin^2\theta. \quad (9.71)$$

9.3.3c. Analysis of the solution in the small-time domain

If now we consider the small-time domain $(t,\ r)$ and the corresponding small-time expansion (9.56), then the solution for $h_0(t,\ r,\ \theta)$ has the form

$$h_0(t,\ r,\ \theta) = h_{00}(t,\ r)\sin^2\theta, \quad (9.72)$$

and we can easily obtain the corresponding equation for $h_1(t,\ r,\ \theta)$, the term proportional to Re in the small time expansion (9.56), namely

$$\left[\frac{\partial}{\partial t} - D^2\right]D^2 h_1 = B(r,t)\sin^2\theta\cos\theta, \quad (9.73)$$

where $h_{00}(t,\ r)$ and $B(r,\ t)$ are unknown functions of r and t. It is easy to show that the solution of equation (9.73) satisfying the initial (at $t = 0$) and boundary conditions (on the surface $r = 1$, and at infinity) has the form

$$h_1(t,\ r,\ \theta) = h_{10}(t,\ r)\sin^2\theta\cos\theta, \quad (9.74a)$$

where $h_{10}(t,\ r)$ is also an unknown function of r and t.

On the other hand, the associated steady solution is:

$$\psi_{S1} = -\frac{3}{32}\left[2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2}\right]\sin^2\theta\cos\theta$$
$$-\frac{3}{32}\left[2r^2 - 3r + \frac{1}{r}\right]\sin^2\theta, \quad (9.74b)$$

according to the solution (9.43) and apparently, (9.74a) cannot approach this above steady solution (9.74b), as $t \to \infty$, meaning that (9.56) is invalid for large t (as a small-time expansion) even in the vicinity of the sphere!

As a consequence, in Bentwich and Miloh (1978), the way of dividing the *(r, t)* plane is correct only if the small-time expansion in the *L*-shaped region contains one term. The purpose of the paper by Sano (1981) is to complement the work of Bentwich and Miloh (1978) by representing a complete procedure for successful matching.

For $T \to 0$, (9.71) gives its steady counterpart (9.74b). Furthermore, it can easily be shown that the first two terms in the large-time inner expansion (9.58) match the small-time expansion (9.56).

Next, if we substitute (9.62) and (9.71),valid in the large-time inner domain, into the large-time inner expansion (9. 58) and recast the latter in terms of *t*, we can obtain the following solution:

$$\psi^* \sim \left\{\left[\frac{1}{2}r^2 - \frac{3}{4}r + \frac{1}{4r}\right] + \left[\frac{1}{4(\pi t)^{1/2}}\right]\left[2r^2 - 3r + \frac{1}{r}\right] + \dots\right\} \sin^2\theta$$
$$+ Re\left\{-\frac{3}{32}\left[2r^2 - 3r + 1 - \frac{1}{r} + \frac{1}{r^2} + \dots\right] \sin^2\theta\cos\theta + O\left(Re^2 \log Re\right)\right\}. \tag{9.75}$$

Now, we can easily verify that the first term in the above solution agrees completely with the asymptotic behaviour of h_0 for large t. Moreover, we can see that the θ dependence of the second term and that of the solution for h_1, as given by (9.74a), are identical. This suggests that these two also match!

The solution of equation (9.61c) for H_2 can be obtained using the condition that there is no term of $O(Re^2 \log Re)$ in the large-time outer expansion and is found to be given by the following steady solution:

$$H_2 = \frac{9}{100}\left[2r^2 - 3r + \frac{1}{r}\right]\sin^2\theta. \tag{9.76}$$

The fact that (9.76) is independent of T suggests (!) that this term matches that of $O(Re^2)$ in the small-time expansion.

Finally, in the large-time region, the inner solution has been obtaind up to the term $O(Re^2 \log Re)$. Therefore, it is desirable to obtain the second term h_1 in the small-time expansion (9.56) in order to make the present analysis complete. However, such a calculation for obtaining h_1 is very tedious!!

It is clear that the first term in the small-time expansion (9.56), $h_0(t, r, \theta)$, satisfies the following unsteady Stokes equation:

$$\left[\frac{\partial}{\partial t}-D^2\right]D^2 h_0=0, \tag{9.77}$$

and when $h_0(t, r, \theta) = h_{00}(t, r) \sin^2\theta$, according to Bentwich and Miloh (1978, p.19), the solution for $h_{00}(t, r)$ is given by:

$$\begin{aligned} h_{00}(t,r)=&\frac{1}{2}r^2H(t)+\frac{1}{2r}\left[H(t)+3\left(\frac{4t}{\pi}\right)^{1/2}+3t\right] \\ &-\frac{3}{2}t^{1/2}\left[\left(\frac{4}{\pi}\right)^{1/2}exp\left(-a^2\right)-2a\,erf(a)\right] \\ &-\frac{3t}{2r}\left[\left(1+2a^2\right)erf(a)-\left(\frac{4}{\pi}\right)^{1/2}a\,exp\left(-a^2\right)\right], \end{aligned} \tag{9.78}$$

where: $a = (r - 1)(4t)^{-1/2}$.

Naturally, the solution of the inhomogeneous unsteady Stokes equation (9.73), for $h_1(t, r, \theta) = h_{10}(t, r) \sin^2\theta\cos\theta$, is not easy to find and the explicit form of $h_{10}(t, r)$ is certainly complicated! But,this term contributes nothing to the drag because of symmetry, and in Sano (1981) this term is not obtained.

In Sano (1981, p. 438) the reader can find the expression of the drag of the sphere for the large-time domain, and also for the small-time domain. From these two expressions, Sano (1981) derives a single composite expansion for the drag which is uniformly valid for all values of time.

Finally, in Sano (1981), the formation of an eddy behind the sphere is discussed, the boundary of which may be calculated by equating to zero the 2-term large-time inner expansion (9.58).

In conclusion, we can say that for the unsteady flow the 2-term (large-time) inner expansion can give information about the eddy only in the final stage near the steady state.

In Bentwich and Miloh (1978) an experimental verification is given: to simulate the model considered in the analysis, a metal sphere of radius 1cm was suspended by an inextensible cord in a tank full of glycerine, and at a certain instant, the cord was pulled sharply (by hand) such that the sphere instantaneously attained a velocity of approximately *0.8 cm/s*. The Reynolds number based on this velocity, the viscosity of the glycerine and the sphere diameter is about $Re = 10^{-2}$.

The glycerine contained air bubbles and the meridional plane was appropriately illuminated. By using these tracers in conjunction with a camera, it was possible to obtain the trajectories (pathlines) of the illuminated bubbles. As a consequence of this experimental verification, the asymptotic analysis is transformed from a piece of speculative mathematics to a realistic fluid mechanical phenomenon!

In Sano (1981) the reader can also find a figure which represents the development with time of the vorticity on the surface of the sphere calculated from the large-time inner solution for $Re = 0.1$.

9.4. A MODEL EQUATION FOR HYDRODYNAMIC LUBRICATION THEORY

The complete omission of the (quasi-linear) inertial terms in the Navier equations results in the so-called creeping motion or Stokes equations and these equations have been applied to the hydrodynamic theory of lubrication. Reynolds (1886) initiated the study of the relative motion of two nearly parallel surfaces and his methods (equation) have since been applied to a variety of lubrication problems by Hays (1959). In addition to the neglect of inertia, it is assumed that the fluid motion is essentially unidirectional. These same simplifications have also been employed, for example, to investigate the axial motion of a sphere in a circular cylinder containing a viscous fluid, where the diameter of the cylinder only slightly exceeds that of the sphere (see, for instance, Christopherson and Dowson (1959)), and to the viscous flow of a fluid through a grating of parallel circular cylinders when the separation between them is small compared with their diameter (Keller (1964)). Good agreement of experiment with theory is obtained in the former case. Many other similar applications have been made.

In the next Chapter in the framework of the theory of thin liquid films subject to the Marangoni effect (which is related to the temperature dependent free-surface tension) the reader can find the derivation of a more complete lubrication model equation which takes into account various (stabilizing/destabilizing) physical effects.

9.4.1. Formulation of the problem

Here, we consider the simple case of a horizontal layer of incompressible strongly viscous fluid, which moves between a fixed flat plane $x_3 = 0$ and a free surface $\Sigma(t)$. We assume that the equation of the free surface is: $x_3 = h(t, x_1, x_2)$.

We start from the Navier equations, for the velocity, $\boldsymbol{u} = \boldsymbol{v} + u_3\,\boldsymbol{k}$, and pressure p, where the unit vector $\boldsymbol{k}$ is in the direction of the upwards normal to the plane $x_3 = 0$ and $\boldsymbol{v} = (u_1, u_2)$. In the equation for u_3 we take into account of the force of gravity (the term g in the 'vertical' equation (9.79c)); thus we write the following three equations:

$$\boldsymbol{D}\cdot\boldsymbol{v} + \frac{\partial u_3}{\partial x_3} = 0; \tag{9.79a}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{D})\boldsymbol{v} + u_3\frac{\partial \boldsymbol{v}}{\partial x_3} + \frac{1}{\rho_0}\boldsymbol{D}p = \nu_0\left[\boldsymbol{D}^2\boldsymbol{v} + \frac{\partial^2 \boldsymbol{v}}{\partial x_3{}^2}\right]; \tag{9.79b}$$

$$\frac{\partial u_3}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{D})u_3 + u_3\frac{\partial u_3}{\partial x_3} + \frac{1}{\rho_0}\frac{\partial p}{\partial x_3} + g = \nu_0\left[\boldsymbol{D}^2 u_3 + \frac{\partial^2 u_3}{\partial x_3{}^2}\right], \tag{9.79c}$$

where $\nu_0 = const$ is the kinematic viscosity and $\boldsymbol{D} = (\partial/\partial x_1, \partial/\partial x_2)$ the surface gradient vector. The boundary conditions relative to x_3 are:

(a) no-slip condition

$$x_3 = 0: \boldsymbol{v} = 0 \text{ and } u_3 = 0, \tag{9.80}$$

(b) at the free surface the kinematic condition:

$$u_3 = \frac{\partial h}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{D}h, \text{ at } x_3 = h(t, x_1, x_3), \tag{9.81a}$$

(c) the complete stress condition, namely:

$$-(p - p_a)\,\boldsymbol{n} + 2\nu_0\rho_0\boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{n} = 2\,\sigma_0\,\mathrm{H}\,\boldsymbol{n}, \text{ at } x_3 = h(t, x_1, x_2), \tag{9.81b}$$

where

$\boldsymbol{D}(\boldsymbol{u}) = (d_{ij})$ is the deformation tenseur

and $d_{ij} = (1/2)[\partial u_i/\partial x_j + \partial u_j/\partial x_i]$, with $i, j = 1, 2, 3$.

In (9.81b), the constant surface tension is denoted by σ_0 and H is the mean free surface curvature, such that

$$2\,H = -\,\boldsymbol{D}\cdot\boldsymbol{n}, \tag{9.82a}$$

with,

$$\boldsymbol{n} = N^{-1/2}\left(-\frac{\partial h}{\partial x_1}, -\frac{\partial h}{\partial x_2}, 1\right) \text{ and } N = 1 + \left(\frac{\partial h}{\partial x_1}\right)^2 + \left(\frac{\partial h}{\partial x_2}\right)^2. \tag{9.82b}$$

Finally, we assume that the deformable free surface is open to ambient passive air at constant pressure p_a.

9.4.2. The long-wave approximation model

The full problem (9.79)-(9.81), is a very complicated problem, but for a thin fluid layer we can apply the long-wave approximation. According to the long-wave approximation, we assume that the characteristic value of the horizontal (in the x_1 and x_2 directions) wave-length: $\lambda >> d$, where d is the thickness of the fluid layer when the fluid is at rest and the layer is limited by an upper surface $z = d$. In this case, we can introduce the following dimensionless variables:

$$x = \frac{x_1}{\lambda},\ y = \frac{x_2}{\lambda},\ z = \frac{x_3}{d},\ \tau = \frac{t}{t_C}, \tag{9.83a}$$

and dimensionless functions:

$$u = \frac{u_1}{U_C},\ v = \frac{u_2}{U_C},\ w = \frac{1}{\varepsilon}\frac{u_3}{U_C},\ P = \frac{p - p_a}{\rho_0 g d}. \tag{9.83b}$$

In (9.83b),

$$\varepsilon = \frac{d}{\lambda} << 1, \tag{9.84}$$

is the long-wave dimensionless, small parameter. Finally, the dimensionless equation of the free surface is:

$$z = H(\tau, x, y), \text{ where } H(\tau, x, y) = \frac{h(t_C\tau, \lambda x, \lambda y)}{d}.$$

First, for the dimensionless functions: $\boldsymbol{v}^* = (u, v) = \boldsymbol{v}/U_C$, w and P as function of dimensionless time-space variables: τ, x, y and z, we derive, in place of equations (9.79a, b, c), the following set of three dimensionless equations:

$$\boldsymbol{D}^* \cdot \boldsymbol{v}^* + \frac{\partial w}{\partial z} = 0; \tag{9.85a}$$

$$S\frac{\partial \boldsymbol{v}^*}{\partial \tau} + (\boldsymbol{v}^* \cdot \boldsymbol{D}^*)\boldsymbol{v}^* + w\frac{\partial \boldsymbol{v}^*}{\partial z} + \frac{1}{Fr^2}\boldsymbol{D}^* P = \frac{1}{Re}\left[\varepsilon^2 \boldsymbol{D}^{*2}\boldsymbol{v}^* + \frac{\partial^2 \boldsymbol{v}^*}{\partial z^2}\right]; \tag{9.85b}$$

$$\varepsilon^2\left[S\frac{\partial w}{\partial \tau} + (\boldsymbol{v}^* \cdot \boldsymbol{D}^*)w + w\frac{\partial w}{\partial z}\right] + \frac{1}{Fr^2}\left[\frac{\partial P}{\partial z} + 1\right] = \frac{\varepsilon^2}{Re}\left[\varepsilon^2 \boldsymbol{D}^{*2} w + \frac{\partial^2 w}{\partial z^2}\right]; \tag{9.85c}$$

where: $\boldsymbol{D}^* = (\partial/\partial x, \partial/\partial y) = \lambda \boldsymbol{D}$ and $S = \lambda/U_C t_C$, $Fr^2 = U_C^2/gd$, $Re = U_C(d^2/\lambda)/\nu_0$.

The choice of the characteristic velocity U_C is such that:

$$\frac{Re}{Fr^2} = F^* = O(1) \Rightarrow U_C \approx \frac{gd^3}{\lambda \nu_0}. \tag{9.86}$$

In place of boundary conditions (9.80) and (9.81a) we obtain the following dimensionless conditions:

$$z = 0: \boldsymbol{v}^* = 0 \text{ and } w = 0, \tag{9.87a}$$

and

$$w = \frac{\partial H}{\partial \tau} + \boldsymbol{v}^* \cdot \boldsymbol{D}^* H, \text{ at } z = H(\tau, x, y). \tag{9.87b}$$

Finally, in place of the free-surface boundary condition (9.81b) we derive (when we take into account the expression for d_{ij} and also (9.82)) the following two approximate dimensionless boundary conditions at $z = H$:

$$\frac{\partial \boldsymbol{v}^*}{\partial z} = O(\varepsilon), \text{ at } z = H(\tau, x, y), \tag{9.87c}$$

which expresses the fact that the shear stresses in the free-surface is zero when $\varepsilon \to 0$, and

$$P = -\, We\, \mathbf{D}^{*2}H + O(\varepsilon), \text{ at } z = H(\tau, x, y), \tag{9.87d}$$

where *We* is the Weber number:

$$We = \frac{\sigma_0}{\lambda^2 \rho_0 g}, \tag{9.88}$$

and $O(\varepsilon)$ in (9.87c, d) denotes the terms with tend to zero as $\varepsilon \to 0$. Now, the long-wave approximation model is derived from the dimensionless equations (9.85a, b, c), with the dimensionless conditions (9.87a, b, c, d), when:

$$\varepsilon \to 0, \text{ with } \tau, x, y, z \text{ and } S, Re, We, F^* \text{ fixed,}$$

under the similarity relation (9.86).Thus, we derive the following long-wave model problem for the unknown functions: $v^*(\tau, x, y, z)$, $w(\tau, x, y, z)$, $P(\tau, x, y, z)$, and $H(\tau, x, y)$:

$$\mathbf{D}^* \cdot v^* + \frac{\partial w}{\partial z} = 0; \tag{9.89a}$$

$$Re\left[S\frac{\partial v^*}{\partial \tau} + \left(v^* \cdot \mathbf{D}^*\right)v^* + w\frac{\partial v^*}{\partial z}\right] + \mathbf{D}^* P = \frac{\partial^2 v^*}{\partial z^2}; \tag{9.89b}$$

$$\frac{\partial P}{\partial z} + 1 = 0, \tag{9.89c}$$

$$z = 0:\ v^* = 0 \text{ and } w = 0, \tag{9.89d}$$

$$w = \frac{\partial H}{\partial \tau} + v^* \cdot \mathbf{D}^* H, \text{ at } z = H(\tau, x, y). \tag{9.89e}$$

$$\frac{\partial v^*}{\partial z} = 0, \text{ at } z = H(\tau, x, y), \tag{9.89f}$$

$$P = -We\boldsymbol{D}^{*2}H \text{ , at } z = H(\tau, x, y). \tag{9.89g}$$

Indeed, from the above problem we can derive a model problem for $\boldsymbol{v}^*(\tau, x, y, z)$, $w(\tau, x, y, z)$ and $H(\tau, x, y)$. First, from the continuity equation (9.89a), with the condition (9.89d) for w and the kinematic condition (9.89e), we derive an equation between $\mathbf{v}^*$ and H, namely:

$$\frac{\partial H}{\partial \tau} + \boldsymbol{D} \cdot \int_0^H \boldsymbol{v}^* dz = 0. \tag{9.90a}$$

Then, from the equation (9.89c), with the free-surface condition (9.89g), we derive a simple solution for P; namely:

$$P = H - z - We\boldsymbol{D}^{*2}H \,. \tag{9.90b}$$

As a consequence, we derive a second relation between $\boldsymbol{v}^*$ and H,

$$Re\left[S\frac{\partial \boldsymbol{v}^*}{\partial \tau} + \left(\boldsymbol{v}^* \cdot \boldsymbol{D}^*\right)\boldsymbol{v}^* + w\frac{\partial \boldsymbol{v}^*}{\partial z}\right] - \frac{\partial^2 \boldsymbol{v}^*}{\partial z^2} = \boldsymbol{D}^* H - We\boldsymbol{D}^*\left(\boldsymbol{D}^{*2}H\right), \tag{9.90c}$$

with

$$w = -\int_0^z \left(\boldsymbol{D}^* \cdot \boldsymbol{v}^*\right) dz\,. \tag{9.90d}$$

For the equation (9.90c) we have as boundary conditions (in z):

$$\boldsymbol{v}^* = 0, \text{ at } z = 0, \ \frac{\partial \boldsymbol{v}^*}{\partial z} = 0 \ \text{ at } z = H. \tag{9.90e}$$

The above problem (9.90) is easier to solve numerically than the above dimensionless problem (9.85), with (9.87).

9.4.3. The lubrication equation for the thickness of the thin film

Now we consider the low-Reynolds number limit:

$$Re \to 0 \text{ with } S \text{ and } We \text{ fixed.} \tag{9.91}$$

In this case, from the equation (9.90c) and boundary conditions (9.90e) we derive for:

$$Lim_{Re\to 0} v^* = V\ , \tag{9.92}$$

the following simple solution:

$$V(z;H) = \frac{1}{2}\{\mathbf{D}^* H - We\mathbf{D}^*[\mathbf{D}^{*2}H]\}[z^2 - 2Hz]. \tag{9.93}$$

As a consequence of the solution (9.93) and equation (9.90a), we derive, for the thickness $H(\tau, x, y)$ of the thin film, the following single equation:

$$\frac{\partial H}{\partial \tau} + \frac{1}{3}\mathbf{D}^* \cdot \{H^3[We\mathbf{D}^*(\mathbf{D}^{*2}H) - \mathbf{D}^* H]\} = 0\,. \tag{9.94}$$

In the above *lubrication equation,* (9.94), the term proportional to Weber number, We, has a stabilizing effect, while the 'gravity' term: $-(1/3)\ \mathbf{D}^* \cdot [H^3\mathbf{D}^*H]$, also stabilizes the evolution of the free surface when the film is supported from below. Finally, we observe that the above lubrication equation (9.94) is valid, in the framework of the present asymptotic derivation, when:

$$\lambda >> \frac{d^2}{\nu_0}[gd]^{1/2}\,, \tag{9.95}$$

where g, d and ν_0 are known data from the physical problem.

9.5. SOME COMPLEMENTARY REMARKS

9.5.1. Low Reynolds number flow past a circular cylinder

Concerning the matching, it is necessary to observe that for the problem of a low-Reynolds number strongly viscous steady flow past a circular cylinder, the idea of matching is based on the introduction of intermediate limits. When there is uniform flow at infinity, and the circular cylinder is at the origin, with no-slip dimensionless boundary condition: $\boldsymbol{u} = 0$ on $r = 1/2$, the inner Stokes equation is simply

$$\nabla^2 \omega_0 = 0, \ \boldsymbol{\omega}_0 = \omega_0 \mathbf{k} \tag{9.96a}$$

where $\boldsymbol{\omega}_0 = curl\ \mathbf{u}_0$, and the inner Stokes expansion ($Re \to 0$ with $r^2 = x^2 + y^2$, fixed) is:

$$\mathbf{u} = (u, v) = \alpha_0(Re)\, \mathbf{u}_0(r, \theta) + \alpha_1(Re)\, \mathbf{u}_1(r, \theta) + \ldots \tag{9.96b}$$

For $\omega_0(r, \theta)$ we easily obtain a solution (see, for instance, pages 193-203, in the book by Cole (1968)), which is the familiar solution of Laplace's equation in cylindrical coordinates (r, θ). As a consequence, we can write, for the leading order of the tangential component of the velocity $u_0(r, \theta)$, the following Stokes solution:

$$u_0(r,\theta) = B_1 \log r + C_1 + B_1 \sin^2 \theta + \frac{D_1}{r^2} \cos 2\theta \,. \tag{9.97a}$$

Unfortunately, if the condition at infinity: $u_0(r, \theta) \to \sin\theta$, when $r \to \infty$, is imposed, then the no-slip condition: $u_0(r, \theta) = 0$, at $r = 1/2$ can not be satisfied (Stokes's paradox). As a consequence, between three constants B_1, C_1, D_1, we have only two relations:

$$\left[\frac{1}{2} \log\left(\frac{1}{2}\right)\right] B_1 + \frac{1}{2} C_1 + 2D_1 = 0, \tag{9.97b}$$

$$\left[\log\left(\frac{1}{2}\right) + 1\right] B_1 + C_1 - 4D_1 = 0. \tag{9.97c}$$

Now, in order to construct the outer (Oseen) expansions, we introduce new variables:

$$x^* = Re\, x \text{ and } y^* = Re\, y \to r^* = Re\, r, \tag{9.98a}$$

and consider the Oseen outer expansion (($Re \to 0$ with $r^{*2} = x^{*2} + y^{*2}$, fixed):

$$u = 1 + \beta_0(Re)\, u^*_0(x^*, y^*) + \beta_1 u^*_1 + \ldots, \tag{9.98b$_1$}$$

$$v = \beta_0(Re)\, v^*_0(x^*, y^*) + \beta_1 v^*_1 + \ldots, \tag{9.98b$_2$}$$

$$p = \beta_0(Re)\, p^*{}_0(x^*, y^*) + \beta_1 u^*{}_1 + \ldots, \qquad (9.98b_3)$$

and derive the equations proposed by Oseen (see, for instance, in Lagerstrom (1964), pages 88-102 and 202-205) as a model for high Reynolds number flow, but they appear here as part of an approximation scheme for low-*Re*:

$$\frac{\partial u^*{}_0}{\partial x^*} + \frac{\partial v^*{}_0}{\partial y^*} = 0, \qquad (9.98c_1)$$

$$\frac{\partial u^*{}_0}{\partial x^*} + \frac{\partial p^*{}_0}{\partial x^*} = \frac{\partial u^*{}_0{}^2}{\partial x^{*2}} + \frac{\partial u^*{}_0{}^2}{\partial y^{*2}}, \qquad (9.98c_2)$$

$$\frac{\partial v^*{}_0}{\partial x^*} + \frac{\partial p^*{}_0}{\partial y^*} = \frac{\partial v^*{}_0{}^2}{\partial x^{*2}} + \frac{\partial v^*{}_0{}^2}{\partial y^{*2}}. \qquad (9.98c_3)$$

The idea of matching of the Stokes and Oseen expansions, according to Cole (1968), is the introduction of a intermediate ('η') limits in which x_η, y_η, are held fixed and:

$$x_\eta = \eta(Re)x,\; y_\eta = \eta(Re)y,\; \eta(Re) \to 0 \text{ with } Re \to 0;\; Re << \eta(Re) << 1.$$

Therefore in this 'η' limit, we know that:

$$r = \frac{r_\eta}{\eta(Re)} \to \infty \text{ but } r^* = \frac{Re}{\eta(Re)} r_\eta \to 0 \text{ with } \eta(Re) \to 0. \qquad (9.99)$$

Now, assuming that the Stokes and Oseen expansions are valid in an (Kaplun) overlap domain, we can compare the intermediate forms of Stokes (inner) and Oseen (outer) expansions according to above relations for $\boldsymbol{u}$.

From (when r_η is fixed):

$$Lim_{Re\to 0}\left[\alpha_0(Re)u_0\left(\frac{r_\eta}{\eta},\theta\right) + \alpha_1(Re)u_1\left(\frac{r_\eta}{\eta},\theta\right) + \ldots \right.$$
$$\left. -1 - \beta_0(Re)u^*{}_0\left(\frac{Re}{\eta}x_\eta, \frac{Re}{\eta}y_\eta\right) - \ldots\right]\frac{1}{(Re)^m} = 0$$

we write, for $m = 0$ and the two term of the Stokes expansion and two terms of the Oseen expansion:

$$Lim_{Re\to 0}\left\{\alpha_0(Re)\left[B_1 \log\frac{r_\eta}{\eta} + C_1 + B_1 \sin^2\theta + \frac{D_1}{\left(\frac{r_\eta}{\eta}\right)^2}\cos 2\theta\right]\right.$$

$$\left.+\alpha_1(Re)u_1\left(\frac{r_\eta}{\eta},\theta\right) - 1 - \beta_0(Re)u_0^*\left(\frac{Re}{\eta}x_\eta, \frac{Re}{\eta}y_\eta\right)\right\} = 0. \quad (9.100)$$

Now, from (9.100), a solution of the Oseen linear equations ($9.98c_1$)-($9.98c_3$) must be found, in which

$$u^*_0(x^*, y^*) \to a \log r^* + \ldots \text{ as } r^* \to 0, \quad (9.101)$$

or

$$u^*_0\left(\frac{Re}{\eta}x_\eta, \frac{Re}{\eta}y_\eta\right) \to a \log\left(\frac{Re}{\eta}r_\eta\right) + \ldots, \text{ under the '}\eta\text{' limit,}$$

if and only if

$$-\alpha_0(Re)B_1 \log\eta - 1 + \beta_0(Re)a \log\left(\frac{1}{Re}\right) + \beta_0(Re)a \log\eta = 0,$$

and matching (in the leading-order) is accomplished if:

$$B_1 = a = 1 \text{ and } \alpha_0(Re) = \beta_0(Re) = \frac{1}{\log\left(\frac{1}{Re}\right)}. \quad (9.102)$$

In this case, with (9.97b) we obtain finally for the Stokes solution (9.97a):

$$B_1 = 1,\ C_1 = -\frac{1}{2}\log\left(\frac{1}{2}\right) - \frac{1}{2} \text{ and } D_1 = \frac{1}{8}. \quad (9.103)$$

Note that the nonlinear effects appear explicitly only in the outer-Oseen expansion and equations for (u^*_1, v^*_1, p^*_1) and these equations are inhomogeneous Oseen equations.

But it is necessary that:

$$\beta_1(Re) = [\beta_0(Re)]^2 = log^2\left(\frac{1}{Re}\right). \tag{9.104}$$

For the incompressible case, it turns out (again) that the outer-Oseen expansion includes the inner -Stokes expansion, and that a uniformly valid (steady) solution is found from the outer-Stokes expansion with a boundary condition satisfied on $r = 1/2$. Naturally, such a result can not be expected in the more general compressible case.

Concerning the unsteady case, in Bentwich and Miloh (1982) the solution for the flow due to an impulsively started cylinder is considered. This solution is expressed in terms of three expansions which represent the flow in three different space-time subdomains: one which holds 'early' in the process 'throughout' the exterior of the cylinder, and two which hold later in the process, a 'late-inner' and a 'late-outer' expansion.

However although it was tacitly assumed by the authors that the later expansion represent the flow everywhere beyond the immediate vicinity of the obstacle, it was subsequently found that this assumption is incorrect.

Indeed, in Bentwich and Miloh (1984), it is shown that there exists a fourth 'late-wake' subdomain and an appropriate additional expansion is developed and matched with the other three. This later expansion represents the flow late in the process in a wake region. This wake extends all the way downstream to infinity and its width is comparable to the diameter of the obstacle.

We observe that the question concerning the validity of the three-expansion solution proposed in Bentwich and Miloh (1982) is whether potential flow is a valid representation of a high Reynolds number flow past an obstacle.

Thus, just as the irrotational solution holds throughout the domain exterior to the obstacle but not on its surface, the three-expansion solution proposed in Bentwich and Miloh (1982) holds almost but not quite throughout to the 'later-outer' space-time subdomain under discussion.

Furthermore, just as the inadequacy of the classical irrotational flow can be mended by appending a boundary-layer, the discontinuity of the 'late-outer' expansion along the half-plane behind the cylinder can be mended by constructing an additional, fourth expansion.

It expresses the flow over this plane and its immediate vicinity late in the process. In the 'late-wake' subdomain we derive a fourth-order significant equation in the coordinate normal to the flow direction.

Two-dimensional flow past a cylindrical body of arbitrary profile at small Reynolds numbers is studied theoretically in Tamada, Miura and Miyagi (1983), who assume that $|logRe|^{-1}$ is *not small* while Re is still small. In this case the outer flow field is governed by the full Navier equations and with the *aid of numerical techniques* the authors fix the asymptotic flow field which matches with the Stokeslet flow at the origin as well as the uniform flow at infinity (an immersed body is equivalent to a Stokeslet at large distance).

The result is used to determine the relation between the force experienced by the cylinder and the Reynolds number for a cylinder of arbitrary cross-section.

In Skinner (1975) the classic singular perturbation problem of low Reynolds number flow past a circular cylinder is again considered, but, higher-order inertial effects, accounting for *asymmetry in the near-cylinder flow,* are determined by matching generalized inner and outer expansions. In fact the author considers a second 'small' parameter:

$$\delta = \frac{1}{l^\circ - \log Re} \quad \text{with } l^\circ = \log 4 - \gamma + \frac{1}{2} \approx 1.309, \qquad (9.105)$$

where $\gamma = 0.577$ is the Euler constant, and the ratio:

$$\mu = \frac{Re}{2\delta}\left[1 - \frac{1}{2}\delta + O(\delta^2)\right] \qquad (9.106)$$

measures the significance of the asymmetry correction to the Stokes expansion (for instance: when $Re = 0.050$ and $\delta = 0.232$, then $\mu(\%) = 9.5$).

9.5.2. *Rigorous mathematical results*

Concerning rigorous mathematical results, related to low Reynolds number flows, we mention the papers by: Finn (1965), Babenko (1976) and Vasiliev (1977).

It is also interesting to note that within the context of Proudman and Pearson's (1957) scheme the solution for the flow past a cylinder is not unique (!), unless steadiness is postulated.

Indeed, according to Bentwich (1985), it seems impossible to obtain a proof of uniqueness for unsteady two-dimensional flow past a cylinder which satifies the requirement that the disturbance flow at infinity should be finite.

Concerning this problem of uniqueness the reader can find a comprehensive study and various references in the review paper by Ladyzhenskaya (1975).

On the other hand in Pogu and Tournemine (1982) the reader can find a study by a variational approach of 2D biharmonic equations acting in unbounded domains. In particular this study is applied to the problem of low-Reynolds number viscous fluid flows and the authors obtain, for an *arbitrary profile,* an *existence* and *uniqueness* theorem for the leading term of the *Stokes inner* expansion.

9.5.3. Final remarks

Concerning the lubrication evolution equation (9.94) (which is, in fact, a Reynolds (1886) type equation for the incompressible case) it is necessary to write an initial condition (at $t = 0$!).

Indeed, this equation (9.94) is derived via two limiting processes:

$$\varepsilon \to 0 \text{ and } Re \to 0, \text{ both with time } \tau \text{ fixed!} \tag{9.107}$$

As a consequence we do not sure that the equation (9.94) remains valid near the time $\tau = 0$? To answer this we must consider two inner-time local expansions and two adjustment problems. Unfortunately, at the present time, these problems are still open.

But in §11.2 of Chapter 11, in the framework of the so-called 'primitive Kibel meteorological' equations, which are derived via a long-wave approximation, we give some information concerning the inner-time local problem near $t = 0$ and the corresponding adjustment process which makes it possible to initialize the primitive equations.

It seems that such a initialization procedure is also adequate for the lubrication equation (9.94), but it is necessary, in this case, to consider a second (relative to $Re << 1$) initialization procedure.

We observe that in the next Chapter 10, in the framework of the theory of thin films, various model equations are also derived via a long-wave approximation and as a consequence it is again necessary to consider the associated inner-time local problem and the adjustment-initialization procedure.

In conclusion, in spite of the fact that, as a topic the investigation of low-Reynolds number flows is not too popular (!), because such flows are rarely encountered, it may have been overlooked that these flows can be relatively easily analyzed and that such analyses raise rather fundamental questions.

For, instance, oscillatory disturbances as admissible solutions for the flow past a cylinder (Bentwich (1985)) and their possible relationship to the classic Von Karman street phenomenon will be addressed elsewhere, the uniqueness problem appears, at present, too complex to be tackled, and the compressible case remains open and, it seems, related to the fluid dynamical limit of the Boltzmann equation.

CHAPTER 10

ASYMPTOTIC MODELLING OF THERMAL CONVECTION AND INTERFACIAL PHENOMENA

In a liquid layer heated from below (Bénard thermal problem), convection and wavy motion of the free surface, open to ambiant passive air at constant temperature T_a and constant pressure p_a, represent one of the most important cases where capillary forces are displayed. The motion induced by tangential gradients of variable (with temperature T) surface tension, $\sigma = \sigma(T)$, is a very important aspect of the modern theory of thin films and is customarily called the Marangoni effect (after one of the first scientists to give an explanation of this effect). Indeed, this Chapter is mainly devoted to an asymptotic modelling of thermal convection (Rayleigh-Bénard type problem) and interfacial-thermocapillary processes in falling thin liquid films (the so-called "Bénard-Marangoni" instability problem). The governing equations and the boundary conditions on the free surface are the full Navier-Stokes-Fourier (NS-F) equations for a viscous, thermally conducting, compressible (expansible) Newtonian fluid (liquid) and associated jump conditions for the stress tensor, the heat flux and the temperature across the free surface. The simplified case we examine involves a simple geometry - the one-layer system, in which there is a weakly expansible liquid layer, whose lower boundary is a heated rigid plate and whose upper boundary is a deformable free surface with a passive gas (having negligible viscosity and density), the so-called "Bénard thermal convection problem".

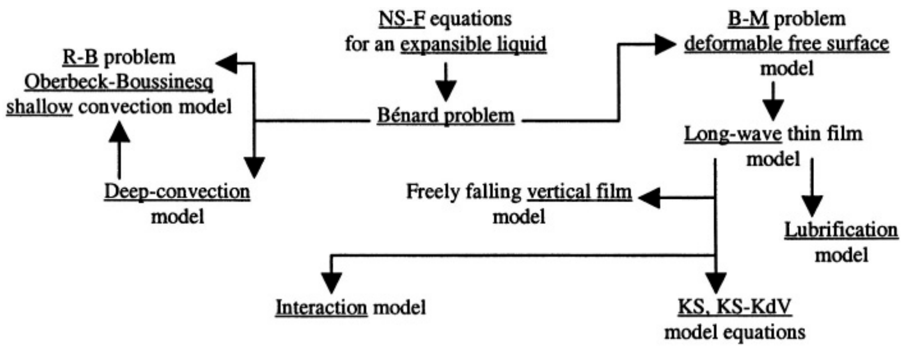


Fig. 10.1 Various models closely linkend with the termal convection and interfacial phenomena

10.1. INTRODUCTION

Consider an infinite horizontal layer of viscous, thermally conducting, expansible liquid of density ρ. On the bounding surface at $x_3 = 0$ this horizontal liquid layer is in contact with a solid wall of constant temperature T_B and at the level $x_3 = d$ is a free boundary with an atmosphere of constant temperature T_a and constant atmospheric pressure p_a, having negligible viscosity and density.

At the free surface Newton's law of heat transfer is invoked (see, for instance, Davis (1987) and Joseph (1976)), and the surface tension $\sigma(T)$ decreases linearly with temperature T, thus:

$$\sigma(T) = \sigma(T_0) - \gamma(T - T_0), \tag{10.1a}$$

where

$$\gamma = -\frac{d\sigma}{dT} \tag{10.1b}$$

is the constant rate of change of surface tension with temperature which is positive for most liquids. In (10.1a) T_0 is the constant temperature of the free surface in the purely static basic state, when:

$$T_S = T_S(x_3) = T_B - \beta x_3, \tag{10.2}$$

since a constant vertical temperature gradient: $-dT_S/dx_3 = \beta > 0$, is imposed on the expansible liquid layer. Obviously: $T_0 = T_B - \beta d$, and when we take into account the Newton's law of heat transfer (see below (10.7)) for β, we can write:

$$T_0 - T_a = \beta \frac{k(T_0)}{q^\circ} = \beta \frac{d}{Bi}, \tag{10.3}$$

where

$$Bi = \frac{dq^\circ}{k(T_0)} \tag{10.4}$$

is the Biot number which accounts for the heat transfer at the interface.

The value of Bi for a very thin layer (whose depth d is at most 0.1 cm, and in a such a case the buoyancy effect can be neglected - i.e. $\rho = const$) is at most 0.1 for the usual liquids and such a small value of Bi does not appreciably affect the results, relative to thermocapillary instability - but the limiting case $Bi \to 0$, certainly poses a problem! The basic temperature state (10.2) describes pure conduction in the liquid at rest, when the undisturbed free surface is at $x_3 \equiv d$.

We assume that the perturbed deformable free surface is represented, in a Cartesian coordinates system $(0; x_i)$, with $i = 1, 2, 3,$ (in which $\mathbf{g} = -g\,\mathbf{k}$ acts in the negative x_3 direction) by the equation:

$$x_3 = d + a\,\eta(t, x_1, x_2) \equiv h(t, x_1, x_2). \tag{10.5}$$

The expansible viscous liquid, with the equation of state:

$$\rho = \rho(T), \tag{10.6}$$

is a Newtonian fluid with viscosities $\lambda(T)$ and $\mu(T)$, specific heat $C(T)$ and thermal conductivity $k(T)$; $\kappa = k/\rho C$ is the thermal diffusivity and $\nu = \mu/\rho$ is the kinematic viscosity. In (10.3) and (10.4), the constant q° is the thermal conductance between the free surface and air.

At the free surface, (10.5), Newton's law of heat transfer is:

$$k(T)\frac{\partial T}{\partial n} + q^\circ (T - T_a) = 0, \tag{10.7}$$

where n is the distance in the direction of the outward normal $\boldsymbol{n}$ to the free surface. Condition (10.7) is, in fact, a so-called 'third-mixed type' (or Robin) condition, embracing the Dirichlet (corresponding to $q^{\circ} \to \infty$) and Neumann (when $q^{\circ} \to 0$) conditions. The justification for such a condition relies on the assumption that heat conduction, within the liquid, is much faster than within air, and that the heat flux on the free surface, considered from the inside of the fluid, may be approximated by such a difference in temperatures. Indeed, the temperature condition (10.7) is a "physically" admissible condition at $x_3 = h(t, x_1, x_2)$.

10.2. GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

10.2.1. Equations

The NS-F equations governing expansible fluid flow for the velocity vector $\boldsymbol{u}$, pressure p and temperature T, are:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \boldsymbol{u} = 0, \tag{10.8a}$$

$$\rho\frac{D\boldsymbol{u}}{Dt} + \nabla p + \rho g k = \nabla \cdot \boldsymbol{S}, \tag{10.8b}$$

$$\rho C(T)\frac{DT}{Dt} + p\nabla \cdot \boldsymbol{u} = \nabla \cdot (k\nabla T) + \Phi(\boldsymbol{u}), \tag{10.8c}$$

with (10.6), where $\boldsymbol{S}$ is the viscous stress (symmetric-second order) tensor of the liquid, thus: $\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}$, with $\boldsymbol{I}$ the unit second-order tensor and $\Phi(\boldsymbol{u})$ the dissipation function:

$$\boldsymbol{S} = \lambda(T)\,(\nabla \cdot \boldsymbol{u})\,\boldsymbol{I} + 2\,\mu(T)\,\boldsymbol{D}, \tag{10.9a}$$

$$\Phi(\boldsymbol{u}) = 2\,\mu(T)\,\boldsymbol{D} : \boldsymbol{D} + \lambda(T)\,(\nabla \cdot \boldsymbol{u})^2;\ \boldsymbol{D} : \boldsymbol{D} = (d_{ij})^2, \tag{10.9b}$$

Since, for the specific internal energy we have: $E = E(T)$, according to (10.6), then:

$$\frac{DE}{Dt} = C(T)\frac{DT}{Dt}, \text{ with } C(T) = \frac{DE}{DT},$$

the specific heat of the expansible liquid.

10.2.2. Boundary conditions

For the above NS-F equations (10.8a, b, c) the relevant boundary conditions are:

$$\boldsymbol{u} = 0,\ T = T_B, \text{ at } x_3 = 0, \tag{10.10}$$

and

$$-(p-p_a)\,\boldsymbol{n} + \boldsymbol{S}\cdot\boldsymbol{n} = 2\,\sigma(T)\,H\,\boldsymbol{n} + \nabla_{//}\sigma(T), \text{ on } x_3 = h(t, x_1, x_2), \tag{10.11a}$$

$$-k(T)\nabla T\cdot \mathbf{n} = q^{\circ}(T-T_a), \text{ on } x_3 = h(t, x_1, x_2). \tag{10.11b}$$

The location of the deformable free surface, $x_3 = h(t, x_1, x_2)$, is determined via the kinematic condition:

$$u_3 = \frac{\partial h}{\partial t} + u_1\frac{\partial h}{\partial x_1} + u_2\frac{\partial h}{\partial x_2}, \text{ on } x_3 = h(t, x_1, x_2). \tag{10.11c}$$

We note that in the above boundary conditions on the free surface, we don't take into account the surface viscosities

10.2.3. Dimensionless dominant equations

For a weakly expansible liquid, when the dimensionless parameter:

$$\beta^* = \beta\, d\, \alpha_T, \text{ with } \alpha_T = -\left[\frac{d(\log\rho)}{dT}\right]_{T=T_0} \tag{10.12}$$

is a *small parameter,* from the above formulated full NS-F model problem, it is possible to derive a simplified dimensionless dominant problem.

Let the coefficients with the subscript "$_0$" be reference values for these coefficients at $T = T_0$. First, we introduce the *dimensionless perturbations* for the pressure and temperature by:

$$\theta = \frac{(T-T_0)}{\beta d}, \tag{10.13a}$$

$$\pi = \frac{1}{Fr^2}\left\{\frac{p - p_a}{gd\rho_0} + [x'_3 - 1]\right\}, \tag{10.13b}$$

where Fr^2 is the square of the Froude number; namely:

$$Fr^2 = \left(\frac{\nu_0}{d}\right)^2 \frac{1}{gd}. \tag{10.14}$$

We observe that the introduction of the pressure perturbation π, according to (19.13b), is deduced from a careful dimensionless analysis of the NS-F model problem written with dimensionless variables (with '):

$$x'_i = \frac{x_i}{d} \text{ and } t' = \frac{t}{\frac{d^2}{\nu_0}}, \tag{10.15}$$

for the dimensionless functions: $v_i = u_i/(\nu_0/d)$, θ and π, which depend on t' and x'_i. The introduction of θ and π makes it possible to derive asymptotically in a consistent way, when β^* tends to zero, the Oberbeck-Boussinesq approximate equations. First, with (10.13a, b), from NS-F equations we derive the following 'dominant' dimensionless equations, when we neglect the terms proportional to the *square* of the small parameter β^*:

$$\frac{\partial v_k}{\partial x'_k} = \beta^* \frac{D\theta}{Dt'}, \tag{10.16a}$$

$$(1 - \beta^*\theta)\frac{Dv_i}{Dt'} + \frac{\partial \pi}{\partial x'_i} - \frac{\beta^*}{Fr^2}\theta k - \nabla'^2 v_i = \beta^* L(v_i, \theta), \tag{10.16b}$$

$$[1 - \beta^*\theta(1 + \Gamma^\circ)]\frac{D\theta}{Dt'} - \frac{1}{Pr}\nabla'^2\theta - 2BqFr^2(d'_{ij})^2 = \beta^* N(v_i, \theta). \tag{10.16c}$$

We note that in (10.16a, b, c) we have:

$$d'_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial x'_j} + \frac{\partial v_j}{\partial x'_i}\right], \quad \frac{D}{Dt'} = \frac{\partial}{\partial t'} + v_i \frac{\partial}{\partial x'_i},$$

$$\nabla'^2 = \frac{\partial^2}{\partial (x'_1)^2} + \frac{\partial^2}{\partial (x'_2)^2} + \frac{\partial^2}{\partial (x'_3)^2},$$

$$L(v_i,\theta) = \left[1 + \frac{\lambda_0}{\mu_0}\right]\frac{\partial}{\partial x'_i}\frac{D\theta}{Dt'} - M^\circ \frac{\partial}{\partial x'_j}\left(2\theta d'_{ij}\right), \tag{10.17}$$

$$N(v_i,\theta) = Bq\left[p'_a + 1 - x'_3 + Fr^2\pi\right]\frac{D\theta}{Dt'} - 2BqFr^2 M^\circ \theta \left(d'_{ij}\right)^2$$
$$- \frac{K^\circ}{Pr}\frac{\partial}{\partial x'_i}\left[\theta \frac{\partial \theta}{\partial x'_i}\right]$$

where $p'_a = p_a/gd\rho_0$, and:

$$\Gamma^\circ = \left[\frac{\dfrac{d\log C}{dT}}{\dfrac{d\log\rho}{dT}}\right]_0, \quad M^\circ = \left[\frac{\dfrac{d\log\mu}{dT}}{\dfrac{d\log\rho}{dT}}\right]_0,$$

$$K^\circ = \left[\frac{\dfrac{d\log k}{dT}}{\dfrac{d\log\rho}{dT}}\right]_0. \tag{10.18}$$

Of course, it is assumed that Γ°, M° and K° are $O(1)$, when $\beta^* \to 0$. For the derivation of the dominant equations (10.16a, b, c) we have also taken into account the following approximate relations, valid with an error of $O(\beta^{*2})$,

$$\rho(T) \cong \rho_0[1 - \beta^*\theta], \; \mu(T) \cong \mu_0[1 - \beta^* M^\circ\theta], \; k(T) \cong k_0[1 - \beta^* K^\circ\theta],$$

$$C(T) \cong C_0 \, [1 - \beta^* \Gamma^\circ \theta]. \tag{10.19}$$

In the above dimensionless equations and relations:

$$Pr = \frac{\nu_0}{\kappa_0}, \text{ and } Bq = \frac{g}{\beta C_0}, \tag{10.20}$$

are the Prandtl number and the analogue of a Boussinesq number.

10.2.4. Dimensionless dominant boundary conditions

For these (dominant) equations (10.16a, b, c) it is necessary to derive a set of dominant dimensionless boundary conditions, from (10.10), (10.11a, b, c), with (10.13a, b), for v_i, θ and π.

We note that, in the free surface boundary condition (10.11a), the operator $\nabla_{//}$ is the surface gradient and H is the mean curvature; namely we can write:

$$\nabla_{//} = \nabla - \boldsymbol{n}\,(\boldsymbol{n} \cdot \nabla) \text{ and } H = -\frac{1}{2}\left(\nabla_{||} \cdot \boldsymbol{n}\right).$$

Now, we write the free surface equation, (10.5), in the dimensionless form:

$$x'_3 = 1 + \delta\,\eta\,(t', x'_1, x'_2) \equiv h', \tag{10.21}$$

where

$$\delta = \frac{a}{d}, \tag{10.22}$$

is the dimensionless amplitude parameter for the dimensionless free surface deformation $\eta(t', x'_1, x'_2)$ relative to the plane $x'_3 = 1$.

Then, we derive the following five dominant dimensionless boundary conditions, at the deformable free surface, $x'_3 = 1 + \delta\,\eta\,(t', x'_1, x'_2) \equiv h'$:

$$\pi = \frac{\delta}{Fr^2}\eta + 2d'_{ij}\,n'_i\,n'_j + [We - Ma\theta]\left(\nabla'_{||} \cdot \boldsymbol{n}'\right) + \beta^* P(v_i, \theta); \tag{10.23a}$$

$$d'_{ij}t'^{(s)}_i n'_j + \frac{1}{2} Mat'^{(s)}_i \frac{\partial \theta}{\partial x'_i} = \beta^* Q(v_i, \theta),\ s = 1, 2; \qquad (10.23b, c)$$

$$\nabla'\theta \cdot \boldsymbol{n} + Bi\theta + 1 = \beta^* K° \theta \nabla'\theta \cdot \boldsymbol{n}'; \qquad (10.23d)$$

$$v_3 = \delta \left[\frac{\partial \eta}{\partial t'} + v_1 \frac{\partial \eta}{\partial x'_1} + v_2 \frac{\partial \eta}{\partial x'_2} \right]. \qquad (10.23e)$$

In the two ($s = 1$ and 2) dimensionless dominant boundary conditions (10.23b, c), $t_i'^{(1)}$ and $t_i'^{(2)}$ are the dimensionless components of two orthonormal tangent vectors to the free surface $x'_3 = h'\ (\equiv 1 + \delta\eta)$, and n'_i are the dimensionless components of the outward normal $\boldsymbol{n}'$ to the free surface; namely:

$$\boldsymbol{t}'^{(1)} = \frac{1}{\sqrt{N'_1}} \left(1; 0; \frac{\partial h'}{\partial x'_1} \right);$$

$$\boldsymbol{t}'^{2} = \frac{1}{\sqrt{N'_1 N'}} \left(-\frac{\partial h'}{\partial x'_1} \frac{\partial h'}{\partial x'_2}; 1 + \left(\frac{\partial h'}{\partial x'_1} \right)^2; \frac{\partial h'}{\partial x'_2} \right); \qquad (10.24)$$

$$\boldsymbol{n}' = \frac{1}{\sqrt{N'}} \left(-\frac{\partial h'}{\partial x'_1}; -\frac{\partial h'}{\partial x'_2}; 1 \right).$$

In (10.24), according to Pavithran and Redeekopp (1994), the tangential and normal unit vectors to the deformed free surface: $x'_3 = h'(t', x'_1, x'_2)$, are written in terms of the (x'_1, x'_2, x'_3) Cartesian system of coordinates. In the boundary condition (10.23a), for $(\nabla'_{//} \cdot \boldsymbol{n}')$ we have the following explicit formula:

$$-(N')^{-3/2} \left\{ N'_2 \frac{\partial^2 h'}{\partial (x'_1)^2} + N'_1 \frac{\partial^2 h'}{\partial (x'_2)^2} - 2 \frac{\partial^2 h'}{\partial x'_1 \partial x'_2} \frac{\partial h'}{\partial x'_1} \frac{\partial h'}{\partial x'_2} \right\} \qquad (10.25a)$$

where

$$N'_1 = 1 + \left(\frac{\partial h'}{\partial x'_1}\right)^2 \; ; \; N'_2 = 1 + \left(\frac{\partial h'}{\partial x'_2}\right)^2 \; ; \qquad (10.25b)$$

$$N' = 1 + \left(\frac{\partial h'}{\partial x'_1}\right)^2 + \left(\frac{\partial h'}{\partial x'_2}\right)^2 . \qquad (10.25c)$$

In (10.23a, b), for the terms $P(v_i, \theta)$ and $Q(v_i, \theta)$ we have:

$$P(v_i, \theta) = \frac{\lambda_0}{\mu_0} \frac{D\theta}{Dt'} - 2M^{\circ}\theta d'_{ij} n'_i n'_j \, ; \qquad (10.26a)$$

$$Q(v_i, \theta) = M^{\circ}\theta d'_{ij} t'^{(s)}_i n'_j . \qquad (10.26b)$$

Finally, instead of (10.10) we write:

$$v_i = 0 \text{ and } \theta = 1, \text{ at } x'_3 = 0, \qquad (10.27)$$

if we take into account (10.13a) and the definition of $T_0 = T_B - \beta d$.

In the dimensionless boundary conditions, (10.23a, b, c, d, e), the dimensionless parameters:

$$We = \sigma_0 \frac{d}{\rho_0 (v_0)^2}, \; Bi = q^{\circ} \frac{d}{k_0}, \; Ma = \gamma d^2 \frac{\beta}{\rho_0 (v_0)^2}, \qquad (10.28)$$

are, respectively, the Weber, Biot and Marangoni numbers (when we take into account (10.1a)). We note also that:

$$Cr = \frac{1}{We} \text{ and } Bn = \frac{Cr}{Fr^2}, \qquad (10.29)$$

are, respectively, the crispation (or capillary) and the Bond numbers.

10.3. FROM THE NS-F DOMINANT DIMENSIONLESS PROBLEM TO RAYLEIGH-BÉNARD (R-B) THERMAL SHALLOW CONVECTION PROBLEM

In the above formulation of the classical Bénard thermal-free surface problem, via the NS-F model problem, there are two mechanisms responsible for driving the instability: the first one is the *density variation* generated by the thermal expansion of the liquid, the second results from the *free-surface tension gradients* due to temperature fluctuations at the upper free surface of the liquid layer.

It is usual in the literature (see, for instance, Drazin and Reid (1981)), to denote as *Rayleigh-Bénard* (R-B) *shallow* thermal convection, the instability problem produced mainly by *buoyancy* (including, possibly, the Marangoni and Biot effects in a *non-deformable free surface).*

Naturally, in the above full mathematical formulation of the dimensionless Bénard thermal, free surface, problem governing by the equations, (10.16a, b, c) and boundary conditions, (10.23a, b, c, d, e), (10.27), for a weakly ($\beta^* << 1$) expansible viscous and heat conducting liquid, *both buoyancy and surface-gradient effects are operative,* so it is important (from our point of view!) to ask:

How are the two effects coupled when $\beta^* \to 0$?

Indeed, in order to extract a tractable problem from the complicated dimensionless system of dominant equations, (10.16a, b, c), and dominant boundary conditions, (10.23a, b, c, d, e), (10.27), which we have just presented above, we will now undertake an asymptotic analysis based mainly upon the small parameter $\beta^* \to 0$!

10.3.1. Boussinesq limiting process

More precisely, from equation (10.16b), when $\beta^* \to 0$, it is obvious that if we want to take into account the buoyancy term: $(\beta^*/Fr^2)\ \theta\mathbf{k}$, then it is necessary to impose, according to Zeytounian (1989), the following "*Boussinesq limiting process*"

$$\beta^* \to 0 \text{ and } Fr^2 \to 0, \text{ with } \frac{\beta^*}{Fr^2} \equiv Gr = O(1), \tag{10.30}$$

where

$$Gr = \frac{\beta \alpha_T d^4 g}{(\nu_0)^2} \tag{10.31a}$$

is the Grashof number and the associated Rayleigh (Ra) number is

$$Ra = Pr\,Gr = \frac{\beta \alpha_T d^4 g}{\nu_0 \kappa_0}. \tag{10.31b}$$

Now, if we consider the free surface boundary condition (10.23a) for π then, as a consequence of $Fr^2 << 1$, the first term, $(\delta/Fr^2)\,\eta$, in the right hand side of (10.23a) is *unbounded* when $Fr^2 \to 0$, if we assume that $\delta = O(1)$.

Thus, when $Fr^2 << 1$:

The free-surface pressure condition (10.23a) is (asymptotically) consistent with the Boussinesq limiting process (10.30), only for a small free surface amplitude parameter δ, such that:

$$\frac{\delta}{Fr^2} = \delta^* = O(1), \text{ when } \delta \to 0 \text{ and } Fr^2 \to 0. \tag{10.32}$$

As a consequence of the condition $Fr^2 << 1$, the Boussinesq limiting process, (10.30), is only valid if the *thickness, d,* of the liquid layer is such that:

$$d >> \left[\frac{(\nu_0)^2}{g}\right]^{1/3} \approx 1mm, \tag{10.33}$$

according to (10.14).

10.3.2. Oberbeck-Boussinesq approximate equations

If now all the dependent and independent variables, the parameters Pr and Bq, the coefficient Γ^0 in (10.16c), and the terms $L(v_i, \theta)$ and $N(v_i, \theta)$ in (10.16b, c) are fixed and of $O(1)$, then we derive asymptotically the following classical shallow convection dimensionless (Oberbeck-Boussinesq) equations (written without the primes) for the limit functions v_{i0}, π_0, θ_0:

$$\frac{\partial v_{k0}}{\partial x_k} = 0, \tag{10.34a}$$

$$\frac{Dv_{i0}}{Dt} + \frac{\partial \pi_0}{\partial x_i} - Gr\theta_0 k = \nabla^2 v_{i0}, \tag{10.34b}$$

$$\frac{D\theta_0}{Dt} = \frac{1}{Pr}\nabla^2 \theta_0 . \tag{10.34c}$$

In these Boussinesq equations (10.34a, b, c) we have:

$$[v_{i0}, \pi_0, \theta_0] = lim[v_i, \pi, \theta], \text{ when } \beta^* \to 0, \; Fr^2 \to 0, \tag{10.35}$$

$$\text{with } \frac{\beta^*}{Fr^2} \equiv Gr = O(1). \tag{10.36}$$

As: $Bq = O(1)$ the above Boussinesq model equations (10.34a, b, c) are valid when:

$$\beta \cong \frac{g}{C_0}. \tag{10.37}$$

10.3.3. Formulation of the R-B rigid-free model problem with the buoyancy, Marangoni and Biot effects

Therefore if we take into account (10.23a) and consider the case of large Weber numbers, $We >> 1$, such that:

$$\delta We = We^* = O(1), \tag{10.38}$$

then (since $\delta << 1$), in conditions (10.23a) and (10.23b, c) the terms: $2\, d'_{ij}\, n'_i\, n'_j$ and $d'_{ij}\, t'^{(s)}_i\, n'_j$, are at least $O(\delta)$ and as a consequence, we derive, in the limit, the following R-B (rigid-free) problem, which takes into account at the non-deformable surface $x_3 = 1$, the buoyancy, Marangoni and Biot effects:

$$\frac{\partial v_{k0}}{\partial x_k} = 0,$$

$$\frac{Dv_{i0}}{Dt} + \frac{\partial \pi_0}{\partial x_i} - Gr\theta_0 k = \nabla^2 v_{i0},\ i = 1, 2, 3, \tag{10.39a}$$

$$\frac{D\theta_0}{Dt} = \frac{1}{Pr}\nabla^2\theta_0,$$

$$v_{10} = v_{20} = v_{30} = 0, \text{ and } \theta = 1, \text{ at } x_3 = 0,$$

$$v_{30} = 0, \text{ and } \frac{\partial^2 v_{30}}{\partial x_3^2} = Ma\left[\frac{\partial^2\theta}{\partial x_1^2} + \frac{\partial^2\theta}{\partial x_2^2}\right], \text{ at } x_3 = 1, \tag{10.39b}$$

$$\frac{\partial\theta}{\partial x_3} + Bi\theta + 1 = 0, \text{ at } x_3 = 1.$$

The deformation of the free surface $\eta(t, x_1, x_2)$ is then determined, when the perturbation of the pressure $\pi_0(t, x_1, x_2, x_3)$ is known at $x_3 = 1$, after the resolution of the R-B problem (10.39a, b), by the equation:

$$\frac{\partial^2\eta}{\partial x_1^2} + \frac{\partial^2\eta}{\partial x_2^2} - \frac{\delta^*}{We^*}\eta = -\frac{1}{We^*}\pi_0(t, x_1, x_2, 1). \tag{10.40}$$

It is important to note that the R-B problem (10.39a, b) is asymptotically derived from the dominant (relative to $\beta^* << 1$) problem formulated in §10.2, when:

$$\beta^* \to 0,\ Fr^2 \to 0,\ \delta \to 0 \text{ and } Cr \to 0, \tag{10.41a}$$

with the following *three similarity* relations:

$$\frac{\beta^*}{Fr^2} = Gr = O(1);\ \frac{\delta}{Fr^2} = \delta^* = O(1);\ \frac{\delta}{Cr} = We^* = O(1), \tag{10.41b}$$

and in a such case

$$Bn = \frac{Cr}{Fr^2} = O(1). \tag{10.42}$$

For a particular case, when:

$$\delta^* = We^* = Bi = Ma \equiv 0,$$

we obtain the classical R-B rigid-free shallow convection problem. The R-B problem (10.39a, b) has been recently considered by Dauby and Lebon (1996).

Concerning the equation of state (10.6), it is necessary to note that, in fact, from the relation (10.13b) we have:

$$\frac{p - p_a}{gd\rho_0} = 1 - x'_3 + Fr^2\pi$$

and it is clear that the presence of the pressure p in a full baroclinic equation of state: $\rho = \rho(T, p))$, in place of (10.6), does not change (in the Boussinesq (10.30) limit) the form of the derived O-B approximate equations (10.34a, b, c), since $Fr^2 \to 0$. Finally, it is necessary to note that [Zeytounian (1997, 1998)]:

It is not consistent (from an asymptotic point of view) to take into account simultaneously in the R-B thermal convection model problem (for a weakly expansible liquid) the buoyancy effect and the deformation of the free surface.

10.4. FROM THE NS-F DOMINANT DIMENSIONLESS PROBLEM TO THE ZEYTOUNIAN DEEP-CONVECTION PROBLEM

The above O-B model equations (10.34a, b, c) are valid only if the Boussinesq-like number Bq is bounded, of the order $O(1)$, and in this case we have for β the relation (10.37). But, $\beta = (T_B - T_0)/d \equiv \Delta T/d$ and as a consequence for the validity of the O-B model equations it is necessary that:

$$d \cong \frac{\Delta T C_0}{g}. \tag{10.43}$$

In Zeytounian (1989) the case corresponding to:

$$BqFr^2 = \beta^* \frac{Bq}{Gr} = O(1), \tag{10.44}$$

is considered. In this case, in the dimensionless equation (10.16c), it is necessary to take into account the following two terms:

$$-2\, Bq\, Fr^2\, (d'_{ij})^2 \text{ and } \beta^*\, Bq\, (p'a + 1 - x'_3)$$

according to the relation (10.17) for $N(v_i, \theta)$.

But when we consider the case of a fluid layer between two rigid boudaries, then we can assume that $p'_a = 0$. According to Zeytounian (1989) the so-called R-B *deep convective* instability dimensionless problem, with a judicious choice of the velocity components V_i, perturbation pressure, Π, and perturbation temperature, Θ, is governed by the following equations:

$$\frac{\partial V_k}{\partial x_k} = 0,$$

$$\frac{1}{Pr}\frac{Dv_i}{D\tau} + \frac{\partial \Pi}{\partial x_i} - \Theta\delta_{i3} = \Delta V_i\,,\ i = 1, 2, 3, \tag{10.45a}$$

$$[1+\delta(1-x_3)]\left\{\frac{D\Theta}{D\tau} - RaV_3\right\} = \Delta\Theta + \frac{1}{2}\delta\left[\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right]^2,$$

where $Ra = Pr\, Gr$ is the Rayleigh number and Gr the Grashof number.

In the equation for Θ,

$$\delta = \frac{\beta^* gd}{\Delta TC_0}, \tag{10.45b}$$

is the "depth" parameter. In the case of deep convection, in place of (10.43) we have:

$$d \cong \frac{\Delta TC_0}{g\beta^*}, \tag{10.45c}$$

with $\Delta T \equiv T_B - T_0$.

For the so-called "*deep convection equations*" (10.45a) we write as boundary conditions relative to x_3:

$$\Theta = 0 \text{ at } x_3 = 0 \text{ and at } x_3 = 1,$$

$$V_i = 0 \text{ at } x_3 = 0 \text{ and } V_3 = 0,\ \frac{\partial^2 V_3}{\partial x_3{}^2} = 0 \text{ at } x_3 = 1. \qquad (10.45d)$$

Concerning the mathematical analysis of the Bénard problem for deep convection, the reader can find various rigorous results in the papers by Charki [see, for instance, Charki (1996)]. For the usual liquids, according to (10.43), the thickness of the liquid layer should be at most of the order of a kilometer! Naturally, for the usual technological applications of the O-B equations this estimate is very good, but for various geophysical applications the deep convection equations (10.45a) are more convenient and the reader can find some numerical calculations in Errafiy and Zeytounian (1991a, 1991b).

It is interesting to note that the parameter $Bq\ Fr^2$ plays the role of the square of a reference Mach number, $M^2 = (\nu^0/d)^2/C_0\Delta T$, based on the temperature difference ΔT, between the rigid lower and upper boundaries and $C_0\Delta T$ plays the role of a characteristic "acoustic speed" for the weakly expansible liquid.

Since $Bq = O(1)$, for the rigorous derivation of the O-B model equations (10.34a, b, c), then the Boussinesq limiting process (10.30) is, indeed, a limiting process relative to $M^2 \approx Fr^2$ and as a consequence the derivation of the O-B model equations, from the full NS-F equations, is really related to the low-Mach number asymptotics.

10.5. FROM NS-F DOMINANT DIMENSIONLESS PROBLEM TO THE BÉNARD-MARANGONI (B-M) PROBLEM

Indeed, the famous Bénard (1900), *convective cells* are primarily induced by the free-surface tension gradients resulting from the temperature variations across the free surface (the so-called Marangoni effect).

Thus, it seems justified to use the term Bénard-Marangoni (B-M) thermocapillary instability problem when, as in Bénard's experiments, the dominant, acting, driving force is the surface-tension gradient on the deformable free surface (without the influence of the buoyancy force).

From (10.23a), obviously, for the derivation of the full B-M model thin film problem, when the free surface deformation:

$\delta = O(1)$ in the equation (10.21),

plays an essential role, it is necessary to assume also that the Froude number Fr is $O(1)$. As a consequence we must consider the following *incompressible limit process:*

$$\beta^* \to 0 \text{ and } Fr^2 = O(1). \tag{10.46a}$$

But in this case, for most liquids: $Bq << 1$, since:

$$\frac{C_0 \beta d}{g} >> d \approx \left(\frac{v_0^{\ 2}}{g}\right)^{1/3}. \tag{10.46b}$$

10.5.1. Formulation of the full B-M model problem

As a consequence of (19.46a) the buoyancy term in equation (10.16b) is negligible for a weakly expansible liquid, and as a consequence, in place of the Boussinesq equations (10.34a, b, c), we derive from the dominant dimensionless equations (10.16a, b, c) the following "*incompressible* model equations":

$$\frac{\partial v_{k0}}{\partial x_k} = 0\,, \tag{10.47a}$$

$$\frac{Dv_{i0}}{Dt} + \frac{\partial \pi_0}{\partial x_i} = \nabla^2 v_{i0}\,, \tag{10.47b}$$

$$\frac{D\theta_0}{Dt} = \frac{1}{Pr}\nabla^2 \theta_0\,, \tag{10.47c}$$

written without the primes and where $D/Dt = \partial/\partial t + v_{i0}\, \partial/\partial x_i$.

For these equations (10.47a, b, c), since $Fr^2 = O(1)$ and $\delta = O(1)$, we must write the *deformable* free-surface conditions. These boundary conditions (see below the dimensuinless free-surface conditions (10.49a, b, c, d)) are written for the unknowns: v_{i0}, π_0, θ_0, and as a consequence we obtain a coupled B-M problem for the limiting functions:

$$[v_{i0},\pi_0,\theta_0]=lim_{\beta^*\to 0, Fr^2=O(1)}[v_i,\pi,\theta], \quad (10.48)$$

and also $h(t, x_1, x_2)$, the variable thickness of the unknown free surface of the film. According to the dominant dimensionless boundary conditions (10.23a, b, c, d) we derive a set of very complicated dimensionless boundary conditions written at the deformable free surface: $x_3 = h(t, x_1, x_2)$, namely:

$$\pi_0=\frac{h-1}{Fr^2}+\frac{2}{N}\left\{\frac{\partial v_{10}}{\partial x_1}\left(\frac{\partial h}{\partial x_1}\right)^2+\frac{\partial v_{02}}{\partial x_2}\left(\frac{\partial h}{\partial x_2}\right)^2+\left[\frac{\partial v_{10}}{\partial x_2}+\frac{\partial v_{20}}{\partial x_1}\right]\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\right.$$
$$\left.+\frac{\partial v_{30}}{\partial x_3}-\left[\frac{\partial v_{10}}{\partial x_3}+\frac{\partial v_{30}}{\partial x_1}\right]\frac{\partial h}{\partial x_1}-\left[\frac{\partial v_{20}}{\partial x_3}+\frac{\partial v_{30}}{\partial x_2}\right]\frac{\partial h}{\partial x_2}\right\}$$
$$-\left[\frac{We-Ma\theta_0}{N^{3/2}}\right]\left\{N_2\frac{\partial^2 h}{\partial x_1{}^2}+N_1\frac{\partial^2 h}{\partial x_2{}^2}-2\frac{\partial^2 h}{\partial x_1\partial x_2}\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\right\}; \quad (10.49a)$$

$$\left[\frac{\partial v_{10}}{\partial x_1}-\frac{\partial v_{30}}{\partial x_3}\right]\frac{\partial h}{\partial x_1}+\frac{1}{2}\left[\frac{\partial v_{10}}{\partial x_2}+\frac{\partial v_{20}}{\partial x_1}\right]\frac{\partial h}{\partial x_2}+\frac{1}{2}\left[\frac{\partial v_{20}}{\partial x_3}+\frac{\partial v_{30}}{\partial x_2}\right]\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}$$
$$-\frac{1}{2}\left[1-\left(\frac{\partial h}{\partial x_1}\right)^2\right]\left[\frac{\partial v_{30}}{\partial x_1}+\frac{\partial v_{10}}{\partial x_3}\right]=\frac{Ma}{2}\left[\frac{\partial\theta_0}{\partial x_1}+\frac{\partial h}{\partial x_1}\frac{\partial\theta_0}{\partial x_3}\right]N^{1/2}; \quad (10.49b)$$

$$\left[\frac{\partial v_{10}}{\partial x_1}-\frac{\partial v_{20}}{\partial x_3}\right]\frac{\partial h}{\partial x_2}\left(\frac{\partial h}{\partial x_1}\right)^2+\left[\frac{\partial v_{10}}{\partial x_3}+\frac{\partial v_{30}}{\partial x_1}\right]\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}$$
$$+\left[\frac{\partial v_{20}}{\partial x_2}-\frac{\partial v_{30}}{\partial x_3}\right]\frac{\partial h}{\partial x_2}+\frac{1}{2}\left[1+\left(\frac{\partial h}{\partial x_1}\right)^2-\left(\frac{\partial h}{\partial x_2}\right)^2\right]\left[\frac{\partial v_{10}}{\partial x_2}+\frac{\partial v_{20}}{\partial x_1}\right]\frac{\partial h}{\partial x_1}$$
$$-\frac{1}{2}\left[1+\left(\frac{\partial h}{\partial x_1}\right)^2-\left(\frac{\partial h}{\partial x_2}\right)^2\right]\left[\frac{\partial v_{20}}{\partial x_3}+\frac{\partial v_{30}}{\partial x_2}\right]$$
$$=\frac{Ma}{2}\left\{-\frac{\partial h}{\partial x_1}\frac{\partial h}{\partial x_2}\frac{\partial\theta_0}{\partial x_1}+\left[1+\left(\frac{\partial h}{\partial x_1}^2\right)\right]\frac{\partial\theta_0}{\partial x_2}+\frac{\partial h}{\partial x_2}\frac{\partial\theta_0}{\partial x_3}\right\}N^{1/2}; \quad (10.49c)$$

$$\frac{\partial\theta_0}{\partial x_3}=\frac{\partial h}{\partial x_1}\frac{\partial\theta_0}{\partial x_1}+\frac{\partial h}{\partial x_2}\frac{\partial\theta_0}{\partial x_2}-(Bi\theta_0+1)N^{1/2} \qquad (10.49d)$$

We note that, according to (10.25b),

$$N_1=1+\left(\frac{\partial h}{\partial x_1}\right)^2;\ N_2=1+\left(\frac{\partial h}{\partial x_2}\right)^2;\ N=1+\left(\frac{\partial h}{\partial x_1}\right)^2+\left(\frac{\partial h}{\partial x_2}\right)^2.$$

Finally, in place of (10.23e) and (10.27) we write:

$$v_{30}=\frac{\partial h}{\partial t}+v_{10}\frac{\partial h}{\partial x_1}+v_{20}\frac{\partial h}{\partial x_2}, \text{ at } x_3=h(t, x_1, x_2), \qquad (10.50)$$

$$v_{10}=v_{20}=v_{30}=0,\ \theta_0=1, \text{ at } x_3=0. \qquad (10.51)$$

A B-M problem similar to [(10.47a, b, c), (10.49a, b, c, d), (10.50), (10.51)] was considered recently by Nepomnyaschy and Velarde (1994), but our boundary conditions (10.49a, b, c) are somewhat different. These conditions (10.49a, b, c) are consistent with the original exact boundary condition (10.11a) or boundary conditions (10.23a, b, c), if we take into account the definition (10.24), with (10.25b), of the unit tangential vectors, $\boldsymbol{t}'^{(1)}$ and $\boldsymbol{t}'^{(2)}$, and normal vector, $\boldsymbol{n}$, to the deformed free surface, under the incompressible limit (10.46a).

The above full B-M model problem is a very complicated problem, but for a thin- film flow we can apply the long-wave approximation. In this case, we derive (as in next Section 10.5.2) a simplified Bénard-Marangoni long-wave thin-film model problem which is a very significant, model problem for the various applications.

10.5.2. The B-M long-wave thin-film model problem

According to the long-wave approximation we assume (as in Section 9.4.2 of the §9.4, Chapter 9), that the characteristic value of the horizontal wave-length λ is *much greater* than the charactristic thickness d of the film.

In this case, instead of dimensionless variables (t', x'_1, x'_2, x'_3), it is judicious to introduce the following new dimensionless variables:

$$x=\varepsilon x'_1,\ y=\varepsilon x'_2,\ z\equiv x'_3,\ \tau=Re\, t', \qquad (10.52a)$$

and news functions:

$$u = \frac{v_{10}}{\frac{Re}{\varepsilon}},\ v = \frac{v_{20}}{\frac{Re}{\varepsilon}},\ w = \frac{v_{30}}{Re},\ \Pi = \frac{\pi_0}{\left(\frac{Re}{\varepsilon}\right)^2}, \qquad (10.52b)$$

in the dimensionless full B-M problem formulated in Section 10.5.1.

In (10.52a, b)

$$\varepsilon = \frac{d}{\lambda} \text{ and } Re = \frac{U^\circ \frac{d^2}{\lambda}}{v_0}. \qquad (10.53)$$

The choice of the characteristic velocity U° is such that:

$$\frac{\varepsilon^2}{ReFr^2} \equiv 1 \Rightarrow U^\circ = \frac{gd^3}{\lambda v_0}, \qquad (10.54)$$

which is compatible with the limiting form $(\varepsilon \to 0)$ of the boundary condition (10.49a), with (10.52b) when $\pi_0 = (Re/\varepsilon)^2 \Pi$.

Finally, in place of the dimensionless parameters *We* and *Ma* we introduce corresponding modified Weber, W^*, and Marangoni, M^* numbers by:

$$W^* = \frac{\sigma_0}{\rho_0 g \lambda^2},\ M^* = \frac{\gamma\beta}{\rho_0 g d}. \qquad (10.55)$$

When: $\varepsilon \to 0$, with Re, W^*, M^* and Pr fixed, (10.56)

in place of the full B-M model problem, we derive for the functions:

$$V = (u(\tau, x, y, z), v(\tau, x, y, z)), w(\tau, x, y, z), \Pi(\tau, x, y, z),$$

$$\theta_0 = \Theta(\tau, x, y, z), h' = H(\tau, x, y),$$

the following approximate B-M long-wave thin-film model problem:

$$\boldsymbol{D}\cdot\boldsymbol{V}+\frac{\partial w}{\partial z}=0,$$

$$\frac{D\boldsymbol{V}}{D\tau}+\boldsymbol{D}\Pi=\frac{1}{Re}\frac{\partial^2\boldsymbol{V}}{\partial z^2},$$

$$\frac{\partial\Pi}{\partial z}=0\Rightarrow\Pi=\frac{H-1}{Re}-\frac{W^*}{Re}\boldsymbol{D}^2H, \tag{10.57a}$$

$$Pr\frac{D\Theta}{D\tau}=\frac{1}{Re}\frac{\partial^2\Theta}{\partial z^2},$$

$$\text{at } z=0:\ \boldsymbol{V}=w=0 \text{ and } \Theta=1, \tag{10.57b}$$

$$\text{at } z=H(\tau,x,y):\ \begin{cases}\dfrac{\partial\boldsymbol{V}}{\partial z}=-M^*\left[\boldsymbol{D}\Theta+(\boldsymbol{D}H)\dfrac{\partial\Theta}{\partial z}\right],\\ \dfrac{\partial\Theta}{\partial z}+1+Bi\Theta=0,\\ w=\dfrac{\partial H}{\partial\tau}+\boldsymbol{V}\cdot\boldsymbol{D}H.\end{cases} \tag{10.57c}$$

In the above B-M long-wave thin-film model problem we have the following operators:

$$\frac{D}{D\tau}=\frac{\partial}{\partial\tau}+\boldsymbol{V}\cdot\boldsymbol{D}+w\frac{\partial}{\partial z},\ \boldsymbol{D}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right),$$

and $\boldsymbol{D}^2=\partial^2/\partial x^2+\partial^2/\partial y^2$.

In the reduced B-M long-wave thin-film model problem (10.57a, b, c) the parameters M^*, Pr, Bi, Re and W^* are $O(1)$, and this model problem is very well suited for a numerical investigation of the thin, slightly viscous thin film instability problem and deserves further consideration. Below, we consider a particular simple case, when Re is very small, and derive a lubrication evolution equation, which generalizes the lubrication equation (9.94) derived in Section 9.4.3 of §9.4 in Chapter 9.

10.5.2a. A lubrication evolution equation for the thickness of the film

When we consider the low-Reynolds number limit:

$$Re \to 0, \text{ with } W^*, M^* \text{ and } Pr \text{ fixed}, \tag{10.58a}$$

and in this case, again,

$$\lambda >> \frac{d^2}{\nu_0}[gd]^{1/2}, \tag{10.58b}$$

then, first, from the problem (10.57a,b,c) for the function Θ we obtain the following reduced linear problem:

$$\frac{\partial^2\Theta}{\partial z^2} = 0\,;\ \Theta = 1, \text{ at } z = 0, \text{ and } \frac{\partial\Theta}{\partial z} + 1 + Bi\Theta = 0, \text{ at } z = H(\tau, x, y),$$

and the solution is very simple:

$$\Theta = 1 - \left[\frac{1+Bi}{1+BiH}\right]z\,. \tag{10.59}$$

On the other hand (even for Re fixed) from the continuity (first) equation of (10.57a), with the conditions:

$$w = 0, \text{ at } z = 0, \text{ and } w = \frac{\partial H}{\partial \tau} + \boldsymbol{V}\cdot\boldsymbol{D}H\,, \text{ at } z = H,$$

we derive the following averaged (evolution) equation:

$$\frac{\partial H}{\partial \tau} + \boldsymbol{D}\cdot\int_0^H \boldsymbol{V}dz = 0\,. \tag{10.60}$$

Now for $\boldsymbol{V}(H, z)$, with (10.58a), we obtain a very simple problem, namely:

$$\frac{\partial^2 \boldsymbol{V}}{\partial z^2} = \boldsymbol{D}H - W^*\boldsymbol{D}\left(\boldsymbol{D}^2 H\right),$$

$$z = 0\text{: } V = 0,\ z = H\text{: } \frac{\partial V}{\partial z} = -M^*\left[D\theta + (DH)\frac{\partial \Theta}{\partial z}\right],$$

and we derive the following limiting solution for the horizontal velocity:

$$V(z;H) = M^* \frac{(1+Bi)}{(1+BiH)^2}(DH)z + \frac{1}{2}\{DH - W^* D[D^2 H]\}[z^2 - 2Hz]. \tag{10.61}$$

As a consequence of (10.60) and (10.61), we derive a single evolution equation for the thickness of the film $H(\tau, x, y)$:

$$\frac{\partial H}{\partial \tau} + \frac{1}{3} D \cdot \left\{H^3[W^* D(D^2 H) - DH] + M^* \frac{1+Bi}{(1+BiH)^2} H^2 (DH)\right\} = 0. \tag{10.62}$$

The lubrication equation (10.62), is very similar to the one derived by Oron and Rosenau (1992). Unfortunately, in the equation derived by these above authors, the term responsible for the Marangoni effect vanishes with vanishing Biot number ($Bi \to 0$)! This is clearly a paradoxical conclusion which has been discussed recently by Velarde and Zeytounian (2000).

The term proportional to W^*, has a stabilizing effect while the term proportional to M^* (which takes into account the Marangoni effect), has, on the contrary, a destabilizing impact. Finally, it should be observed that, again, the 'gravity' term: - $(1/3)D \cdot [H^3 DH]$), in (10.62), stabilizes the evolution of the free-surface when the film is supported from below.

10.5.3. The B-M problem for a freely falling vertical film

We consider now the thermocapillary instabilities on a freely falling vertical two-dimensional (2D) film, since most experiments and theories have focused on the latter problem. Using dimensionless variables and functions, the governing equations and boundary conditions for the (incompressible but thermally conducting) liquid motion in a vertical plane are as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{10.63a}$$

$$\frac{Du}{D\tau}+\frac{\partial \Pi}{\partial x}-\frac{1}{\varepsilon}F^2=\frac{1}{\varepsilon Re^{\dagger}}\left[\frac{\partial^2 u}{\partial z^2}+\varepsilon^2\frac{\partial^2 u}{\partial x^2}\right], \tag{10.63b}$$

$$\varepsilon^2\frac{Dw}{D\tau}+\frac{\partial \Pi}{\partial z}=\frac{\varepsilon}{Re^{\dagger}}\left[\frac{\partial^2 w}{\partial z^2}+\varepsilon^2\frac{\partial^2 w}{\partial x^2}\right], \tag{10.63c}$$

$$Pr\frac{D\Theta}{D\tau}=\frac{1}{\varepsilon Re^{\dagger}}\left[\frac{\partial^2 \Theta}{\partial z^2}+\varepsilon^2\frac{\partial^2 \Theta}{\partial x^2}\right], \tag{10.63d}$$

with $D/D\tau = \partial/\partial\tau + u\partial/\partial x + w\partial/\partial z$;

$$\text{at } z=0\text{: } u=w=0 \text{ and } \Theta=1; \tag{10.64}$$

$$\frac{\partial u}{\partial z}=-\varepsilon M\left[\frac{\partial \Theta}{\partial x}+\frac{\partial H}{\partial x}\frac{\partial \Theta}{\partial z}\right]-\varepsilon^2\left[\frac{\partial w}{\partial x}+4\frac{\partial H}{\partial x}\frac{\partial w}{\partial z}\right]$$
$$-\frac{3}{2}\varepsilon^3 M\left(\frac{\partial H}{\partial x}\right)^2\left[\frac{\partial \Theta}{\partial x}+\frac{\partial H}{\partial x}\frac{\partial \Theta}{\partial z}\right] \tag{10.65a}$$

$$\Pi=2\frac{\varepsilon}{Re^{\dagger}}\left[\frac{\partial w}{\partial z}-\frac{\partial H}{\partial x}\frac{\partial u}{\partial z}\right]-\varepsilon^2 W\frac{\partial^2 H}{\partial x^2}+\frac{\varepsilon^2 M}{Re^{\dagger}}\frac{\partial^2 H}{\partial x^2}\Theta$$
$$+2\frac{\varepsilon^3}{Re^{\dagger}}\left[\left(\frac{\partial H}{\partial x}\right)^3\frac{\partial u}{\partial z}-2\left(\frac{\partial H}{\partial x}\right)^2\frac{\partial w}{\partial z}-\frac{\partial H}{\partial x}\frac{\partial w}{\partial x}\right], \tag{10.65b}$$
$$\text{at } z=H(\tau,x).$$

$$\frac{\partial \Theta}{\partial z}=-(1+Bi\Theta)+\varepsilon^2\left[\frac{\partial H}{\partial x}\frac{\partial \Theta}{\partial x}-\frac{1}{2}\left(\frac{\partial H}{\partial x}\right)^2(1+Bi\Theta)\right]; \tag{10.65c}$$

and

$$w=\frac{\partial H}{\partial \tau}+u\frac{\partial H}{\partial x}, \text{ at } z=H(\tau,x). \tag{10.65d}$$

We note that the deformable free surface boundary conditions (at $z = H(\tau, x)$), (10.65a, b, c), are written with an error of $O(\varepsilon^4)$.

In the equations (10.63b, c, d) and condition (10.65b): $Re^{*} = U^\circ d/\nu_0$, in the equation (10.63b): $F^2 = U^{\circ 2}/gd$, in condition (10.65a): $M = \gamma\beta d/\rho_0\nu_0 U^\circ$, and in condition (10.65b): $W = \sigma_0/d\rho_0 U^{\circ 2}$.

10.6. THE KS AND KS-KDV MODEL EQUATIONS

10.6.1. The KS equation

We start with the equations (10.63a, b, c, d), and boundary conditions (10.64) and (10.65a, b, c, d), governing the B-M problem for a freely falling vertical film.

We assume that $Re^{*} = O(1)$, and consider that ε is the main small parameter occuring within the equations and conditions, for the full two-dimensional B-M problem related with a freely falling vertical film, formulated in Section 10.5.3.

As characteristic velocity U° we choose the interface velocity: $U^\circ = g\, d^2/\nu_0$, and in this case: $Re^{*}/F^2 = 1$. Usually the Weber number W is large and we introduce a 'Weber similarity parameter':

$$We^* = \varepsilon^2 W = O(1).$$

Concerning the Prandtl, Marangoni and Biot numbers in (10.63d) and in the boundary conditions (10.65a, b, c) we suppose that:

$$Pr = O(1), M = O(1) \text{ and } Bi = O(1).$$

We look the solution of (10.63a, b, c, d)) with (10.64) and (10.65a, b, c, d), in the form proposed by Benney (1966); namely:

$$U = [u, w, \Pi, \Theta]^T = U_0 + \varepsilon U_1 + O(\varepsilon^2), \text{ when } \varepsilon \to 0, \tag{10.66}$$

but we do not expand the thickness of the film $H(\tau, x)$ for the moment.

The solution for U_0 is:

$$u_0 = -z\left[\frac{1}{2}z - H\right]; w_0 = -\frac{1}{2}\left(\frac{\partial H}{\partial x}\right)z^2; \tag{10.67a}$$

$$\Pi_0 = -We^* \frac{\partial^2 H}{\partial x^2}; \tag{10.67b}$$

$$\Theta_0 = 1 - (1+Bi)\frac{z}{(1+BiH)}. \tag{10.67c}$$

We get also:

$$\frac{\partial H}{\partial \tau} + H^2 \frac{\partial H}{\partial x} = O(\varepsilon). \tag{10.68}$$

Now, writing out the full set of equations and boundary conditions for U_1, and again, assuming that $H(\tau, x)$ is not expanded, it is easy to get an analytic expression for u_1, as a function of z. Using u_0 and u_1, we may compute q_0 and q_1 the expansion of:

$$q(\tau,x) = \int_0^{H(\tau,x)} u(\tau,x,z)dz = q_0 + \varepsilon q_1 + O(\varepsilon^2). \tag{10.69}$$

Concerning $q_0 = (1/3)\ H^3$, this has already been taken into account with (10.68), while for q_1 we get the following expression:

$$q_1(\tau,x) = \frac{Re^{\dagger}}{3} We^* H^3 \frac{\partial^3 H}{\partial x^3} + \frac{M}{2}\frac{H^2}{(1+BiH)^2}\frac{\partial H}{\partial x} + \frac{2}{15} Re^{\dagger} H^6 \frac{\partial H}{\partial x}. \tag{10.70}$$

Now, by analogy with (10.60), from the equation: $\partial H/\partial\tau + \partial q/\partial x = 0$, we may get an equation (with an error of $O(\varepsilon^2)$) for the thickness of the film $H(\tau, x)$, involving the $O(\varepsilon)$ term occuring in (10.68):

$$\frac{\partial H}{\partial \tau} + H^2 \frac{\partial H}{\partial x}$$
$$+ \varepsilon \frac{\partial}{\partial x}\left\{ \frac{Re^{\dagger}}{3} We^* H^3 \frac{\partial^3 H}{\partial x^3} + \frac{2Re^{\dagger}}{15} H^6 \frac{\partial H}{\partial x} + \frac{M}{2}\frac{H^2}{(1+BiH)^2}\frac{\partial H}{\partial x} \right\} = 0. \tag{10.71}$$

This evolution equation (10.71), of the "Benney type", contains the small parameter ε, due to the fact that the thickness of the film $H(\tau, x)$ has not been expanded as it should be in a fully consistent asymptotic approach through an expansion with respect to ε.

Of course we may expand $H(\tau, x)$ in different ways and we shall investigate here only the same kind of phenomenon which led the Kuramoto-Sivashinsky (so-called, 'KS') equation. In order to achieve this, we put in (10.71):

$$\theta = \varepsilon\,\tau,\ \xi = x - \tau,\ H(\tau, x) = 1 + \delta\,\eta(\theta, \xi) + \ldots, \tag{10.72a}$$

and we assume that

$$\varepsilon = \delta. \tag{10.72b}$$

Since:

$$\frac{\partial H}{\partial \tau} = -\varepsilon\frac{\partial \eta}{\partial \xi} + \varepsilon^2\frac{\partial \eta}{\partial \theta} \text{ and } \frac{\partial H}{\partial x} = \varepsilon\frac{\partial \eta}{\partial \xi},$$

if we let $\varepsilon \to 0$ within the transformed version of equation (10.71) we derive the following KS equation:

$$\frac{\partial \eta}{\partial \theta} + 2\eta\frac{\partial \eta}{\partial \xi} + \alpha\frac{\partial^2 \eta}{\partial \xi^2} + \gamma\frac{\partial^4 \eta}{\partial \xi^4} = 0, \tag{10.73}$$

where

$$\alpha = \frac{2}{15}Re^{\ast} + \frac{1}{2}\frac{M}{1+Bi};\ \gamma = \frac{1}{3}Re^{\ast}We^{*}. \tag{10.74}$$

10.6.1a. Some properties of solutions of the KS equation

The KS equation (10.73) is asymptotically consistent, when $\varepsilon = \delta \to 0$, and is an approximate equation valid with an error of $O(\varepsilon)$. Now, if we introduce the amplitude: $A(\theta, \xi) = 2\eta(\theta, \xi)$, then we obtain, in place of (10.73), the following, canonical, KS equation

$$\frac{\partial A}{\partial \theta} + A\frac{\partial A}{\partial \xi} + \alpha\frac{\partial^2 A}{\partial \xi^2} + \gamma\frac{\partial^4 A}{\partial \xi^4} = 0. \tag{10.75}$$

When $\alpha = 0$ and $\gamma = 0$, we obtain the well-known equation:

$$\frac{\partial A}{\partial \theta} + A\frac{\partial A}{\partial \xi} = 0,$$

and along characteristics (defined by $d\xi/d\theta = A(\theta, \xi)$) the solution $A(\theta, \xi(\theta))$ is constant.

When $\gamma = 0$ ($We^* = 0$), the surface tension term is removed and (10.75) reduces to Burgers's equation. In this case, the Cole-Hopf transformation further reduces it to the heat equation. Since $\alpha > 0$ the Cole-Hopf transformation produces a heat equation backwards in time and initial disturbances will then grow without limit. Hence, we shall include the surface tension term and discuss equation (10.75) when: $\alpha > 0$ and $\gamma > 0$. The full KS equation (10.75) is capable of generating solutions in the form of irregularly fluctuating quasi-periodic waves. This KS model equation provides a mechanism for the saturation of an instability, in which the energy in long-wave instabilities is transferred to short-wave modes which are then damped by surface tension. In the full KS equation (10.75), the two terms: $\partial A/\partial \tau + A\, \partial A/\partial \xi$, lead to steepening and wave breaking in the absence of stabilizing terms. The term: $\alpha \partial^2 A/\partial \xi^2$ destabilizes shorter wave-length modes preferentially and therefore aggravates wave steepening (since M and Bi are both positive. For small Bi the Marangoni effect aggravates this destabilization). Finally, the term: $\gamma \partial^4 A/\partial \xi^4$ is required for saturation. Unfortunately, explicit analytic solutions of the KS equation are not available!

A very naïve *linear stability* analysis shows that for the KS equation (10.75) there exists a cutoff wave number. Indeed, if:

$$\eta(\theta, \xi) \sim exp\,[\omega\,\theta + i\,k\xi],$$

then for ω we derive the following dispersion relation

$$\omega - \alpha k^2 + \gamma k^4 = 0. \tag{10.76}$$

The curve $\omega = 0$ corresponds to neutral linear stability; in this case the phase velocity, $\omega/k = c = 0$, where the wavenumber k is assumed to be real, and as a consequence we obtain a cutoff wavenumber k^* such that:

$$\left(k^*\right)^2 = \frac{\alpha}{\gamma} = \frac{\left[\frac{2}{5} + \frac{3}{2}\frac{M}{Re^*(1+Bi)}\right]}{We^*} \qquad (10.77)$$

The linear dispersion relation (10.76) shows that short waves are stable, and long waves are unstable. The critical wavenumber is $k^* = \sqrt{\alpha/\gamma}$ which ought to be small for the long wave analysis to make sense. The maximum growth rate is $(\alpha^2/4\gamma)$ and occurs at $k^*/\sqrt{2}$. It is anticipated that the effect of the the nonlinear term in the canonical KS equation (10.75) will be to allow energy exchange between a wave with wavenumber k and its harmonics with the end result being nonlinear saturation. The final state may be either chaotic oscillatory motion or a state involving only a few harmonics. The energy equation, corresponding to (10.75), is obtained by multiplying (10.75) by A and integrating by parts, assuming A is periodic with period *2L*:

$$\frac{1}{2}\frac{\partial}{\partial\theta}\left[\int_0^{2L} A^2 d\xi\right] = \int_0^{2L}\left[\alpha\left(\frac{\partial A}{\partial\xi}\right)^2 - \gamma\left(\frac{\partial^2 A}{\partial\xi^2}\right)^2\right]d\xi . \qquad (10.78)$$

The minimization of the right hand side of (10.78), over all periodic functions, shows that this right hand side will be negative for: $\pi/L > k^*$; and therefore the nonlinear KS equation (10.75) is globally stable for an initial condition with a wavenumber satisfying the linear stability criterion.

In other words, if we put in an initial disturbance (e.g. $sin(k\xi)$) with a wavenumber k' greater than k^*, then the nonlinear term in (10.75) creates higher harmonics, but it will not create waves with wavenumbers smaller than k', so there will be stability.

If we want to generate a component with a wavenumber in the unstable region, we have to put in an initial condition with a wavenumber less than k^*. Hence, we need to consider only the case $k < k^*$. The periodic boundary conditions allow A to be written as the Fourier series:

$$A = \sum_{-\infty}^{+\infty} A_n(\theta)exp(ink\xi), \; A_{-n} = A^*_n, \qquad (10.79)$$

where A^*_n is the complex conjugate of A_n. Since $A_0 = const$ we may put $A_0 = 0$ and substitution of (10.79) into the KS equation (10.75) gives:

$$\frac{\partial A_n}{\partial \theta} - \sigma_n A_n + inkB_n = 0 , \tag{10.80a}$$

where

$$B_n = \sum_{r=1}^{\infty} A^*_r A_{r+n} + \frac{1}{2}\sum_{r=1}^{n-1} A_r A_{n-r} , \quad \sigma_n = \alpha(nk)^2 - \gamma(nk)^4 . \tag{10.80b}$$

The significant feature of the system of equations (10.80a) is that: for any given k, only a finite number of Fourier modes, $A_1, A_2, \ldots$, say, are unstable ($\sigma_n > 0$), and all higher modes are stable. Note that the nth mode has a critical wavenumber of k^*/n, and a maximum growth rate of $(\alpha^2/4\gamma)$ - independent of n - at $k^*/n\sqrt{2}$. This implies that unstable modes will be stabilized by energy transfer to higher harmonics. The simplest case amenable to some analysis is when: $k^*/2 < k < k^*$. Only $n = 1$ is unstable and in the following it is assumed that it is sufficient to consider just the interaction between $n = 1$ and $n = 2$. The approximate version of (10.80a) is then:

$$\frac{\partial A_1}{\partial \theta} - \sigma_1 A_1 + ikA_2 A^*_1 = 0 , \tag{10.81a}$$

$$\frac{\partial A_2}{\partial \theta} - \sigma_2 A_2 + ik(A_1)^2 = 0 , \tag{10.81b}$$

and we note that A_1 is unstable ($\sigma_1 > 0$) but A_2 ($\sigma_2 < 0$) is stable. Note also that

$$\sigma_1 |A_1|^2 + \sigma_2 |A_2|^2 = 0 ,$$

reflecting the required energy balance in the approximate version of

$$\frac{\partial}{\partial \theta}\left(\sum |A_n|^2\right) = 2\sum \sigma_n |A_n|^2 , \ n = 1 \text{ to } \infty,$$

as a consequence of (10.78) with (10.79). Equation (10.81a) has the steady solution:

$$|A_1| = \left[-\frac{1}{k^2} \sigma_1 \sigma_2 \right]^{1/2}, \tag{10.82a}$$

since from (10.81b)

$$A_2 = \frac{ik}{\sigma_2} (A_1)^2. \tag{10.82b}$$

Here, A_1 is growing and A_2 is stabilizing. However, as k is decreased, the hypothesis that only two modes are involved becomes more suspect! Indeed, as k is decreased the steady solution, given approximately by (10.82a,b), is at first modified by the presence of a small correction due to A_3 and then when:

$$\frac{k^*}{3} < k < \frac{k^*}{2} \text{ (i.e. } \sigma_2 > 0, \text{ but } \sigma_3 < 0),$$

is replaced by another "two-mode equilibrium" in which A_2 and A_4 are the dominant components. Further decrease in k then leads to a succession of states, alternating between "two-mode equilibria" and "bouncy states". If the steady solution for A_2 in (10.82b) is substituted into the equation (10.81a) a Landau-Stuart (LS) equation is obtained for A_1; namely:

$$\frac{\partial A_1}{\partial \theta} = \sigma_1 A_1 + \frac{k^2}{\sigma_2} |A_1|^2 A_1, \tag{10.83}$$

and this LS equation (10.83) is, in fact, valid only for k close to k^*. If in (10.83) we assume that $A_1 = |A_1| \exp(i\phi)$, then $\phi = const$ and for $|A_1|$ we derive a classical Landau equation:

$$\frac{\partial |A_1|}{\partial \theta} = \sigma_1 |A_1| + \lambda |A_1|^3, \tag{10.84}$$

with: $\lambda = (k^2/\sigma_2) < 0$, since $\sigma_2 < 0$. The solution of (10.84) gives:

$$|A_1| \sim A_1^\circ \exp(\sigma_1 \theta), \text{ as } \theta \to -\infty,$$

where $A_1°$ is the initial value at $\theta = 0$ and $\sigma_1 > 0$, which decays like the linearized theory. However,

$$|A_1|^2 \to -\frac{2\sigma_1}{\lambda}, \text{ as } \theta \to +\infty,$$

for all value of $A_1°$ - this case is called the *supercritical stability*.

If now we introduce a small perturbation parameter, κ, defined by:

$$\kappa^2 \mu = k^2 \left[\frac{\alpha}{\gamma} - k^2\right] > 0, \tag{10.85}$$

and a slow time scale: $T = \kappa^2 \theta$, then for the slowly varying amplitude of the fundamental wave: H(T) such that $|A_1| = \kappa H$, from (10.84), with (10.80b) for σ_1 and σ_2, we derive the following canonical Landau equation for $H(T)$:

$$\frac{\partial H}{\partial T} = \gamma\mu H - \lambda H^3, \tag{10.86}$$

where the (positive) Landau constant is: $\lambda = 1/16\gamma[k^2 - (\alpha/4\gamma)] > 0$.

10.6.2. The KS-KdV equation

The problem considered below is interesting when we look at (10.71). As a matter of fact this equation (10.71) is a singular perturbation of the hyperbolic equation:

$$\frac{\partial H}{\partial \tau} + H^2 \frac{\partial H}{\partial x} = 0.$$

Curiously, we get again a singular pertubation of this same equation, but of an another type, if we make the assumption (the low Reynolds number case):

$$Re^* << 1, M << 1, \tag{10.87a}$$

such that

$$\frac{Re^{\dagger}}{\varepsilon} = R^{\circ} \text{ and } \frac{M}{\varepsilon} = M^{\circ}, \tag{10.87b}$$

and again assume that:

$$\varepsilon^2 W = We^* \text{ and } \frac{Re^{\dagger}}{F^2} = 1, \tag{10.87c}$$

in the full problem (10.63a, b, c, d), (10.64) and (10.65a, b, c, d). In this case, in place of this starting problem, we obtain the following problem:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{10.88a}$$

$$\frac{\partial^2 u}{\partial z^2} + 1 = \varepsilon^2 R^{\circ}\left[\frac{Du}{D\tau} + \frac{\partial \Pi}{\partial x} - \frac{1}{R^{\circ}}\frac{\partial^2 u}{\partial x^2}\right], \tag{10.88b}$$

$$\frac{\partial \Pi}{\partial z} - \frac{1}{R^{\circ}}\frac{\partial^2 w}{\partial z^2} = \varepsilon^2\left[\frac{1}{R^{\circ}}\frac{\partial^2 w}{\partial x^2} - \frac{Dw}{Dt}\right], \tag{10.88c}$$

$$\frac{\partial^2 \Theta}{\partial z^2} = \varepsilon^2\left[Pr\, R^{\circ}\frac{D\Theta}{D\tau} - \frac{\partial^2 \Theta}{\partial x^2}\right]. \tag{10.88d}$$

$$\text{At } z = 0\text{: } u = w = 0 \text{ and } \Theta = 1. \tag{10.89}$$

$$\frac{\partial u}{\partial z} = -\varepsilon^2 M^{\circ}\left[\frac{\partial \Theta}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial \Theta}{\partial z}\right] - \varepsilon^2\left[\frac{\partial w}{\partial x} + 4\frac{\partial H}{\partial x}\frac{\partial w}{\partial z}\right] + O(\varepsilon^4), \tag{10.90a}$$

$$\Pi = -We^*\frac{\partial^2 H}{\partial x^2} + \frac{2}{R^{\circ}}\left[\frac{\partial w}{\partial z} - \frac{\partial H}{\partial x}\frac{\partial u}{\partial z}\right] + O(\varepsilon^2), \tag{10.90b}$$

$$\frac{\partial \Theta}{\partial z} = -(1 + Bi\,\Theta) + O(\varepsilon^2), \tag{10.90c}$$

$$w = \frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x}, \tag{10.90d}$$

at $z = H(\tau, x)$.

Obviously in this case, the formal Benney expansion in ε is modified. Here it is necessary to write:

$$U = (u, w, \Pi, \Theta)^T = U_0 + \varepsilon^2 U_2 + .., \text{ when } \varepsilon \to 0. \tag{10.91}$$

The solution U_0 is obtained in a straightforward way. In this case, when $\varepsilon \to 0$ in the problem, (10.88a, b, c, d), (10.89), (10.90a, b, c, d), we obtain the following leading-order solution:

$$u_0 = -\frac{1}{2}z^2 + Hz, \; w_0 = -\frac{1}{2}\frac{\partial H}{\partial x}z^2, \tag{10.92a}$$

$$\Pi_0 = -We^*\frac{\partial^2 H}{\partial x^2} - \frac{1}{R^\circ}\frac{\partial H}{\partial x}(H + z), \tag{10.92b}$$

$$\Theta_0 = 1 - (1 + Bi)\frac{z}{1 + BiH}. \tag{10.92c}$$

We get, this time, again:

$$\frac{\partial H}{\partial \tau} + H^2\frac{\partial H}{\partial x} = O(\varepsilon^2), \tag{10.93}$$

since $q_0 = (1/3)\, H^3$ - but valid with an error of $O(\varepsilon^2)$. Writting out the set of equations and boundary conditions at the order ε^2, from (10.88b) and (10.90a), with (10.89) for u_2, and assuming that $H(\tau, x)$ is not yet expanded, we may get an awkward expression for u_2 that may be integrated with respect to z in order to obtain an explicit expression for q_2 in the equation:

$$\frac{\partial H}{\partial \tau} + H^2\frac{\partial H}{\partial x} + \varepsilon^2\frac{\partial q_2}{\partial x} = O(\varepsilon^4).$$

The final result is analogous to (10.71), but with some additional terms:

$$\frac{\partial H}{\partial \tau}+H^2\frac{\partial H}{\partial x}+\varepsilon^2\frac{\partial}{\partial x}\left\{\frac{H^3}{3}\left[R^\circ We^*\frac{\partial^3 H}{\partial x^3}+7\left(\frac{\partial H}{\partial x}\right)^2\right]\right.$$

$$\left.+H^4\frac{\partial^2 H}{\partial x^2}+\frac{2H^6}{15}\frac{\partial H}{\partial x}+\frac{M^\circ(1+Bi)}{2}H^2\frac{\partial H}{\partial x}\frac{1}{(1+BiH)^2}\right\}. \quad (10.94)$$

This above evolution equation (10.94), for $H(\tau, x)$, is valid with an error of $O(\varepsilon^4)$. Now, we intend to play, with this evolution equation (10.94), the same game as the one considered for the reduction of (10.71) to a KS equation (10.73). Thus we use:

$$\theta=\delta\tau,\ \xi=x-\tau,\ H=1+\frac{1}{\phi}\varepsilon^2\eta(\theta,\xi)+\ldots, \quad (10.95a)$$

and

$$\delta=\frac{1}{\phi}\varepsilon^2, \quad (10.95b)$$

where ϕ is the dispersion similarity parameter. Carrying out again the limiting process $\varepsilon \to 0$, we find, in place of (10.94), an equation which combines the features of the KdV equation on the one hand and the KS equation on the other hand:

$$\frac{\partial \eta}{\partial \theta}+2\eta\frac{\partial \eta}{\partial \xi}+\alpha\frac{\partial^2 \eta}{\partial \xi^2}+\phi\frac{\partial^3 \eta}{\partial \xi^3}+\gamma\frac{\partial^4 \eta}{\partial \xi^4}, \quad (10.96)$$

where

$$\alpha=\phi\left[\frac{2}{15}R^\circ+\frac{1}{2}\frac{M^\circ}{1+Bi}\right],\ \gamma=\phi\frac{1}{3}R^\circ We^*. \quad (10.97)$$

The evolution KS-KdV equation (10.96) is again a significant model equation valid for large time with an error of $O(\delta)$. The coefficients α, γ and ϕ are all positive constants characterizing: *instability*, *dissipation* and *dispersion.*, respectively.

10.6.2a. Some properties of solutions of the KS-KdV equation

As a consequence of the derivation of the KS-KdV equation (10.96), valid for *low Reynolds numbers,* we conclude that the features of a thin film of a strongly viscous liquid are quite different:

The dispersive term, $\phi(\partial^3\eta/\partial\xi^3)$, *changes the behavior of the thickness* $\eta(\theta,\ \xi)$ *in space and in time.*

The above derivation of the KS-KdV model equation was first published in Zeytounian (1995). For $\phi = 0$, equation (10.96) reduces to a self-exciting dissipative KS equation (10.73) which exhibits turbulent (chaotic) behavior. On the other hand, for $\alpha = \gamma = 0$ (limiting case when: $R° = 0$ and $M° = 0$ - non-viscous liquid film, without the Marangoni effect), the equation (10.96) reduces to the classical KdV equation which is known to admit soliton solutions instead of chaos! Thus, in the general case of non zero α, γ and ϕ, increasing the value of ϕ is expected to change the character of the solution of the equation (10.96) from an irregular wave train to a regular row of solitons (a row of pulses of equal amplitude). The trend is amplified at larger values of ϕ, and the asymptotic state of the solution for large ϕ takes the form of a row of the KdV solitons. There seems to exist a critical value of about unity for the dimensionless parameter:

$$\mu = \frac{\phi}{(\gamma\alpha)^{1/2}}$$

which represents the relative importance of dispersion corresponding to the transition from an irregular wave train to a regular row of solitons. We note that the complicated evolution of solutions of (10.96) is described by the weak interaction of pulses, each of which is a steady solution of (10.96). When the dispersion is strong, pulse interactions become repulsive, and the solutions tend, in fact, to form stable lattices of pulses.

The linear dispersion relation of the KS-KdV equation (10.96) for the wave:

$$\eta(\theta,\xi) \approx \exp\,[ik\xi + \sigma\theta]$$

is expressed as

$$\sigma = \alpha k^2 - \gamma k^4 + i\,\phi k^3. \tag{10.98}$$

For $Real(\sigma) > 0$ we have instability and for $Real(\sigma) < 0$, stability, and $Real(\sigma) = 0$, if

$$k = k_C = \left(\frac{\alpha}{\gamma}\right)^{1/2} . \tag{10.99}$$

Consequently, the cut-off wavenumber for the KS-KdV equation (10.96), satisfies the relation

$$k_C^{\ 2} = \frac{2}{5We^*} + \frac{3}{2}\frac{M^\circ}{R^\circ We^*(1+Bi)} . \tag{10.100}$$

Thus waves of small wavenumber are amplified while those of large wavenumber are damped.

To demonstrate the competition between the stationary waves and the non-stationary (possibly chaotic) attractors of the KS-KdV equation (10.96), we convert (10.96), with $\alpha = \gamma = 1$, into a dynamical system by the Galerkin projection in a periodic medium with wavelength $2\pi/k$:

$$\eta(\theta,\xi) = \frac{1}{2}\sum A_p(\theta)cos(pk\xi) + B_p(\theta)sin(pk\xi),\ p \geq 1. \tag{10.101}$$

For a qualitative analysis of projections of the chaotic phase trajectory onto the plane it seems sufficient (!) to consider a dynamical system truncated at the third harmonics. This system can easily be written in an explicit form. First we make a simple linear transformation of the coordinate: $k\ \xi \to x$, $k\ \theta \to t$, with the initial $\eta(0, \xi) = \eta^\circ(\xi)$, and periodic boundary conditions: $\eta(\theta, \xi) = \eta(\theta, \xi + 2\pi/k)$, the spatial period (wavelength) of the equation is then equal to 2π. Next, substituting (10.101) into the KS-KdV equation (10.96), we derive for the amplitudes $A_1(t)$, $B_1(t)$ and $B_2(t)$, the following reduced dynamical system:

$$\begin{cases} \dfrac{dA_1}{dt} = \sigma_1 A_1 + k^2\phi B_1 - 2A_1B_2, \\ \dfrac{dB_1}{dt} = \sigma_1 B_1 - k^2\phi A_1 + 2B_1B_2, \\ \dfrac{dB_2}{dt} = 2\sigma_2 B_2 + 2\left[(A_1)^2 - (B_1)^2\right], \end{cases} \tag{10.102}$$

where

$$\sigma_1 = k(1 - k^2) \text{ and } \sigma_2 = 2\,k(1 - 4k^2).$$

The phase flow of the above dynamical system (10.102) is dissipative if the following relation is satisfied: $\sigma_1 + \sigma_2 < 0$, and because of this dissipative effect, the corresponding strange attractors have zero phase volume and dimensionality smaller than 3 (when time t tends to infinity) for the waveaumber k such that:

$$0.58 < k < 1. \tag{10.103}$$

This three-amplitude DS (10.102) can be studied qualitatively and numerically.

Another way to derive of a three-amplitude DS for KS-KdV equation (10.96) is via Fourier series. In this case we assume that:

$$\eta(\theta,\xi) = \frac{1}{2}\sum A_n(\theta)exp[in(\omega_1{}^\circ\theta - k_1\xi)], \tag{10.104}$$

where $A_n(\theta)$ is the complex amplitude of the n-th spatial harmonic, and where $\omega_1{}^o$ is the linear angular frequency (in fact, the angular frequency of the fundamental harmonic, with k_1 as wavenumber, at the first stage of its growth).

It must be stressed that, if the wavenumber $k_n = n\ k_1$ is the actual wavenumber of the n-th harmonic, on the contrary, the frequency: $n\ \omega_1{}^\circ$, cannot be considered as its actual frequency ω_n (the latter may vary a little, owing to possible small dispersive effects). The slow variation $\psi_n(\theta)$ of the phase corresponding to this small frequency shift is taken into account in the complex amplitude:

$$A_n(\theta) = |A_n(\theta)| exp(i\psi_n(\theta)). \tag{10.105}$$

Inserting (10.104) into (10.96), for the first three harmonics we derive the following three-amplitude DS (again, with $\alpha = \gamma = 1$):

$$\frac{dA_1}{d\theta} = \gamma_1 A_1 + ik_1 A_1^* A_2\,, \tag{10.106a}$$

$$\frac{dA_2}{d\theta} = \left(\gamma_2 - 6ik_1{}^3\phi\right)A_2 + ik_1A_1{}^2\,, \qquad (10.106b)$$

$$\frac{dA_3}{d\theta} = \left(\gamma_3 - 24ik_1{}^3\phi\right)A_3 + 3ik_1A_1A_2\,, \qquad (10.106c)$$

where

$$\gamma_n = (nk_1)^2\,[1 - (nk_1)^2],\ n = 1, 2, 3.$$

Near criticality, where the mode A_1 is the only unstable mode, while the others are linearly strongly damped, the dynamics are controlled by the marginally unstable mode A_1, to which the other two modes are *slaved*. As a consequence, from (10.106b), we have that the dynamics of the harmonics A_2 are slaved to the dynamics of the fundamental harmonic, A_1, according to:

$$A_2 = -\left[\frac{ik_1}{\gamma_2 - 6ik_1{}^3\phi}\right]A_1{}^2\,. \qquad (10.107)$$

From (10.106a), with (10.107), the fundamental harmonic A_1 obeys the following Stuart-Landau type equation:

$$\frac{dA_1}{d\theta} = \gamma_1A_1 + \lambda A_1{}^*A_1{}^2\,, \qquad (10.108a)$$

where

$$\lambda = \frac{\gamma_2k_1^2}{a^2}\left[1 + i\frac{6k_1{}^3\phi}{\gamma_2}\right], \qquad (10.108b)$$

with: $a^2 = \gamma_2^2 + 36\,k_1{}^6\,\phi^2$. The real part (positive) of the complex Landau constant corresponds to nonlinear dissipation; and its imaginary part to nonlinear frequency correction (due to dispersive effects).

The dispersive character of the waves plays a crucial role (via the parameter ϕ in (10.96)) in the occurence of amplitude collapses and frequency locking.

This may be understood within the framework of the DS (10.106) after separation of modulus and phase of the complex amplitudes (according to (10.105)).

For the simple case, of $|A_1(\theta)|$ and $|A_2(\theta)|$ and phase difference:

$$\Theta(\theta) = \psi_2 - 2\psi_1,$$

we derive the following DS of three equations in place of (10.106a, b, c):

$$\frac{d|A_1|}{d\theta} = \gamma_1|A_1| - k_1|A_1||A_2|\sin\Theta, \tag{10.109a}$$

$$\frac{d|A_2|}{d\theta} = \gamma_2|A_2| + k_1|A_1|^2 \sin\Theta, \tag{10.109b}$$

$$\frac{d\Theta}{d\theta} = -6(k_1)^3\phi + k_1\left\{\frac{|A_1|^2 - 2|A_2|^2}{|A_2|}\right\}\cos\Theta. \tag{10.109c}$$

This above DS (10.109a, b, c) deserves a further careful numerical investigation.

10.7. THE INTERACTION BETWEEN SHORT-SCALE MARANGONI CONVECTION AND LONG-SCALE DEFORMATIONAL INSTABILITY

It is important to note that in the case of the Bénard-Marangoni convection in a liquid layer with a deformable interface, as was previously shown by Takashima (1981a) through a linear stability analysis, there exist *two monotonous modes* of surface tension driven instability.

One, a *short-scale* mode, is caused by surface tension gradients alone, without surface deformation, and it leads to the formation of stationary convection with a characteristic scale of the order of the liquid layer depth.

The other, a *long-scale* mode, is influenced also by gravity and capillary (Laplace) forces, and surface deformation plays a crucial role in its development. This instability mode results in large-scale convection and in the growth of long surface deformations for which the characteristic scale is large in comparaison with the thickness of the liquid layer. As shown in the paper by Golovin, Nepomnyaschy and Pismen (1994), these two types of the Marangoni convection, having different scales, can interact with each other in the course of their nonlinear evolution.

There are *two mechanisms* of the coupling between them. On the one hand, *surface deformation changes* locally the Marangoni number, which depends on the depth of the liquid layer - this leads to a space-dependent growth rate of the short-scale convection and, hence, its intensity also becomes space dependent. On the other hand, the *short-scale convection* generates an additional mean mass/heat flux from the bottom to the free surface, which is proportional to its intensity (square of the amplitude).

When the intensity is not uniform, this leads to additional long-scale surface tension gradients affecting the evolution of the long-scale mode. Indeed, the coupling effects are most pronounced in the case when the long- and the short-scale modes have instability thresholds close to each other. In the above cited paper, Golovin, Nepomnyaschy and Pismen (1994), have studied these effects analytically (via a weakly nonlinear analysis) in the vicinity of the instability thresholds.

Close to the bifurcation point, the mean long-scale flow generated by the short-scale convection is very weak, of the order of ε^3, and it will considerably affect the long-scale surface deformations only if the latter are also small, of the order of ε^2, where ε is the amplitude of the deformationless convective mode - this happens when the surface tension is sufficiently large.

According to Golovin, Nepomnyaschy and Pismen (1994), near the instability threshold, the nonlinear evolution and interaction between the two modes can be described by a system of two coupled nonlinear equations, namely:

$$\frac{\partial A}{\partial T} = (\pm A) + \frac{\partial^2 A}{\partial x^2} + A|A|^2 + \eta A, \tag{10.110a}$$

$$\frac{\partial \eta}{\partial T} = -(\pm m)\frac{\partial^2 \eta}{\partial x^2} - w\frac{\partial^4 \eta}{\partial x^4} + s\frac{\partial^2 |A|^2}{\partial x^2}, \tag{10.110b}$$

where the parameters *m, w, s* are all positive.

The parameter *m* characterizes the effect of surface tension gradients and gravity, the parameter *w* corresponds to the Laplace pressure and *s* is the interaction parameter characterizing the coupling between the two modes of Marangoni convection.

The complex amplitude of the short-scale convection, *A,* undergoes a long-scale evolution, described by the *Ginsburg-Landau equation* (10.110a), but the latter, however, contains an additional nonlinear (quadratic) term, ηA, connected with the surface deformation η - this term adds to the linear growth rate of the amplitude *A* and describes, in fact, a non-uniform space-dependent supercriticality. It plays a stabilizing role when the surface is elevated ($\eta > 0$), and suppresses the short-scale convection under the surface deflections.

The surface deformation η is governed by the *nonlinear evolution equation* of fourth order (10.110b). However, the only nonlinear term in this equation is the coupling term proportional to $\partial^2|A|^2/\partial x^2$, describing the effect of the mean flow generated by the short-scale convection - this term always plays a stabilizing role. In the two equations (10.110a, b) the various signs of the terms correspond to the four cases described by Golovin, Nepomnyaschy and Pismen (1994).

In fact:

(i) when in (10.110a) we have $+A$ and in (10.11 0b) $+m$, both the short-scale deformational mode and the long-scale deformational one are unstable;

(ii) if in (10.110a) we have $+A$ and in (10.110b) $-m$, only the short-scale mode is unstable;

(iii) when in (10.110a) we have $-A$ and in (10.110b) $+m$, the deformational mode is unstable;

(iv) if in (10.110a) we have $-A$ and in (10.110b) $-m$, both modes are linearly stable, but their nonlinear interaction may lead to an instability.

The typical neutral stability curve, $Ma(k)$, has two minima: first Ma_l corresponding to $k = 0$, which describes the long-wave instability, and then Ma_S, related to $k_C \neq 0$, which indicates the threshold of short-scale convection. In a layer with an undeformable interface only one minimum exist, Ma_S.

The surface deformation η is a real quantity, whereas the amplitude A of the small-scale convection is complex. If we assume, for the sake of simplicity, that A is also real, thus considering the evolution of the short-scale convective structure with the fixed wave number $k = k_C$, then we write for the derivation of a three-mode truncated model:

$$A(T) = A_0\,(T) + A_1(T)\cos(k_m x) + \ldots;\ \eta\ = B_1(T)\cos(k_m x) + \ldots, \qquad (10.111)$$

with: $k_m = (m/2w)^{1/2}$, corresponding to the maximum linear growth rate of the first harmonic. In this case, substituting (10.111) into (10.1106a, b) and considering the third case $(-A,\ +m)$, after appropriate rescaling, we obtain the following dynamical system for $A_0(T)$, $A_1(T)$, and $B_1(T)$:

$$\frac{dA_0}{dT} = -A_0\left[1 + A_0^2 + \frac{3}{2}A_1^2\right] + \frac{1}{2}A_1B_1, \qquad (10.112a)$$

$$\frac{dA_1}{dT} = -A_1\left[1 + \frac{m}{2w} + 3A_0^2 + \frac{3}{4}A_1^2\right] + A_0B_1, \qquad (10.112b)$$

$$\frac{dB_1}{dT} = -\mu B_1 - \sigma A_0 A_1, \tag{10.112c}$$

where:

$$\mu(k_m) = m\, k_m^2 - w\, k_m^4,\ \sigma = 2\, k_m^2\, s. \tag{10.112d}$$

The system (10.112) describes the time evolution of a periodic surface deformational mode (B_1), which can generate not only a periodic mode of short-scale convection (A_1) following the surface deformation, but also a uniform zero mode (A_0). The coupling between the short-scale convection and large-scale deformations of the interface can lead to *stochastization* of the system and be one of the causes of interfacial turbulence of the thin film. The reader can find in Kazhdan *et al.* (1995), a numerical analysis of the system of two nonlinear coupled equations (10.110a, b), which confirms the predictions of weakly nonlinear analysis and shows the existence of either standing or travelling waves in the proper parametric regions, at low supercriticality. With increasing supercriticality, the waves undergo various transformations leading to the formation of pulsating travelling waves and nonharmonic standing waves as well as irregular wavy behavior resembling "interfacial turbulence". According to the numerical study of Kazhdan *et al.* (1995), it can be seen that oscillations of η and A are highly correlated: the amplitude of the short-scale convection follows the surface oscillations, being large beneath surface elevations and small under surface depressions.

The motion is apparently chaotic. In the spatial Fourier spectrum, due to the strong damping of the higher harmonics caused by the fourth derivative in the evolution equation (10.110b) for η, only a small number of modes are excited (less than eight), and the number of modes does not change significantly with an increase of the coupling parameter. Hence the observed irregular pattern can be characterized as a low mode chaotic system

With increasing s [coupling parameter in equation (10.110b)], the growth of the surface deformations is suppressed by the short-scale convection and the coupling between the two modes gives rise to long-scale standing waves modulating the short-scale roll convection pattern.

As the coupling becomes stronger, these oscillations decay and a stationary large-scale structure appears instead. This structure consists of narrow well separated depressions, with the rest of the interface being almost flat - under surface depressions the fluid is almost quiescent while in flat regions convection has an almost constant amplitude.

With further increase of the coupling parameter s, the stationary pattern becomes unstable and both long-scale surface deformations and the amplitude of the short-scale roll convection undergo irregular oscillations.

10.8. CONCLUDING REMARKS

Although significant understanding has been achieved, surface tension gradient-driven (B-M) convection flows still deserve further study.

Indeed, as a paradigmatic form of a spontaneous self-organizing system, the doctrine about the original Bénard problem has not reached the degree of sophistication, in theory and experimentation, attained in buoyancy-driven (R-B) convection. There are still challenging problems like relative stability of patterns (hexagons, rolls, squares,..., labyrinthine convection flows!), higer transitions and interfacial turbulence (possibly at low Marangoni number), a case of space-time chaos with high dissipation. It should be pointed out, also, that the ever increasing number of industrial applications of thin film flows and the richness of behaviour of the governing equations make this area a particularly rewarding one for mathematicians, engineers, and industrialists alike.

Although Bénard was aware of the role of surface tension and surface tension gradients in his experiments, it took five decades to unambiguously assess, experimentally and theoretically [see, for instance, the papers by Block (1956) and Pearson (1958)], that indeed the surface tension gradients rather than buoyancy were the cause of Bénard cells in thin liquid films. Only in 1997 was this almost evident physical fact rigorously proved, through an asymptotic approach, by Zeytounian (1997):

"Either the buoyancy is taken into account and in this case the free surface deformation effect is negligible and we can the only partially take into account the Marangoni effect, or this free surface deformation effect is taken into account and in this case the buoyancy does not play a significant rôle in the Bénard-Marangoni full thermocapillary problem".

It seems that the first author to explain the effect of the surface tension gradients on Bénard convection was Pearson (1958). The review articles by Normand, Pomeau and Velarde (1977) and by Davis (1987) consider the role of both buoyancy and surface tension gradients in triggering convective instability. Recently, a review article by Cross and Hohenberg (1993) was devoted to non-equilibrium pattern formation, with a sketchy section dealing with genuine Bénard cells i. e. Bénard-Marangoni convection. Koschmieder (1993), who has been a key figure in the experimental investigation of the Bénard problem for decades, wrote recently a very valuable monograph.

The more recent review article by Bragard and Velarde (1997), is comprised of salient findings, old and recent, about Bénard convection flows in a liquid layer heated from below and open to the ambient air.

In Zeytounian (1998) I have considered, for a thin film problem, three main situations in relation to the magnitude of the characteristic Reynolds number and discussed various model equations - these model equations are analyzed from various points of view but the central intent of my review paper is to elucidate the role of the Marangoni number on the evolution of the free-surface in space-time.

The recent paper by Myers (1998) is a review of work on thin films when (high) surface tension is a driving mechanism. Its aim is to highlight the substantial amount of literature dealing with relevant physical models and also analytic work on the resulting equations. The recent paper of Ida and Miksis (1998a) considers also the dynamics of a general 3D thin film subject to van der Waals forces, surface tension, and surfactants. Using an asymptotic analysis based upon the thinness of the film with respect to the lateral extent, evolution equations for the leading-order film thickness, tangential velocities, and surfactant concentrations are obtained. The scaling has been chosen by the above authors such that the surface tension effects occur at leading order in the dynamical model of the thin film. Note that the analysis applies to the breaking off of a thin liquid film from a stable center surface. Unfortunately, the model equations as presented form a complicated set of evolution equations and cannot be solved until the center surface is prescribed. In Part II of their work, Ida and Miksis (1998b), consider a series of special center surfaces and in each case consider the linear stability and solve the resulting nonlinear equations numerically. In particular, it is shown that increasing surface tension is stabilizing, while increasing the effects of van der Waals forces is destabilizing. The effects of surfactants, although irrelevant in the determination of the neutral stability curves, is stabilizing. The results obtained by solving the full evolution equations numerically agree with the stability results obtained analytically. Again I stress that when the liquid layer is subjected to a thermal gradient orthogonal to its open surface, the attention is focused on the role played by surface tension non-uniformity, which induces surface stresses and (Marangoni) convective motions (beyond an instability threshold). Long waves in a highly viscous Bénard layer placed on an almost stress-free support are also considered by various authors and a KdV- like equation is derived [see, for instance, Garazo and Velarde (1991)], but augmented with dissipative terms entering with a smallness parameter - travelling periodic and solitary waves of this equation are discussed. In particular, the dissipative solitary wave has been shown to possess a nonmonotonic tail. It is important to note that the presence of the free surface introduces additional interesting effects of surface tension and gravity, which change the character of the instability dramatically in a parallel flow. While the instability of the parallel flow between two rigid walls takes the form of short shear waves the instability in a liquid film takes the form of long gravity-capillary waves at relatively small Reynolds number.

Another interesting feature of the film instability is that there exists no finite critical wavelength according to the linear theory, in contrast to the case of rigid boundaries. The linear theory predicts the instability to take place in the form of an infinitely long wave.

On the other hand, surface waves of finite wavelengths were observed, as a consequence of the instability, by Kapitza and Kapitza (1949) and Binnie (1957). This led to the conjecture that the observed waves are the most amplified waves with the wavelength λ_m predicted by the linear theory. However, referring to the case of a freely falling vertical film, Benjamin (1957) pointed out that:

"One can scarcely expect waves to appear with a strictly uniform and distinct periodicity, because under all conditions infinitesimal waves with a wide range of wavelengths are unstable, and the wave with length λ_m comes into prominence only through a rather uncritical selection process depending on differences in the rates of amplification of different wavelengths. The ultimate state of the amplified waves is, of course, determined largely by nonlinear effects which remain unknown".

This statement is consistent with the experiment of Kapitza and Kapitza (1949), who found that distinctively periodic waves could not be observed unless the disturbances were introduced at precisely controlled frequencies. Thus Lin (1969, 1970) was led to investigate the nonlinear evolution of the so-called Benjamin (1957) - Yih (1963) wave of a given mode.

In Lin (1974), the nonlinear instability to disturbances of a finite frequency bandwidth is studied, in the case of a layer of an incompressible viscous fluid flowing down a plane inclined at an angle β to the horizontal. Indeed, Lin considers the case of weakly nonlinear wave motion which pertubs the free surface only slightly, and derives by a multiple-scale asymptotic expansion an amplitude equation for the leading-wave envelope. This Lin (1974, eq.(18)) equation is the appropriate equation for the description of the weakly nonlinear evolution of relatively short waves near the upper branch of the neutral curve where the amplification rate c_i is $O(\varepsilon^2)$ - see in Lin (1974) the amplitude equation (18) and figure1 [which represents in the (Re, α) plane, the stability curves for water at $15°C$ with $\beta = 90°$ and Weber number $We = 463.3$]. Near the lower branch of the neutral curve the wave number α (in a distance $2\pi d$, d is the constant film thickness) is zero, even where $c_i < O(\alpha^2) = O(\varepsilon)$ the modal interaction is stronger, and the Lin (1974) equation, (25) or (26) - which are very similar to the KS-KdV equation (10.96), but with $\gamma = 0$ - **without** the diffusion term - is then the governing equation of the nonlinear evolution. Farther away from the neutral curve where $c_i = O(\varepsilon)$ the diffusion term in the Lin equation (25) or (26), becomes important.

Unfortunately, the Lin (1974) film stability study does not take into account the Marangoni effect. In a recent paper by Wilson and Thess (1997), explicit analytical expressions for the linear growth (and decay) rates of long-wave modes in Bénard-Marangoni convection are derived and discussed. These analytical predictions are shown to be in good agreement with experimental observations [of VanHook *et al.* (1995)] and are used to estimate the minimum experimental time necessary in order to observe the long-wave instability under microgravity conditions. This work is a natural extension of previous linear stability studies by Pérez - Garcia and Carneiro (1991) which concentrated on the determination of the marginal stability curves, the nonlinear analyses of Marangoni convection in a thin layer of fluid by Kopbosynov and Pukhnachev (1986) and Davis (1987) and the recent investigation of the linear growth rates of the Marangoni problem near the onset of convection by Regnier and Lebon (1995). It is interesting to note that long-wave instability (L-WI) occurs even in highly viscous fluids, but its growth rate becomes so small as to be experimentally undetectable! On the other hand in the limit of small Biot number (which is typical of experimental situations), neither the dimensional growth rate S nor the corresponding critical temperature difference ΔT_L (for the onset of the L-WI at which $S = 0$), depends on the thermal diffusivity of the fluid, nor does S depend on the absolute value of the surface tension.

The results which are obtained by Regnier and Lebon (1995) indicate that the influence of surface deformation on the relaxation time and the correlation length is weak for the non-zero wavenumber instability mode; in contrast, the zero mode which is related to the surface deformation, exhibits a high sensitivity to the crispation number Cr. The dispersion relation for the zero mode was found analytically by using an appropriate scaling, as a main result of the analysis of Regnier and Lebon (1995), it is shown that the presence of the zero mode can only be detected in very large aspect ratio boxes and for very thin fluid layers - the results for zero mode provide a first step towards a (weakly) nonlinear analysis.

Just such a weakly nonlinear analysis of coupled surface-tension and gravitationally driven instability in a thin fluid layer (but, again, with a flat upper ‘free’ surface!) is presented in Parmentier, Regnier and Lebon (1996).

In a weakly nonlinear analysis, it is sufficient to take into account the modes that are critical at the linear threshold and as a consequence for the critical modes, in Parmentier, Regnier and Lebon (1996), a system of three coupled Ginzburg-Landau type equations for the three amplitudes A_1, A_2, A_3, is derived:

$$\tau\frac{\partial A_i}{\partial t} = \varepsilon A_i + aA_j^* A_k^* - bA_i\left[|A_j|^2 + |A_k|^2\right] - cA_i|A_i|^2 \qquad (10.113)$$

with, $i = 1$ and $j = 2$, $k = 3$; $i = 2$ and $j = 3$, $k = 1$; $i = 3$ and $j = 1$, $k = 2$, wherein the coefficients τ, a, b and c depend generally on the Prandtl and the Biot number and also on the ratio α (the ratio of buoyancy effect with regard to thermocapillary effect; $\alpha = 0$ corresponds to pure thermocapillarity and $\alpha = 1$ to pure buoyancy). The relative distance from the threshold is:

$$\varepsilon = 1 - \frac{\lambda}{\lambda_C}, \text{ with } \lambda = \frac{Ra}{Ra^\circ} + \frac{Ma}{Ma^\circ}, \qquad (10.114)$$

where the wave number k corresponding to λ_C is the critical wave number k_C; Ra° is the critical Rayleigh number for pure buoyancy and Ma° is the critical Marangoni number for pure thermocapillarity. According to Parmentier, Regnier and Lebon (1996), when buoyancy is solely responsible to the convection, only rolls will be observed.

As soon as capillary effects become significant, however, it appears that a hexagonal structure is preferred at the linear threshold. The more the thermocapillary forces are dominant with respect to the buoyancy forces the larger the size of the region where hexagons are stable. It is shown that the direction of the motion inside the hexagons is directly linked to the value of the Prandtl number and for $Pr > 0.23$, the fluid moves upward at the centre of the hexagons, in accord with experiments.

A subcritical region where hexagons are stable has also been displayed by these above authors - the region is the largest when buoyancy does not act and in this case, the value found for the subcritical parameter is in excellent agreement with direct numerical simulations performed by Thess and Orszag (1995).

But, all these above results correspond to the case when the upper free (!) surface is flat. A detailed analysis of the above system (10.113) can be found in Cross (1980, 1982) and in Cross and Hohenberg (1993).

Recently, the instability of a liquid layer beneath a rigid surface has been considered by Limat (1993) in a short note, according to a lubrication equation derived in Kopbosynov and Pukhnachev (1986), - this author discuss the influence of the initial thickness on the instability and the results are summarized by a diagram giving the different possible regimes. This diagram allows one to predict which of two different thickness dependences will be selected by the physical properties of the liquid.

For a vertical film, the recent review paper by Chang (1994) gives an excellent survey concerning mostly the various transition regimes on a freely falling vertical film and for an extension of this review, see Chang and Demekhin (1996) - but in both these review papers any discussion concerning the Marangoni effect is absent. The experimental investigation of three-dimensional instabilities of film flows is presented in the paper by Liu, Schneider and Gollub (1995) and several distinct transverse instabilities are found to deform the travelling waves: a synchronous mode (in which the deformations of adjacent wave fronts are in phase) and a subharmonic mode (in which the modulations of adjacent wave fronts are out of phase - in this case the herringbone patterns result).

The 3D subharmonic weakly nonlinear instability is due to the resonant excitation of a triad of waves consisting of the fundamental two-dimensional wave and two oblique waves. The evolution of wavy films after the onset of either of these 3D instabilities is complex - however, sufficiently far downstream, large-amplitude solitary waves absorb the smaller waves and become dominant. In Liu, Schneider and Gollub (1995) a detailed study of these instabilities is then presented, along with a qualitative treatment of the further evolution toward an asymptotic "turbulent" regime.

In the recent review paper by Oron, Davis and Bankoff (1997), the long-scale evolution of thin (macroscopic) liquid films is considered. By means of long-scale evolution equations, many interesting cases are discussed, giving the reader both an overview and representative behaviours typical of thin films. The first topic is 'bounded films', which have one free surface and one interface with a solid phase. The second topic concerns 'bounded films with slowly varying spatial nonuniformities at the boundaries'. Next, the related problem of the spreading of liquid drops on substrates is considered; the case in which the substrate is inclined to the horizontal and gravity drives a mean flow is also discussed. Finally, an overview is given - the results discussed in this paper motivate the development of careful experiments which can be used to test the theories and exhibit new phenomena.

Nonlinear dynamics and breakup of free-surface flows is reviewed in the recent paper by Eggers (1997). Thin-film rupture is considered by Ida and Miksis (1996). A wavy free surface flow of a viscous film down a cylinder is considered by A.L. Frenkel (1993).

Concerning the flow and stability of thin films on a rotating disk we mention the papers by: Charwat, Kelly and Gazley (1972), and Higgins (1986). This last author derives an asymptotic solution that describes the thinning of a fluid layer on a rotating disk when the Reynolds number for the flow is small - the solution results from matching a long-time scale expansion that ignores the initial acceleration of the fluid layer with a short-time-scale expansion that accounts for fluid inertia.

As more recent papers, concerning the thin films on a rotating disk, we mention the papers by Needham and Merkin (1987), Sisoev and Shkadov (1987, 1988, 1990) and the Doctoral Thesis by Christel Bailly (1995).

Nonlinear evolution of waves on a vertically falling film (but unfortunately, without the Marangoni effect!) is considered by Chang, Demekhin and Kopelevich (1993). In the above cited paper by Thess and Orszag (1995), devoted to the limit of infinite Prandtl number, the case of high Marangoni number is also considered - these authors note that “the kinematically possible velocity fields can have remarkable complexity when $Ma >> 1$ ”, and it is a challenging problem for future studies to understand Bénard-Marangoni convection in the limit $Ma \to \infty$. Viscous thermocapillary convection at high Marangoni number is also considered by Cowley and Davis (1983), where a boundary-layer analysis is performed that is valid for large Ma and Pr.

The onset of steady Marangoni convection, in a spherical shell of fluid with an outer free surface surrounding a rigid sphere, is analyzed by Wilson (1994) using a combination of analytical and numerical techniques.

In the Doctoral thesis by Jean-Marc Vince (1994), the propagating waves in convective systems subject to surface tension effects are studied very accuratly, a dynamical systems approach is also performed, in particular via amplitude equations ‘à la Ginsburg-Landau’.

In Sisoev and Shkadov (1997a, b), recently, dominant waves in a viscous liquid flowing in a thin sheet are analyzed; the principle of selection of the periodic solutions realized experimentally as regular waves is justified and it is shown that, if several periodic solutions exist at a given wavenumber, a regime characterized by the maximum values of both amplitude and phase velocity is realized - variation of the wave number causes a jump-like transition of the attractor to another regime, which gives rise to two periodic solutions in the vicinity of the bifurcation points.

The unsteady spreading of a thin layer of an incompressible viscous liquid over an impermeable curved surface, which occurs under the action of the gravity force, is considered by Grigorian and Khairetdinov (1998) - the solution of the boundary-value problem which arises reduces to solving a

Cauchy problem for equations with a small number of independent variables [see also: Grigorian and Khairetdinov (1996)].

An interesting and well documented overview concerning drops, liquid layers and the Marangoni effect, is the recent paper by Velarde (1998). Spatio-temporal instability in free ultra-thin films was considered, recently, by Shugai and Yakubenko (1998) - the analysis shows that even in the linear approximation, the long-range intermolecular force strongly affects the evolution of initially localized disturbances, but linear theory always overestimates the film lifetime, due to the explosive nonlinear growth of disturbances at later stages of evolution.

The recent paper by Regnier, Dauby and Lebon (2000) is devoted to linear and nonlinear R-B-M instability with surface deformations but, unfortunately, mathematical model which is used is not clearly and explicitly formulated!

It is necessary to observe here that the derivation of the Boussinesq approximate equations for the gas (see, for instance, the §4.7 in Chapter 4) is very different than for the liquids.

An interesting IUTAM Symposium Proceedings (held in Haifa, Israel, 17-21 March 1997) and published in 1999 (edited by D. Durban and J.R.A. Pearson) concerns the 'Non-linear singularities in deformation and flow' and contains various papers related with the interfacial effects in fluids and also capillary breakup and instabilities.

Finally, we mention some books related to the thin-film theory: T.S. Sorensen (1978, Ed), J. Zierep and H. Oertel (1982, Eds.), R.E. Meyer (1983, Ed.), J.K. Platten and J.C. Legros (1984), M.G. Velarde (1987 and 1988, Ed.), I.B. Ivanov (1988, Ed.), S.V. Alekseenko, V.E. Nakoryakov and B.T. Pokusaev (1992), B. Straughan (1992), D.D. Joseph and Yu.R. Renardy (1993, Part I), J.B. Simanovskii and A.A. Nepomnyaschy (1993), R.F. Probstein (1994).

CHAPTER 11

METEO-FLUID-DYNAMICS MODELS

First, in the §11.2 we derive the NS-F atmospheric equations and in §11.3 the large-scale hydrostatic non-tangent model equations and various simplified forms, of these equations for atmospheric motions are considered. The so-called "Kibel meteorological primitive" (tangent, with β-effect) equations are considered in the §11.4.. From these Kibel equations, in the §11.5, we derive the quasi-geostrophic model, when the Rossby/Kibel number is a small parameter. The §11.6 is devoted, again, to low-Mach number asymptotics and, in particular, to discussion of the so-called quasi-nondivergent model. In §11.7, the infuence of a local relief is investigated and we consider, asymptotically, lee-waves and primitive model problems as inner and outer expansions. In §11.8 a new system of model equations for the lee-waves is derived - the so-called "deep atmospheric convection model", since the Boussinesq approximation is not adequate for the lee-waves problem in the whole of the troposphere.

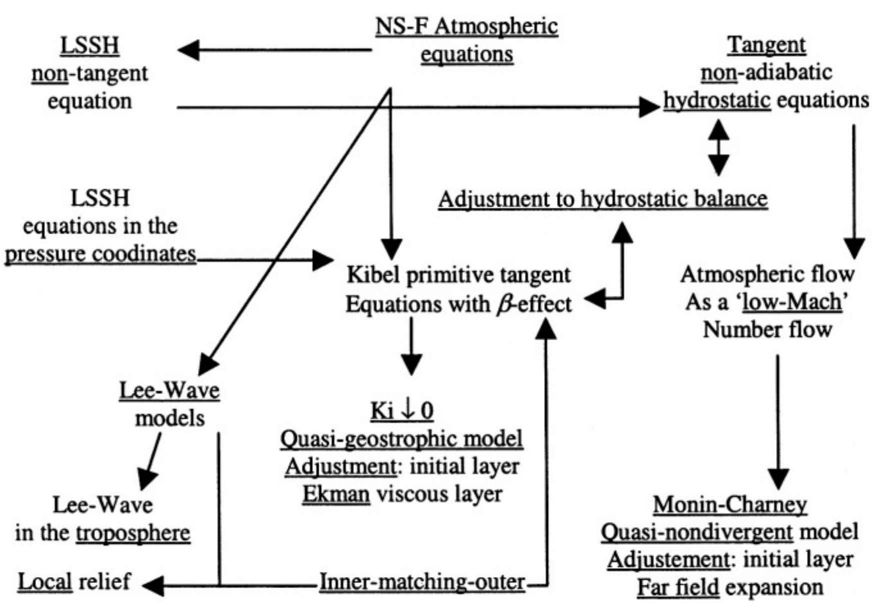


Fig. 11 Asymptotic models derived from the NS-F atmospheric equations

11.1. A FLUID DYNAMICIST'S VIEW OF METEOROLOGY

The method of derivation of various asymptotic models in this Chapter has been strongly infuenced by the author's own conception of Meteorology as a Fluid Dynamical discipline and a privileged area for the application of asymptotic modelling.

The author has in mind the derivation of: "a set of consistent rational asymptotic models for various atmospheric phenomena, which are useful in Meteorology".

In fact, our goal is a "return of Meteorology to the family of Fluid Dynamics". This program is very ambitious and actually only a "small part" of this program is realized, and, for instance, the reader is invited to consult our review paper: Zeytounian (1985), and also our two books: Zeytounian (1990, 1991).

The main idea is, in fact, that the careful asymptotic analysis of 'extended' NS-F equations - NS-F Atmospheric equations, when we take into account the Coriolis and gravitational forces and the effect of thermal radiation - via the various dimensionless (singular perturbation) parameters (Mach, Rossby/Kibel, Boussinesq, Ekman, Froude, hydrostatic,...), makes it possible to derive in a consistent way not only the approximate models utilized in Meteorology, but also to resolve various problems related to their well-posedness and validity.

It is obvious that atmospheric motions are low-Mach number flows and as a consequence the Mach number is the fundamental small parameter in the Meteo-fluid-dynamics! But, most intriguing features are found with low-Mach number atmospheric flows: they are due to interactions of gravitational and Coriolis forces, with pressure forces. As a consequence one encounters a number of fascinating problems some of which have been discussed in Zeytounian (1990). But, one cannot avoid some feeling of frustration, the reason being that we do not encounter a model playing such a central role as the one played by incompressible flows (considered in Chapter 5). The reason seems to be that blocking (in the horizontal planes) is overwhelming and induces a number of unpleasant features which lead the sceptical reader to believe more in numerical codes than in asymptotic modelling. Indeed, it is somewhat discouraging to realize (see, for instance, Zeytounian (1990, Chapter 12)) that the low-Mach number approximation looks, in the atmosphere (and ocean), so good from the outset and that very few useful results have been obtained by using it.

When the Mach number tends to zero, in the atmosphere, a strong tendancy towards blocking in the horizontal planes (or for the constant hydrostatic pressure levels) occurs! Something analogous to acoustics occurs (see Chapter 8) and corresponds to non uniformity of the limiting process at infinity in the horizontal directions. As a matter of fact the model which emerges from that is the one for adaptation to quasi-non divergent (Monin-Charney) flows. The quasi-non divergent (main) approximation and its higher order companion are also unable to cope with the full set of initial conditions adequate for the Kibel primitive model equations (a basic model in Meteorology) and it is necessary to consider an appropriate initial layer expansion which may deal with the unsteady adjustment process.

The reader can find in Chapter I ('*The rotating earth and its atmosphere*') and Chapter II ('*Dynamical and thermodynamical equations for atmospheric motions*') of my book; Zeytounian (1991, pp. 1 to 35) a short introduction to 'Meteorological Fluid Dynamics (MFD)'. We note also that the 'Geophysical Fluid dynamics (GFD)' is the term used to describe a body of mathematical problems that arises primarily in meteorology and oceanography and in its recent book Salmon (1998) gives a very readable description of major problems in this topic.

Obviously, the advent of powerful computers has allowed some progress in the past half decade toward understanding and predicting the behaviour of the ocean and atmosphere. However, even the most powerful computer cannot answer some crucially important questions such as those relating to precise, long-term weather prediction. It is therefore important, from a practical point of view, that attention be paid to the mathematical models of geophysical fluid motions. In the 1980's, Pedlosky (1987) published a major book soley devoted to GDF, and it has now become a standard text for the subject, while Salmon's recent book (1998) overlaps with that of Pedlosky, the former contains some of the important developments of the past decade and discuss more modern mathematical techniques that have proved to be useful in fluid dynamics.

11.2. THE NS-F ATMOSPHERIC EQUATIONS

In a coordinate frame rotating with the earth Newton's classical momentum equation is:

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla p + \rho \boldsymbol{g} - \rho(2\boldsymbol{\Omega} \wedge \boldsymbol{u}) + \boldsymbol{F}(\boldsymbol{u}), \tag{11.1}$$

where $\boldsymbol{u}$ is the (relative) velocity vector as observed in the earth's frame rotating with the angular velocity $\boldsymbol{\Omega}$, ρ is the atmospheric density, p is the atmospheric pressure, $\boldsymbol{g}$ is the gravitational acceleration (modified by the centrifugal force) and $\boldsymbol{F}(\boldsymbol{u})$ is the frictional force in the atmosphere.

As the coefficient of eddy viscosity μ varies then it is necessary to take into account the following expression:

$$F(u) = \nabla \cdot \{\mu[\nabla u + (\nabla u)^T]\} - \frac{2}{3}\nabla(\mu\nabla \cdot u), \qquad (11.2)$$

when the Stokes hypothesis (which amounts to neglecting the bulk viscosity) is adopted. The conservation law of mass is expressed as usual by the equation of continuity:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot u) = 0, \qquad (11.3)$$

and the first law of thermodynamics by an energy equation, namely:

$$\rho\frac{DE}{Dt} + p(\nabla \cdot u) = \nabla \cdot (k\nabla T) + \chi(u) + \rho_S Q, \qquad (11.4)$$

where T is the absolute temperature of the atmosphere and E its internal energy per unit mass, k being the coefficient of thermal eddy conductivity. Finally, the function $\chi(u)$ is expressible as:

$$\chi(u) = (\nabla u) \cdot \left\{\mu[\nabla u + (\nabla u)^T] - \frac{2}{3}\mu(\nabla \cdot u)I\right\}. \qquad (11.5)$$

we note that ρ_S in equation (11.4) is the so-called standard density function only of the altitude z, and Q, which is a heat source (thermal radiation), is assumed to depend only on the 'mean' standard atmosphere. Doing this we consider only a mean, standard distribution for Q, and also for μ and k, and ignore variations therefrom for the perturbed atmosphere in motion.

The thernodynamic functions, p_S, ρ_S and T_S, for the standard atmosphere, satisfy the following equations:

$$\frac{dp_S}{dz} + g\rho_S = 0, \; p_S = R\rho_S T_S, \qquad (11.6a)$$

$$k\frac{dT_S}{dz} + R(T_S) = 0, \; \frac{dR}{dz} = \rho_S Q. \qquad (11.6b)$$

For 'dry' atmospheric air we assume as equations of state:

$$p = R\rho T, E = C_V T, \qquad (11.7)$$

with $C_V = const.$

Now, as the vector of rotation of the earth $\boldsymbol{\Omega}$ is directed from south to north according to the axis of the poles, it can be expressed as follows:

$$\boldsymbol{\Omega} = \Omega^{\circ}\, \boldsymbol{e}, \text{ with } \boldsymbol{e} = (\boldsymbol{k} \sin\phi + \boldsymbol{j} \cos\phi), \tag{11.8}$$

where ϕ is the algebraic latitude of the point P° of the observation on the earth's surface, around which the atmospheric flow is analyzed ($\phi > 0$ in the northen hemisphere). The unit vectors directed to the east, north, and zenith, in the opposite direction from $\boldsymbol{g} = -g\,\boldsymbol{k}$ (the force of gravity), are denoted by $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ respectively. Let $\boldsymbol{u}(t, \boldsymbol{x}) = u\,\boldsymbol{i} + v\,\boldsymbol{j} + w\,\boldsymbol{k}$, in this case the term, $2\,\boldsymbol{\Omega} \wedge \boldsymbol{u}$, becomes:

$$2\Omega \wedge \boldsymbol{u} = \left(2\Omega^{\circ} \cos\phi\, w - 2\Omega^{\circ} \sin\phi\, v\right)\boldsymbol{i} + \left(2\Omega^{\circ} \sin\phi\, u\right)\boldsymbol{j} - \left(2\Omega^{\circ} \cos\phi\, u\right)\boldsymbol{k} \tag{11.9}$$

with $\Omega^{\circ} = |\boldsymbol{\Omega}|$.

11.2.1. The dimensionless dominant equations

It is helpful to employ spherical coordinates λ, ϕ, r, and in this case u, v, w denotes again the corresponding relative velocity components in these directions, respectively - increasing azimuth (λ), latitude (ϕ), and radius (r).

For the gradient operator ∇, divergence of the velocity $\boldsymbol{u}$, and associated scalar product $\boldsymbol{u} \cdot \nabla$, we can write:

$$\nabla = \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}\boldsymbol{i} + \frac{1}{r}\frac{\partial}{\partial\phi}\boldsymbol{j} + \frac{\partial}{\partial r}\boldsymbol{k}\,, \tag{11.10a}$$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, v) + \frac{\partial w}{\partial r} + \frac{2w}{r}, \tag{11.10b}$$

$$\boldsymbol{u} \cdot \nabla = \frac{u}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v}{r}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial r}. \tag{11.10c}$$

If now, we take into account the changes in unit vectors during the differentiation, we obtain the following formula for the material derivative:

$$\frac{Du}{Dt}=\left\{\frac{\partial u}{\partial t}+\frac{u}{r\cos\phi}\frac{\partial u}{\partial \lambda}+\frac{v}{r}\frac{\partial u}{\partial \phi}+w\frac{\partial u}{\partial r}+u\frac{w}{r}-v\frac{u}{r}\tan\phi\right\}\boldsymbol{i}$$
$$+\left\{\frac{\partial v}{\partial t}+\frac{u}{r\cos\phi}\frac{\partial v}{\partial \lambda}+\frac{v}{r}\frac{\partial v}{\partial \phi}+w\frac{\partial v}{\partial r}+v\frac{w}{r}+u\frac{u}{r}\tan\phi\right\}\boldsymbol{j}$$
$$+\left\{\frac{\partial w}{\partial t}+\frac{u}{r\cos\phi}\frac{\partial w}{\partial \lambda}+\frac{v}{r}\frac{\partial w}{\partial \phi}+w\frac{\partial w}{\partial r}-u\frac{u}{r}-v\frac{v}{r}\right\}\boldsymbol{k}. \qquad (11.11)$$

Next, we introduce the following transformations:

$$x = a_0 \cos\phi^{\circ}\, \lambda;\ y = a_0\,(\phi - \phi^{\circ});\ z = r - a_0, \qquad (11.12)$$

where ϕ° is a reference latitude and for $\phi^{\circ} \approx 45^{\circ}$ we have $a_0 \approx 6300$ km. It follows immediately that:

$$\frac{\partial}{\partial \lambda}=a_0\cos\phi^{\circ}\frac{\partial}{\partial x};\ \frac{\partial}{\partial \phi}=a_0\frac{\partial}{\partial y};\ \frac{\partial}{\partial r}=\frac{\partial}{\partial z}. \qquad (11.13)$$

The origin of this right handed curvilinear coordinates system lies on the earth's surface (for a flat ground, where $r = a_0$) at latitude ϕ° and longitude $\lambda = 0$. We assume therefore that the atmospheric motion occurs in a mid-latitude region, distant from the equator, around some central latitude ϕ° and therefore, $\sin\phi^{\circ}$, $\cos\phi^{\circ}$ and $\tan\phi^{\circ}$ are all of order unity.

Although x and y are, in principle, new longitude and latitude coordinates in terms of which the basic Euler equations may be rewritten without approximation, they are obviously introduced in the expectation that for small:

$$\delta=\frac{L^{\circ}}{a_0}, \qquad (11.14)$$

they will be the Cartesian coordinates of the so-called "*f°-plane approximation*". For this, it is necessary to introduce nondimensional variables and functions. For example:

$$t'=\frac{t}{t^{\circ}},\ x'=\frac{x}{L^{\circ}},\ y'=\frac{y}{L^{\circ}},\ z'=\frac{z}{H^{\circ}},\ r'=\frac{r}{a_0},\ u'=\frac{u}{U^{\circ}},\ v'=\frac{v}{U^{\circ}},$$

$$w' = \frac{w}{\varepsilon U^\circ}, \tag{11.15a}$$

and

$$p' = \frac{p}{p^\circ},\ \rho' = \frac{\rho}{\rho^\circ},\ T' = \frac{T}{T^\circ}, \tag{11.15b}$$

where

$$\varepsilon = \frac{H^\circ}{L^\circ} \tag{11.16a}$$

is, the so-called "*long-wave* or *hydrostatic*" nondimensional parameter and

$$Re = \frac{U^\circ L^\circ}{\mu^\circ \rho^\circ}, \tag{11.16b}$$

is the Reynolds number. Above, the reference constant values: μ°, ρ°, T°, p°, are the corresponding values of μ, ρ_S, T_S, p_S on the ground $r = a_0$.

Using the above relations, we can write the material derivative operator: *D/Dt,* in the following form:

$$\frac{D}{Dt} = \frac{U^\circ}{L^\circ}\left\{St\frac{\partial}{\partial t'} + \frac{\cos\phi^\circ/\cos\phi}{1+\varepsilon\delta z'}u'\frac{\partial}{\partial x} + \frac{1}{1+\varepsilon\delta z'}v'\frac{\partial}{\partial y'} + w'\frac{\partial}{\partial z'}\right\}, \tag{11.17}$$

where

$$St = \frac{L^\circ}{t^\circ U^\circ} \tag{11.18}$$

is the Strouhal number which characterizes the unsteady effects (in fact, below, we assume that: $St = 1$ and in this case $t^\circ = L^\circ/U^\circ$).

Finally, dropping the primes we readily derive, after a careful dimensional analysis, in place of the NS-F Atmospheric equations (11.1), (11.3) and (11.4) a set of dimensionless dominant equations for the horizontal velocity $\boldsymbol{v} = (u, v)$, vertical velocity w and thermodynamic functions, p, ρ, T, namely:

$$\rho\left\{\frac{Dv}{Dt}+\left[\frac{1}{Ro}\frac{\sin\phi}{\sin\phi^{\circ}}+\delta\frac{\tan\phi}{1+\varepsilon\delta z}\right](k\wedge v)\right\}+\frac{1}{1+\varepsilon\delta z}\frac{1}{\gamma M^2}Dp$$

$$=\frac{1}{\varepsilon^2 Re}\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right)+O(\varepsilon); \qquad (11.19a)$$

$$\frac{\partial p}{\partial z}+Bo\rho=\gamma M^2 O(\varepsilon^2); \qquad (11.19b)$$

$$\frac{D\rho}{Dt}+\rho\left\{\frac{\partial w}{\partial z}+\frac{1}{1+\varepsilon\delta z}(D\cdot v-\delta\tan\phi\, v+\varepsilon\delta w)\right\}=0; \qquad (11.19c)$$

$$\rho\frac{DT}{Dt}-\frac{\gamma-1}{\gamma}\frac{Dp}{Dt}=\frac{1}{\varepsilon^2 Re\, Pr}\left\{\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)\right.$$

$$\left.+\frac{Pr\,\mu(\gamma-1)M^2}{1+\varepsilon\delta z}\left|\frac{\partial}{\partial z}\left[\frac{v}{1+\varepsilon\delta z}\right]\right|^2+Bo\sigma\frac{d\mathrm{R}}{dz}\right\}+O(\varepsilon^2),$$

$$(11.19d)$$

with $p=\rho T$.

In the above equations

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\frac{1}{1+\varepsilon\delta z}v\cdot D+w\frac{\partial}{\partial z},$$

$$D=\frac{\cos\phi^{\circ}}{\cos\phi}\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j,\ k\cdot D=0.$$

The Rossby number is $Ro = U^{\circ}/f^{\circ}L^{\circ}$, with $f^{\circ} = 2\Omega^{\circ}\sin\phi^{\circ}$, the Mach number is $M = U^{\circ}/(\gamma RT^{\circ})^{1/2}$, and the Prandtl number is $Pr = C_P\mu^{\circ}/k^{\circ}$, where k° is the value of k on the ground. In equation (11.19d) $\sigma = (R/gk^{\circ})\mathrm{R}^{\circ}$, with R° a characteristic value of the radiative (standard) flux $\mathrm{R}(T_S)$. Finally, we observe that: $\phi = \phi^{\circ} + \delta y$.

11.3. LARGE-SYNOPTIC SCALE, HYDROSTATIC (LSSH), NON-TANGENT MODEL EQUATIONS

Obviously, this above set of dimensionless dominant equations (11.19a, b, c, d) is complicated and below we consider an approximate form of this set, which is derived when we consider the following limiting process:

$$\varepsilon \to 0 \text{ and } Re \to \infty, \text{ such that: } \varepsilon^2 Re \equiv Re_\perp = O(1). \tag{11.20}$$

With (11.20) the set of dimensionless dominant equations (11.19a, b c, d) is significantly simplified.

Namely, we obtain for the limiting values of $\boldsymbol{v} = (u, v)$, w and p, ρ, T, the following, "non-tangent", with $\delta \neq 0$, large-synoptic scale, hydrostatic, (LSSH), model equations:

$$\rho\left\{\frac{D\boldsymbol{v}}{Dt}+\left[\frac{1}{Ro}\frac{\sin\phi}{\sin\phi^\circ}+\delta\tan\phi\right](\boldsymbol{k}\wedge\boldsymbol{v})\right\}+\frac{1}{\gamma M^2}\boldsymbol{D}p=\frac{1}{Re_\perp}\frac{\partial}{\partial z}\left(\mu\frac{\partial\boldsymbol{v}}{\partial z}\right); \tag{11.21a}$$

$$\frac{\partial p}{\partial z}+Bo\rho=0; \tag{11.21b}$$

$$\frac{D\rho}{Dt}+\rho\left\{\frac{\partial w}{\partial z}+\boldsymbol{D}\cdot\boldsymbol{v}-\delta\tan\phi v\right\}=0; \tag{11.21c}$$

$$\rho\frac{DT}{Dt}-\frac{\gamma-1}{\gamma}\frac{Dp}{Dt}$$
$$=\frac{1}{Re_\perp Pr}\left\{\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+Pr\,\mu(\gamma-1)M^2\left|\frac{\partial\boldsymbol{v}}{\partial z}\right|^2+Bo\,\sigma\frac{dR}{dz}\right\} \tag{11.21d}$$

with $p = \rho T$ and $D/Dt = \partial/\partial t + \mathbf{v}.\mathbf{D} + w\,\partial/\partial z$, with $\boldsymbol{D} = (\cos\phi^\circ/\cos\phi)\,(\partial/\partial x)\,\boldsymbol{i} + (\partial/\partial y)\,\boldsymbol{j}$.

These LSSH, non-tangent equations (11.21a, b, c, d) constitute a very significant system for large scale atmospheric motions in a thin layer, such as the troposphere, around the earth. For these LSSH equations we write, on the flat ground $z = 0$, the following boundary conditions:

$$v = 0,\ w = 0, \text{ and } k\frac{dT}{dz} + Bo\sigma R = 0, \text{ on } z = 0. \tag{11.22}$$

Concerning the initial conditions for the above LSSH system of equations (11.21a, b, c, d) with (11.22), we must give only the initial values for $\boldsymbol{v}$ and T (or p) and they (in general) have nothing to do with the corresponding initial conditions for the full NS-F Atmospheric equations! Consequently, it is necessary to formulate for the boundary value problem, (11.2la, b, c, d), (11.22), an unsteady adjustment problem analogous to the one that was considered by Guiraud and Zeytounian (1982) for the primitive Kibel equations (see §11.4).

Actually, the derivation and the analysis of this unsteady, local in time, adjustment problem to the LSSH main model is an open problem. We leave unspecified, here, the behaviour conditions at high altitude when $z \uparrow \infty$, and in the horizontal directions for x and y tend to infinity.

11.3.1. The LSSH equations in pressure coordinates

The above hydrostatic equation (11.21b) makes it possible to write the relation:

$$\frac{\partial}{\partial z} = -Bo\rho\frac{\partial}{\partial p}, \tag{11.23}$$

and this allows us to make a standard way to change of variables from, t, x, y and z, to new (pressure) variables, τ, ξ, η and p, with:

$$\tau \equiv t,\ \xi \equiv x,\ \eta \equiv y,\ z = H(\tau, \xi, \eta, p), \tag{11.24}$$

where H has to be considered as one of the unknown functions. We keep the notation $\boldsymbol{D}$ for the horizontal gradient on the isobaric $p = const$ surface (with the components, $(\cos\phi^0/\cos\phi)\ (\partial/\partial\xi)$ and $\partial/\partial\eta$), and we set $D/D\tau$ for the material derivative operator: $\partial/\partial\tau + \boldsymbol{v}\cdot\boldsymbol{D} + \omega\,\partial/\partial p$, where:

$$\omega = \frac{Dp}{D\tau} = Bo\rho\left[\frac{\partial H}{\partial\tau} + \boldsymbol{v}\cdot\boldsymbol{D}H - w\right], \tag{11.25}$$

is the vertical pseudo-velocity that is the rate of variation of pressure following the air particles.

11.3.1a. The hydrostatic, large-scale non-adiabatic, viscous, non-tangent model equations

Finally, as a result we obtain for v, ω, H, T and ρ, functions of τ, ξ, η, p, the following "hydrostatic, large-scale non-adiabatic", non-tangent model equations, in place of the LSSH equations (11.21a, b, c, d):

$$\frac{Dv}{D\tau}+\left[\frac{1}{Ro}\frac{\sin\phi}{\sin\phi^{\circ}}+\delta\tan\phi\right](k\wedge v)+\frac{Bo}{\gamma M^2}DH=\frac{Bo^2}{Re_{\perp}}\frac{\partial}{\partial p}\left(\rho\mu\frac{\partial v}{\partial p}\right); \tag{11.26a}$$

$$\frac{\partial\omega}{\partial p}+D\cdot v-\delta\tan\phi v=0;\ T=-Bop\frac{\partial H}{\partial p}; \tag{11.26b, c}$$

$$\frac{DT}{D\tau}-\frac{\gamma-1}{\gamma}\frac{T}{p}\omega;$$

$$=\frac{Bo^2}{Re_{\perp}}\frac{1}{Pr}\left\{\frac{\partial}{\partial p}\left(\rho k\frac{\partial T}{\partial p}\right)+Pr\,\mu(\gamma-1)M^2\left|\rho\frac{\partial v}{\partial p}\right|^2-\sigma\rho\frac{dR}{dp}\right\}, \tag{11.26d}$$

where $\rho = p/T$ and we assume that μ, k and R are known functions of p only.

For the system of equations (11.26a, b, c, d) it is necessary to impose, in place of (11.22), the following boundary conditions on the flat ground, namely:

$$v=0,\ \omega=Bop\frac{\partial H}{\partial\tau},\ \text{and}\ \rho k\frac{dT}{dp}=\sigma R,\ \text{on}\ H=0. \tag{11.27}$$

We do not specify the boundary conditions that must be applied at the upper end of the troposphere, $p = 0$, and at infinity in the horizontal planes $H = const.$

These may be, for instance, that:

$$\frac{p^2}{K_S(p)}\left\{\left|\frac{\partial H}{\partial p}\right|^2+|DH|^2\right\}\equiv L(H), \tag{11.28}$$

decays sufficiently rapidly at infinity, where by definition

$$K_S(p)=T_S(p)\left\{\frac{\gamma-1}{\gamma}-\frac{p}{T_S(p)}\frac{dT_S}{dp}\right\}>0. \qquad (11.29)$$

11.3.1b. The non-tangent adiabatic nonviscous, primitive equations

If, now, we assume that in the above equations (11.26a, b, c, d): $Re_\perp \uparrow \infty$, with *Bo, Pr, Ro* and M^2 fixed, then we find, non-tangent adiabatic nonviscous, primitive equations for $\boldsymbol{v}$, ω and H. Namely:

$$\frac{D\boldsymbol{v}}{D\tau}+\left[\frac{1}{Ro}\frac{\sin\phi}{\sin\phi^\circ}+\delta\tan\phi\right](\boldsymbol{k}\wedge\boldsymbol{v})+\frac{Bo}{\gamma M^2}\boldsymbol{D}H=0, \qquad (11.30a)$$

$$\frac{\partial\omega}{\partial p}+\boldsymbol{D}\cdot\boldsymbol{v}-\delta\tan\phi\,\boldsymbol{v}\cdot\boldsymbol{j}=0; \qquad (11.30b)$$

$$\frac{D}{D\tau}\left(p\frac{\partial H}{\partial p}\right)-\frac{\gamma-1}{\gamma}\frac{\partial H}{\partial p}\omega=0. \qquad (11.30c)$$

In this case as boundary condition at the flat ground we can only impose the slip condition:

$$\omega=Bo\rho\left[\frac{\partial H}{\partial\tau}+\boldsymbol{v}\cdot\boldsymbol{D}H\right]=0,\ \text{on}\ H=0. \qquad (11.31)$$

We note that according to El Mabrouk and Zeytounian (1984), the condition at infinity in altitude:

$$\lim_{p\downarrow 0}\left\{\frac{p^2}{K_S(p)}H\frac{\partial H}{\partial p}\right\}=0, \qquad (11.32)$$

is a necessary condition for the well-posedness of the initial ($\tau = 0$: $\boldsymbol{v} = \boldsymbol{v}^0$, $H = H^0$) - **boundary** value problem (11.30a, b, c), (11.31), (11.32). But, an unsteady adjustment problem is necessary to obtain, the initial values $\boldsymbol{v}^0$ and H^0?

11.4. THE KIBEL PRIMITIVE TANGENT EQUATIONS WITH THE β-EFFECT

For the derivation of the famous Kibel primitive (tangent, adiabatic, nonviscous, but with the so-called β-effect) equations, according to Kibel (1957, pp. 370-272, original Russian edition), from the above equations (11.30a, b, c) it is necessary, first, to observe that:

$$\frac{\sin\phi}{\sin\phi^\circ} = 1 + \frac{\delta}{\tan\phi^\circ} y + O\left(\delta^2\right), \text{ since } \phi = \phi^\circ + \delta y,$$

and as a consequence

$$\frac{1}{Ro}\frac{\sin\phi}{\sin\phi^\circ} = \frac{1}{Ro} + \beta y, \tag{11.33a}$$

with an error of $O(\delta^2)$, where

$$\beta = \frac{\delta}{Ro \tan\phi^\circ}. \tag{11.33b}$$

on the other hand

$$\tan\phi = \tan\phi^\circ[1 + O(\delta)], \tag{11.33c}$$

with, for instance, $\tan\phi^\circ \approx 1$ for $\phi^\circ \approx 45°$. Now, if we consider the following limiting process:

$$\delta \to 0, \text{ with } \beta = O(1), \tag{11.34}$$

then we derive the Kibel primitive tangent equations with β-effect:

$$\frac{D\boldsymbol{v}}{D\tau} + \left[\frac{1}{Ro} + \beta y\right]\left(\boldsymbol{k} \wedge \boldsymbol{v}\right) + \frac{Bo}{\gamma M^2} \boldsymbol{D}H = 0, \tag{11.35a}$$

$$\frac{\partial \omega}{\partial p} + \boldsymbol{D} \cdot \boldsymbol{v} = 0; \tag{11.35b}$$

$$T = -Bop\frac{\partial H}{\partial p}; \tag{11.35c}$$

$$\frac{DT}{D\tau} - \frac{\gamma - 1}{\gamma}\frac{T}{p}\omega = 0. \tag{11.35d}$$

Obviously in equation (11.35a) the Rossby number *Ro* and the Mach number *M* are both small parameters (for $\phi^\circ \approx 45°$), but $Bo = O(1)$.

For equations (11.35a) and (11.35d) it is necessary to impose the initial condition:

$$\tau = 0: \boldsymbol{v} = \boldsymbol{v}^0, T = T^0, \tag{11.36}$$

and again a slip condition on the flat ground:

$$\omega = Bo\rho\left[\frac{\partial H}{\partial \tau} + \boldsymbol{v}\cdot\boldsymbol{D}H\right] = 0, \text{ on } H = 0. \tag{11.37}$$

But we note (again!) that the initial values $\boldsymbol{v}^0$ and T^0, are, in fact, derived by matching via an *unsteady adjustment problem to the primitive equations* (see, for instance, Guiraud and Zeytounian (1982)). As $Ro << 1$ (or $Ki << 1$) and $M << 1$, it is judicious in the equation (11.35a) to introduce the following similarity parameter (when the Strouhal number $St = 1$, the Kibel number *Ki* = Rossby number *Ro*):

$$\lambda_0 = \frac{1}{\gamma}\left[\frac{Ki}{M}\right]^2, \tag{11.38}$$

and rewrite this equation (11.35a) in the following form:

$$Ki\frac{D\boldsymbol{v}}{D\tau} + [1 + \beta Ki y](\boldsymbol{k} \wedge \boldsymbol{v}) + \lambda_0 Bo\frac{1}{Ki}\boldsymbol{D}H = 0, \tag{11.35a'}$$

where $Ki = 1/f^o/(L^\circ/U^\circ) << 1$ is the (singular perturbation) Kibel number.

The equation (11.35a') can be used (see below §11.5) for the derivation of the so-called quasi-geostrophic main model equation, when *Ki* tends to zero with λ_0 fixed.

11.5. THE QUASI-GEOSTROPHIC ASYMPTOTIC MODEL

In the paper by Guiraud and Zeytounian (1980) the reader can find an asymptotic derivation of the quasi-geostrophic equation (as first approximation) and also the ageostrophic equation (as second approximation), from the Kibel primitive tangent equations and then the solution of the unsteady adjustment problems to geostrophy and ageostrophy.

Concerning the steady and unsteady Ekman layer problems, in Guiraud and Zeytounian (1980) these problems have been also considered in the framework of hydrostatic, non-adiabatic, viscous, but tangent model equations with the terms proportional to $1/Re_{\perp}$.

11.5.1. The tangent hydrostatic non-adiabatic viscous equations

The tangent hydrostatic non-adiabatic equations are derived from the model equations (11.26a, b, c, d) when we take into account the β-effect (as in §11.4) but with $Re_{\perp} = O(1)$. In this case with $Bo \equiv 1$, we obtain the following system of equations which generalize the Kibel primitive tangent equations (11.35a, b, c, d). These equations take the following form:

$$\frac{D\boldsymbol{v}}{D\tau}+\left[\frac{1}{Ro}+\beta y\right](\boldsymbol{k}\wedge\boldsymbol{v})+\frac{1}{\gamma M^2}\boldsymbol{D}H=\frac{1}{Re_{\perp}}\frac{\partial}{\partial p}\left(\rho\mu\frac{\partial\boldsymbol{v}}{\partial p}\right); \quad (11.39a)$$

$$\frac{\partial\omega}{\partial p}+\boldsymbol{D}\cdot\boldsymbol{v}=0; \quad (11.39b)$$

$$T=-p\frac{\partial H}{\partial p}; \quad (11.39c)$$

$$\frac{DT}{D\tau}-\frac{\gamma-1}{\gamma}\frac{T}{p}\omega$$
$$=\frac{1}{Re_{\perp}}\frac{1}{Pr}\left\{\frac{\partial}{\partial p}\left(\rho k\frac{\partial T}{\partial p}\right)+Pr\,\mu(\gamma-1)M^2\left|\rho\frac{\partial\boldsymbol{v}}{\partial p}\right|^2-\sigma\rho\frac{dR}{dp}\right\}. \quad (11.39d)$$

These above equations (11.39a, b, c, d) constitute a very consistent model for the atmospheric tangent motions in a thin layer as the troposphere.

Below we start from the above equations (11.39a, b, c, d) but, as a simplified case, *without* the β-effect.

First, it is necessary to write a similarity relation between $Re_\perp$ and Ki. A judicious and consistent choice is:

$$\frac{1}{Re_\perp} = \kappa_0 Ki, \text{ with } \kappa_0 = O(1), \tag{11.40}$$

and we observe that

$$\frac{Ki}{Re_\perp} = Ek_\perp = \frac{\mu\rho}{f^\circ (H^\circ)^2}, \tag{11.41}$$

is the Ekman number associated with $Re_\perp$ which according to (11.40) is a small parameter of the order $O(Ki^2)$.

Then, with (11.40) and (11.38) we obtain the following set of equations, from (11.39a, b, c, d) when the β-effect is *neglected*; namely:

$$Ki\left\{\frac{\partial v}{\partial \tau} + (v \cdot D)v + \omega\frac{\partial v}{\partial p}\right\} + (k \wedge v) + \lambda_0 \frac{1}{Ki} DH = \kappa_0 Ki^2 \frac{\partial}{\partial p}\left(\rho\mu\frac{\partial v}{\partial p}\right); \tag{11.42a}$$

$$\frac{\partial \omega}{\partial p} + D \cdot v = 0; \; T = -p\frac{\partial H}{\partial p} \tag{11.42b, c}$$

$$Ki\left\{\frac{\partial T}{\partial \tau} + v \cdot DT + \omega\left[\frac{\partial T}{\partial p} - \frac{\gamma - 1}{\gamma}\frac{T}{p}\right]\right\}$$
$$= \frac{1}{Pr}\kappa_0 Ki^2\left\{\frac{\partial}{\partial p}\left(\rho k \frac{\partial T}{\partial p}\right) + Pr\,\mu(\gamma - 1)\frac{Ki^2}{\gamma\lambda_0}\left|\rho\frac{\partial v}{\partial p}\right|^2 - \sigma\rho\frac{dR}{dp}\right\} \tag{11.42d}$$

with $\rho = p/T$.

The similarity relation (11.40) is motivated by the fact that it corresponds to the least degeneracy of equations (11.42a, b, c d) when $Ki \downarrow 0$.

11.5.2. The limit Ki tends to zero

Our main small parameter is the Kibel number Ki and we expand $\boldsymbol{H}$, $\boldsymbol{v}$, ω, ρ and T according to the following scheme:

$$(H, \rho, T) = (H_0, \rho_0, T_0) + Ki\,(H_1, \rho_1, T_1) + Ki^2\,(H_2, \rho_2, T_2) + O(Ki^3), \tag{11.43a}$$

$$(\boldsymbol{v}, \omega) = (\boldsymbol{v}_0, \omega_0) + Ki\,(\boldsymbol{v}_1, \omega_1) + Ki^2\,(\boldsymbol{v}_2, \omega_2) + O(Ki^3). \tag{11.43b}$$

We find first that (H_0, ρ_0, T_0) do not depend on the horizontal variables and then we assume that they do not depend on time τ either. Although this does not follow directly from the above equations it will be found to be consistent with the constancy of $dR/dp \equiv Q(p)$, which has been assumed previously. Of course, we have:

$$T_0 = -p\frac{\partial H_0}{\partial p}, \quad \rho_0 = \frac{p}{T_0}, \tag{11.43c}$$

but we dont know yet how T_0 depends on p. Then, assuming:

$$-p\left[\frac{\partial T_0}{\partial p} - \frac{\gamma - 1}{\gamma}\frac{T_0}{p}\right] \equiv K_0(p) \neq 0, \tag{11.43d}$$

we find, from (11.42d)

$$\omega_0 = 0. \tag{11.44a}$$

But, the same equation (11.42d) leads for T_1 to:

$$\frac{\partial T_1}{\partial \tau} + \boldsymbol{v}_0 \cdot \boldsymbol{D}T_1 - \frac{K_0(p)}{p}\omega_1 = \frac{\kappa_0}{Pr}\left\{\frac{\partial}{\partial p}\left(\rho_0 k_0 \frac{\partial T_0}{\partial p}\right) - \sigma\rho_0 Q(p)\right\}. \tag{11.44b}$$

Of course, from the equation (11.42a) we derive the geostrophic balance:

$$(\boldsymbol{k} \wedge \boldsymbol{v}_0) + \lambda_0 \boldsymbol{D}H_1 = 0, \tag{11.44c}$$

and continuity equation (11.42a) leads to

$$D \cdot v = 0. \tag{11.44d}$$

11.5.3. The main model quasi-geostrophic equation

Going to higher order, we find:

$$\frac{\partial v_1}{\partial \tau} + (v_0 \cdot D)v_1 + k \wedge v_1 + \lambda_0 D H_2 = 0,$$

$$\frac{\partial \omega_1}{\partial p} + D \cdot v_1 = 0; \; T_1 = -p\frac{\partial H_1}{\partial p}, \tag{11.45}$$

$$v_0 = \lambda_0 (k \wedge D H_1).$$

Now, through the elimination of v_1, ω_1 and T_1, from the equations (11.44b), (11.45), we get the following quasi-geostrophic potential vorticity equation:

$$\frac{D_{qg}(\Lambda H_1)}{D\tau} = G_0(p) \tag{11.46}$$

where

$$\frac{D_{qg}}{D\tau} = \frac{\partial}{\partial \tau} + \lambda_0 (DH_1 \wedge D), \; \Lambda = \lambda_0 D^2 + \frac{\partial}{\partial p}\left\{\frac{p^2}{K_0(p)}\frac{\partial}{\partial p}\right\}, \tag{11.47a}$$

and

$$G_0(p) = \kappa_0 \frac{1}{Pr}\frac{d}{dp}\left\{\frac{p}{K_0(p)}\left[\frac{d}{dp}\left(\rho_0 k_0 \frac{\partial T_0}{\partial p}\right) - \sigma \rho_0 Q(p)\right]\right\}. \tag{11.47b}$$

We observe that (11.46) is the equation for the quasi-geostrophic model, at least whenever $G_0(p) = 0$! We leave aside the question of boundary and initial conditions and investigate the value of $G_0(p)$.

We consider a 3D bounded domain D in space-time at $p = const$, and set ∂D for its boundary, with unit normal $N = (\alpha, \beta, \gamma)$ and area element ds. Integrating the equation (11.46) over D we find:

$$G_0(p) = \frac{1}{|D|} \iint\limits_{\partial D} [\alpha u_0 + \beta v_0 + \gamma] \Lambda H_1 ds\,, \tag{11.48}$$

where $|D|$ stands for the volume of D and $\boldsymbol{v}_0 = (u_0, v_0)$ is given by the geostrophic relation (the third equation of (11.45) as consequence of (11.44c)). If we let the dimensions of D go to infinity then, from the assumed boundedness of ΛH_1, we find: $G_0(p) = 0$, and this confirms that the equation of the quasi-geostrophic model is:

$$\frac{D_{qg}(\Lambda H_1)}{D\tau} = 0\,, \tag{11.49}$$

even in a non-adiabatic atmosphere.

But, as a consequence, it is seems that the standard atmosphere must satisfy the relation:

$$\frac{d}{dp}\left(\rho_0 k_0 \frac{dT_0}{dp}\right) - \sigma\rho_0 Q(p) = 0\,. \tag{11.50}$$

To be sure we may conclude that the left hand side of (11.50) is constant, but we now present an argument which shows that the constant must be zero (as in (11.22)). We go back to equation for T_1, (11.44b) - integrating over D on the ground and use the fact that, on the ground, outside of the Ekman layer, we have the slip condition:

$$\omega_1 = \rho_0\left[\frac{\partial H_1}{\partial \tau} + \boldsymbol{v}_0 \cdot \boldsymbol{D} H_1\right], \tag{11.51}$$

then we get a relation analogous to (11.48) involving a constant on its left hand side - this constant is precisely the one which occurs on the right hand side of (11.50). Again letting the dimensions of D go to infinity we find that this constant must be zero.

Obviously, the physical meaning of (11.50) is that, the standard atmosphere is realized as a heat balance averaged over time and space. We check that the constancy of $Q(p)$ implies that T_0 does not depend on time τ as was assumed beforehand.

We observe that the quasi-geostrophic model equation (11.49) contains one derivation with respect to time τ and, as a consequence, only one initial condition must be supplied for H_1, via an unsteady adjustment problem. We observe also that when Ki tends to zero the flat ground with the equation $H = 0$, leads to equation $H_0(p) = 0$, and below it is assumed that $p = 1$ (in dimensionless form) is the solution of this equation. At $p = 1$, via the Ekman boundary-layer problem, a boundary condition must be derived.

11.5.4. Adjustment to geostrophy (11.44c)

It is not difficult to verify, that the unsteady adjustment is satisfied by setting:

$$t^* = \frac{\tau}{Ki}, \tag{11.52a}$$

and applying the initial limiting process:

$$Ki \downarrow 0, \text{ with } t^* \text{ fixed.} \tag{11.52b}$$

Let us set f^* for any quantity f considered as a function of t^* instead of τ. We expand the functions according to:

$$(v, \omega, H, \rho, T) = (v^*_0, \omega^*_0, H^*_0, \rho^*_0\ T^*_0) + Ki\,(v^*_1, \omega^*_1, H^*_1, \rho^*_1, T^*_1) + \ldots, \tag{11.53}$$

and we substitute into the equations (11.42a, b, c, d) where $\partial/\partial\tau = (1/Ki)\,\partial/\partial t^*$.

One finds first:

$$(H^*_0, \rho^*_0, T^*_0) = (H_0(p), \rho_0(p), T_0(p)),$$

with

$$T^*_0 = -Bop\frac{\partial H_0}{\partial p}, \text{ and } \rho^*_0 = \frac{p}{T_0(p)}.$$

In order to find equations for $(v^*_0, \omega^*_0, H^*_1, T^*_1)$ we have to go to higher order:

$$\frac{\partial v^*_0}{\partial t^*} + k \wedge v^*_0 + \lambda_0 DH^*_1 = 0,$$

$$\frac{\partial \omega^*{}_0}{\partial p} + \boldsymbol{D} \cdot \boldsymbol{v}^*{}_0 = 0\,;\; T^*{}_1 = -p\frac{\partial H^*{}_1}{\partial p}, \tag{11.54}$$

$$\frac{\partial T^*{}_1}{\partial t^*} - \frac{K_0(p)}{p}\omega^*{}_0 = 0.$$

The system of equations (11.54) is the system governing the unsteady process of adjustment to geostrophy. For the solution of the equations (11.54) we may, without loss of generality the analysis, set:

$$\boldsymbol{v}^*{}_0 = \boldsymbol{D}\phi^*{}_0 + \boldsymbol{k} \wedge \boldsymbol{D}\psi^*{}_0 \tag{11.55}$$

and we derive at once from the first equations (11.54), for $\boldsymbol{v}^*{}_0$,

$$\boldsymbol{D}\left[\frac{\partial \phi^*{}_0}{\partial t^*} + \lambda_0 H^*{}_1 - \psi^*{}_0\right] + \boldsymbol{k} \wedge \boldsymbol{D}\left[\phi^*{}_0 + \frac{\partial \psi^*{}_0}{\partial t^*}\right] = 0.$$

This last equation implies that the two expressions in [ ] are, as functions of two horizontal coordinates x and y, *harmonic* expressions and are related by the classical Cauchy-Riemman relations.

Then the *Liouville theorem* tells us that they should be *polynomials* and physical evidence suggests that they are indeed zero.

As a matter of fact, physical evidence should be replaced by matching with a solution valid on the whole sphere. If we consider the primitive equations on the sphere (for instance, the equations (11.30a, b, c)) and carry over the analysis which follows from relations (11. 52a, b) and (11.53), then we shall find a result analogous to the one concerning the above expressions in [ ] and then it will not be necessary to call for physical evidence in order to justify:

$$\frac{\partial \phi^*{}_0}{\partial t^*} + \lambda_0 H^*{}_1 - \psi^*{}_0 = 0, \tag{11.56a}$$

$$\phi^*{}_0 + \frac{\partial \psi^*{}_0}{\partial t^*} = 0. \tag{11.56b}$$

On the other hand from the last two equations of the system (11.54) we derive:

$$\omega^*{}_0 = -\frac{p^2}{K_0(p)}\frac{\partial^2 H_1}{\partial p \partial t^*}. \tag{11.57}$$

Finally, going back to the first two equations of the system (11.54), we find a pair of equations for $\boldsymbol{v}^*{}_0$ and $\boldsymbol{H}^*{}_1$, namely:

$$\frac{\partial \boldsymbol{v}^*{}_0}{\partial t^*} + \boldsymbol{k} \wedge \boldsymbol{v}^*{}_0 + \lambda_0 \boldsymbol{D} H^*{}_1 = 0, \tag{11.58a}$$

$$\boldsymbol{D} \cdot \boldsymbol{v}^*{}_0 - \frac{\partial}{\partial p}\left\{\frac{p^2}{K_0(p)}\frac{\partial^2 H^*{}_1}{\partial p \partial t^*}\right\} = 0, \tag{11.58b}$$

and it is necessary to give initial condition for $\boldsymbol{v}^*{}_0$ and for $\boldsymbol{H}^*{}_1$. Concerning $\boldsymbol{v}^*{}_0$ we may use the initial value, $\boldsymbol{v}^0$, of the horizontal velocity $\boldsymbol{v}$ imposed to equation (11.42a).

Concerning the initial value for $H^*{}_1$ we may use the initial value for H, which is related to the initial value of T, imposed on equation (11.42d), by the relationship (11.42c) but this works only if the initial values (T^0, H^0), for T and H, may be set in the form,

$$(H^0, T^0) = (H_0(p), T_0(p)) + Ki\,(H^0{}_1, T^0{}_1). \tag{11.58c}$$

In this case we get:

$$t^* = 0: \boldsymbol{v}^*{}_0 = \boldsymbol{v}^0, H^*{}_1 = H^0{}_1. \tag{11.58d}$$

Whenever the initial value appropriate to the equation (11.42d) cannot be put into such a form, we must expect that another adjustment process holds! Now, the equation (11.58b), according to (11.55), can be written as:

$$\boldsymbol{D}^2 \phi^*_0 = \frac{\partial}{\partial p}\left\{\frac{p^2}{K_0(p)}\frac{\partial^2 H^*{}_1}{\partial p \partial t^*}\right\},$$

but from (11.56a, b) we obtain also:

$$\lambda_0 \frac{\partial H^*{}_1}{\partial t^*} = -\left(\frac{\partial^2 \phi^*{}_0}{\partial t^{*2}} + \phi^*{}_0\right).$$

Finally, a single equation for ϕ^*_0 is derived, namely:

$$\frac{\partial^2}{\partial t^{*2}}\left\{\frac{\partial}{\partial p}\left[\frac{p^2}{K_0(p)}\frac{\partial \phi^*_0}{\partial p}\right]\right\}+\lambda_0 D^2\phi^*_0+\frac{\partial}{\partial p}\left[\frac{p^2}{K_0(p)}\frac{\partial \phi^*_0}{\partial p}\right]=0. \quad (11.59)$$

When $K_0(p) = 1$, the equation (11.59) is the one derived by I. A. Kibel (see §4.2 in his (1963) book). Ilya Afanasievitch was able (in 1955) to settle the main issue of the adjustment problem which is to know whether or not v^*_0 and H^*_1 evolve towards the geostrophic balance (11.44c) when t^* tends to infinity. As a matter of fact one has:

$$lim_{t^*\uparrow\infty}(v^*_0, H^*_1) = [v_0(\tau=0,\xi,\eta,p), H_1(\tau=0,\xi,\eta,p)], \quad (11.60a)$$

and

$$\text{at } \tau = 0\text{: } (k\wedge v_0) + BoDH_1 = 0. \quad (11.60b)$$

There is an important observation which was known to Ilya Afanasievitch and which concerns the way in which, $lim_{t^*\uparrow\infty} H^*_1$ is related to the initial values v^0, H^0_1, according to (11.58d).

If we start from:

$$k\cdot\left\{D\wedge\left(\frac{\partial v^*_0}{\partial t^*}+k\wedge v^*_0\right)\right\}=0,$$

which follows from (11.58a), and transform it thanks to the (11.58b) and the last two equations in the set (11.54), then we obtain the following equation:

$$\frac{\partial}{\partial t^*}\left\{k\cdot(D\wedge v^*_0)+\frac{\partial}{\partial p}\left\{\frac{p^2}{K_0(p)}\frac{\partial H^*_1}{\partial p}\right\}\right\}=0.$$

Now, if we integrate this last equation between $t^* = 0$ and $t^* = \infty$ and if we use the geostophic balance for the limiting values of v^*_0 and H^*_1 when $t^* \uparrow \infty$, we get:

$$\left(\Lambda \mathrm{H}_1\right)_{\tau=0} = \boldsymbol{k} \cdot \left\{\boldsymbol{D} \wedge \boldsymbol{v}^0\right\} + \frac{\partial}{\partial p}\left\{\frac{p^2}{K_0(p)}\frac{\partial H^0{}_1}{\partial p}\right\}, \tag{11.61}$$

where Λ is the operator (see (11.47a)) which appear in the quasi-geostrophic main model equation (11.49).

We observe that:

The initial condition for the quasi-geostrophic operator ΛH_1, in (11.49), is related, according to (11.61), to the initial values $\boldsymbol{v}^0$ and $H^*{}_1$ *for the hydrostatic, non-adiabatic and tangent, model equations (11.42a, b, c, d).*

Indeed, these (hydrostatic) initial values $\boldsymbol{v}^0$ and $H^*{}_1$ must be derived from an unsteady adjustment problem to hydrostatic balance (11.21b) (which is a consequence of the hydrostatic (long-wave) limiting process (11.20).

For the Kibel primitive equations (see the §11.4) this adjustment problem has been considered by Guiraud and Zeytounian (1982).

11.5.5. The Ekman layer and the boundary condition at p=1 for the quasi-geostrophic main model equation (11.49)

Below, the investigation of the steady Ekman layer which leads to the classical Ackerblom problem, makes it possible to derive, via matching, the boundary condition at $p = 1$ for the quasi-geostrophic main model equation (11.49).

We consider near $p = 1$ a local inner-region which corresponds to:

$$|p-1| = O(Ki) \Rightarrow p^* = \frac{1-p}{Ki}, \tag{11.62}$$

and we call it the steady Ekman region. Within this region the independent variables are τ, ξ, η and p^* and we consider the following inner limiting process:

$$Ki \to 0 \text{ with } \tau, \xi, \eta \text{ and } p^* \text{ fixed.} \tag{11.63}$$

With (11.63) we consider the following inner expansion,

$$(\boldsymbol{v}, \omega, H, T, \rho) = (\boldsymbol{v}_{Ek}, 0, 0, T_0(1), \rho_0(1)) + Ki\,(\boldsymbol{v}^1{}_{Ek}, \omega_{Ek}, H_{Ek}, T_{Ek}, \rho_{Ek}) + \ldots, \tag{11.64}$$

and we obtain from the starting equations (11.42a, b, c, d) the following set of equations, for $\boldsymbol{v}_{Ek}$, ω_{Ek}, H_{Ek} and T_{Ek},

$$(\boldsymbol{k} \wedge \boldsymbol{v}_{Ek}) + \lambda_0 \boldsymbol{D} H_{Ek} = \kappa_0 \frac{\partial}{\partial p^*}\left(\frac{\partial \boldsymbol{v}_{Ek}}{\partial p^*}\right); \tag{11.65a}$$

$$\frac{\partial \omega_{Ek}}{\partial p^*} = \boldsymbol{D} \cdot \boldsymbol{v}_{Ek}; \tag{11.65b}$$

$$T_0(1) = \frac{\partial H_{Ek}}{\partial p^*}; \tag{11.65c}$$

$$\frac{1}{Pr}\kappa_0 \frac{\partial}{\partial p^*}\left(\frac{\partial T_{Ek}}{\partial p^*}\right) = 0, \tag{11.65d}$$

when we assume that $\rho_d(1) = 1$, $k(1) = 1$ and $\mu(1) = 1$. From the boundary conditions on the flat ground (11.27) with $Bo = 1$ and (11.62) we have:

$$\text{on } H_{Ek} = 0\text{: } \boldsymbol{v}_{Ek} = 0,\ \omega_{Ek} = \frac{\partial H_{Ek}}{\partial \tau},\ \frac{\partial T_{Ek}}{\partial p^*} = \sigma R(1), \tag{11.66}$$

if we assume that $T_0(1) = 1$ and that the main radiative transfer does not have an Ekman structure.

Indeed, the flat ground in the Ekman layer theory is characterized by:

$$p^* = p_{w0}^* + Ki\, p_{w1}^* + \ldots,$$

and from matching with the main quasi-geostrophic region we can write:

$$T_{Ek} = T_1(\tau,\xi,\eta,1) + \left(\frac{dT_0}{dp}\right)_{p=1} p^*, \text{ with } T_1\big|_{p=1} = -\left(\frac{\partial H_1}{\partial p}\right)_{p=1},$$

but from (11.65c), for $T_0(1) = 1$:

$$\frac{\partial H_{Ek}}{\partial p^*} = 1 \Rightarrow H_{Ek} = H_1(\tau,\xi,\eta,1) + p^*, \tag{11.67a}$$

and $H_{Ek} = 0$ implies

$$p_{w0}^{*} = - H_{1}(\tau, \xi, \eta, 1). \tag{11.67b}$$

11.5.5a. Ackerblom's problem

Now, in equation (11.65a) for $\boldsymbol{v}_{Ek}$ we set:

$$\boldsymbol{v}_{Ek} = \boldsymbol{v}_{0}(\tau, \xi, \eta, 1) + \boldsymbol{v}_{Ek}'.$$

From matching with the main quasi-geostrophic region, when p^{*} tends to infinity, we have

$$\lim_{p^{*}\uparrow\infty} \boldsymbol{v}_{Ek}' = 0 \text{ and } [\boldsymbol{k} \wedge \boldsymbol{v}_{0} + \lambda_{0}\boldsymbol{D}H_{1}]_{p=1} = 0, \tag{11.68}$$

and we obtain the following relationship: $\lambda_{0}\boldsymbol{D}H_{Ek} + \boldsymbol{k} \wedge \boldsymbol{v}_{Ek} = \boldsymbol{k} \wedge \boldsymbol{v}_{Ek}'$. Finally, for $\boldsymbol{v}_{Ek}'$ we derive the following classical Ackerblom problem:

$$\kappa_{0}\frac{\partial}{\partial p^{*}}\left(\frac{\partial \boldsymbol{v}_{Ek}'}{\partial p^{*}}\right) - \boldsymbol{k} \wedge \boldsymbol{v}_{Ek}' = 0,$$

$$\boldsymbol{v}_{Ek}' = -\boldsymbol{v}_{0}\big|_{p=1}, \text{ on } p^{*} = - H_{1}(\tau, \xi, \eta, 1), \tag{11.69}$$

$$\lim_{p^{*}\uparrow\infty} \boldsymbol{v}_{Ek}' = 0.$$

The solution of (11.69) is obtained in a standard way, namely:

$$\boldsymbol{v}_{Ek}' - i\boldsymbol{k} \wedge \boldsymbol{v}_{Ek}' = -\left(\boldsymbol{v}_{0}\big|_{p=1} - i\boldsymbol{k} \wedge \boldsymbol{v}_{0}\big|_{p=1}\right)E^{*}, \tag{11.70}$$

where

$$E^{*} = exp\left\{-\frac{1+i}{(2\kappa_{0})^{1/2}}\left[p^{*} - H_{1}\big|_{p=1}\right]\right\}. \tag{11.71}$$

Now, from (11.65b), we obtain

$$\omega_{Ek} = \int_{p_{w0}^*}^{p^*} (D \cdot v_{Ek}) dp^* + \left(\frac{\partial H_{Ek}}{\partial \tau}\right)_{p^* = p_{w0}^*},$$

according to (11.66) for ω_{Ek} and p_{w0}^* is given by (11.67b). Therefore, when $p^* \uparrow \infty$, we obtain, after a straigtforward calculation, the following limit relation, for

$$lim_{p^*\uparrow\infty}\, \omega_{Ek} \equiv \omega_{Ek}^{\infty} = \frac{\partial \left(H_1|_{p=1}\right)}{\partial \tau} - \lambda_0 K_0^{1/2} D^2 H_1\Big|_{p=1}. \quad (11.72)$$

Finally, by matching between the Ekman inner region ($p^* \uparrow \infty$) and the main quasi-geostrophic outer region ($p \downarrow 0$), considered in Section 11.5.3, we derive the following boundary condition at $p = 1$ for the main quasi-geostrophic model equation (11.49); namely:

$$\left\{\frac{\partial}{\partial \tau} + \frac{1}{K(1)}\left[\frac{\partial}{\partial \tau} + v_0 \cdot D\right]\frac{\partial}{\partial p} - \lambda_0 K_0^{1/2} D^2\right\} H_1 = 0, \text{ on } p = 1, \quad (11.73)$$

where $v_0 = \lambda_0 k \wedge DH_1$ and when we take into account (11.51).

This last boundary condition at flat ground $p = 1$ takes into account the influence of the Ekman steady boundary-layer on the main quasi-geostrophic flow which is governed by equation (11.49).

11.6. ATMOSPHERIC FLOWS AS LOW-MACH NUMBER FLOW

In any realistic meteorological situation the Mach number M is rather small and we may consider the atmospheric process as a so-called "hyposonic flows" (see, for instance, Zeytounian (1983)). But the asymptotic theory of atmospheric flows at low Mach number is presently very incomplete despite the new results obtained by Guiraud and Zeytounian in connection with the CISM Course (Von Karman Session, 3-5 October 1983; Coordinator R. Kh. Zeytounian): "Models for atmospheric flows", at Udine (Italy). Further researchs are necessary in this direction; in particular a careful asymptotic analysis of the LSSH system of equations subject to the small parameter $M << 1$, interacting with the parameters, $Re_\perp$, Ro, Bo and δ, is a challenging problem.

Below, we begin with the primitive Kibel equations (11.35a, b, c, d) and we consider the simplified case (with $Bo = 1$), when the Coriolis term may be neglected ($Ro = 0$ and $\beta = 0$) in equation (11.35a). First, we set up an expansion with respect to M, proceeding as follows:

$$U = U_0 + M^2 U_2 + M^4 (logM) U_{4,0} + M^4 U_4 + \ldots + M^\alpha U_\alpha + \ldots, \quad (11.74)$$

where $U = (v, \omega, H, T)$ and the term $M^\alpha U_\alpha$ enters in (11.74) if we add dissipation, of $O(M^\alpha)$, in the right hand side of the energy equation (11.35d). In fact, in this case it is necessary to write a similarity relation between $M << 1$ and $Re_\perp >> 1$, for example,

$$\frac{1}{Re_\perp} = \nu_0 M^\alpha, \text{ with } \nu_0 = O(1), \text{ and } \alpha > 0, \quad (11.75)$$

and in place of equation (11.35d) to consider the equation (11.39d).

Then, it is found that H_0 and T_0 depend only on p, and ω_0 that is zero, and the function $H_0(p)$ would be determined by the consideration of the equation for T_α with the result that $H_0(p)$ is such that the standard atmosphere in pressure coordinates is energetically balanced (see (11.50)).

11.6.1. The Monin-Charney quasi-nondivergent model

For the functions v_0 and H_2 it is found that they are determined by equations governing incompressible two dimensional motions; the so-called "Monin-Charney" quasi-nondivergent equations (see, for instance, §8 in Monin (1972) book). Namely:

$$\frac{\partial v_0}{\partial \tau} + (v_0 \cdot D) v_0 + D\left(\frac{H_2}{\gamma}\right) = 0, \; D \cdot v_0 = 0. \quad (11.76)$$

The equations (11.76) correspond to a kind of Froude blocking within isobaric surfaces as discovered by Drazin (1961). Such a blocked flow is unable to ride over any relief and must turn over it. This is a serious drawback of such an approximation as is the fact that flows within two isobaric surfaces, close to each other, are apparently disconnected. At the level of the quasi-nondivergent (or quasi-solenoidal, according to Monin (1961)) model equations (11.76) the only way by which the flow in two isobaric surfaces may be coupled is through lateral, upper or initial condition, via an unsteady adjustment problem close to $\tau = 0$ where the equations (11.76) are not valid.

11.6.1 a. Equations for a slightly perturbed incompressible flow with a source

Some connection is uncovered at higher approximations but only in a parametric way. As a matter of fact, one we know the values of $\boldsymbol{v}_0$ and H_2 have been computed we may determine the values of T_2 and ω_2 by the relations:

$$T_2 = -p\frac{\partial H_2}{\partial p};\ \omega_2 = -\frac{p}{K_0(p)}\left\{\frac{\partial T_2}{\partial \tau} + (\boldsymbol{v}_0 \cdot \boldsymbol{D})T_2\right\}, \qquad (11.77\text{a, b})$$

and $(\boldsymbol{v}_2, H_4)$ correspond to a slightly perturbed incompressible flow with a source, namely:

$$\frac{\partial \boldsymbol{v}_2}{\partial \tau} + (\boldsymbol{v}_0 \cdot \boldsymbol{D})\boldsymbol{v}_2 + (\boldsymbol{v}_2 \cdot \boldsymbol{D})\boldsymbol{v}_0 + \omega_2 \frac{\partial \boldsymbol{v}_0}{\partial p} + \boldsymbol{D}\left(\frac{H_4}{\gamma}\right) = 0, \quad (11.77\text{c})$$

$$\boldsymbol{D} \cdot \boldsymbol{v}_2 = -\frac{\partial \omega_2}{\partial p}. \qquad (11.77\text{d})$$

It is obvious that some vertical (thanks to the derivative $\partial/\partial p$) coupling is recovered through the terms:

$$\frac{\partial \omega_2}{\partial p} = \frac{\partial}{\partial p}\left(\frac{p^2}{K_0(p)}\left[\frac{\partial}{\partial \tau} + (\boldsymbol{v}_0 \cdot \boldsymbol{D})\right]\frac{\partial H_2}{\partial p}\right),$$

$$\omega_2 \frac{\partial \boldsymbol{v}_0}{\partial p} = \frac{\partial}{\partial p}\left\{\frac{p^2}{K_0(p)}\left[\frac{\partial}{\partial \tau} + (\boldsymbol{v}_0 \cdot \boldsymbol{D})\right]\frac{\partial H_2}{\partial p}\right\}\frac{\partial \boldsymbol{v}_0}{\partial p},$$

in (11.77c, d). We observe also, that when $Bo = 1$, then:

$$Fr^2 = \gamma M^2, \text{ where } Fr^2 = \frac{U^{\circ 2}}{gH^\circ}$$

is the **(Froude number)**^{2} based on H°, and as a consequence the above expansion (11.74) is in fact a low-Froude number expansion (à la Drazin

(1961)). For a complete discussion of Drazin's theory and its extension for rotating flow, see Brighton (1977) and Hunt and Snyder (1980).

11.6.2. The behaviour of H_4 at infinity with respect to horizontal variables and, the far field expansion

A much stronger coupling with (11.76) is obtained in the far field. In fact, in (11.77c) the term H_4 is unbounded at infinity with respect to the horizontal variables (ξ, η) and as a consequence it is necessary to consider the matching with the corresponding H'_4 of the outer far field expansion for H (see (11.79a, b) and this is the reasons for the presence in the outer expansion (11.74) of the term proportional to $M^4(\log M)$. Therefore, the quasi-nondivergent approximation is embedded within a consistent expansion scheme and the intriguing decoupling is no longer a mystery and above all we have learnt to be cautions with the unexpected $M^4(\log M)U_{4,0}$ term.

In the far field we put for the horizontal position: $(\xi, \eta) = (1/M)(X, Y)$ and we let:

$$M \downarrow 0 \text{ while } (X, Y) \text{ and } \tau \text{ remains fixed, } O(1). \tag{11.78}$$

In this case, we find that:

$$\boldsymbol{v} = \boldsymbol{v}_\infty(p) + M\,\boldsymbol{v}',\ \omega = M^2\,\omega',\ H = H_S(p) + M^2\,H', \tag{11.79a}$$

and $H_S(p)$ is the standard distribution of altitude and $\boldsymbol{v}_\infty(p)$ is a horizontal wind with an arbitrary dependence on pressure p or, which is the same through $H_S(p)$, on altitude. We note that:

$$\boldsymbol{v} = \boldsymbol{v}_\infty(p),\ H = H_S(p),\ T = T_S(p) \text{ and } \omega = 0,$$

is an exact solution of the primitive equations (11.35), for the far field when the Coriolis term may be neglected.

In the case when there is no relief at all, even at finite horizontal distance (flat ground), we set up as far-field expansion:

$$\boldsymbol{v}' = \boldsymbol{v}'_1 + M\boldsymbol{v}'_2 + M^2\boldsymbol{v}'_3 + \ldots,\ \omega' = M^2\omega'_4 + \ldots,\ H' = M^2 H'_4 + \ldots, \tag{11.79b}$$

and the following results emerge. The horizontal velocity $\boldsymbol{v}'_1$ is a potential vortex which depends on the distribution of vertical vorticity associated with the quasi-nondivergent velocity field $\boldsymbol{v}_0$. As a matter of fact, Drazin's model is unable to explain how the vorticity is created but it may describe how such a vorticity evolves once created. If one assumes that this distribution of vorticity is localized it generates the potential vortex $\mathbf{v}'_1$ in the far field.

Then we find that $\mathbf{v}'_2$ is a pair of potential vortices and neither $\mathbf{v}'_1$ nor $\mathbf{v}'_2$ affords any new information which is not contained in the quasi-nondivergent field $\mathbf{v}_0$.

At the level of $\mathbf{v}'_3$ and H'_4 new features occur. From the system of equations for $\mathbf{v}'_3$, ω'_4 and H'_4, with $T'_4 = -p\partial H'_4/\partial p$, it is straightforward to extract an equation for $H'_4(\tau, X, Y, p)$. Namely, we obtain the following far-field equation:

$$\left\{\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\gamma\frac{\partial^2}{\partial \tau^2}\left\{\frac{\partial}{\partial p}\left[\frac{p^2}{K_0(p)}\frac{\partial}{\partial p}\right]\right\}\right\}H'_4$$
$$=-\frac{2\gamma}{\left(X^2+Y^2\right)^2}\left\{\frac{<\Omega_0(p)>}{2\pi}\right\}^2. \tag{11.80}$$

This equation plays, with respect to the adiabatic nonviscous primitive equations, a role analogous to the one played, by the acoustic equations, with respect to the equations governing Euler compressible inviscid fluid flow. As far as aerodynamics is concerned (see Chapter 8), acoustics governs the phenomenon of adjustment to incompressible flow, here the far-field equation (11.80) governs the phenomenon of adjustment to the quasi-solenoidal approximation.

We observe that in (11.80), $<\Omega_0(p)>$ is the total amount of vertical vorticity which is contained in the isobaric surface $p = const$ and which drives the potential vortex $\mathbf{v}'_1$.

The solution of the equation (11.80) may be found with the following conditions:

$$lim_{(X,Y)\downarrow 0}\, H'_4 = lim_{(\xi,\eta)\uparrow\infty}\, H_4\,,\; lim_{(X,Y)\uparrow\infty}\, H'_4 = 0\,, \tag{11.81a, b}$$

and we recover coupling between all the isobaric surfaces.

Concerning the upper boundary condition at $p \to 0$ it has been discussed in §11.3 above (see (11.32)) and we state again here the result:

$$lim_{p\downarrow 0}\left\{\frac{p^2}{K_0(p)}H'_4\frac{\partial H'_4}{\partial p}\right\}=0. \tag{11.81c}$$

In §12.3 of Chapter 12 of Zeytounian's (1990) book the reader can find a detailed exposition of the above Guiraud and Zeytounian results relative to low-Mach number asymptotics of the primitive Kibel meteorological equations.

11.6.3. The unsteady adjustment problem

It is not surprisingly that the homogeneous equation corresponding to (11.80) (when $<\Omega_0(p)> = 0$) is identical to the equation describing the unsteady adjustment problem to the quasi-nondivergent model (11.76) and a little reflexion suggests that it should be so.

For the unsteady adjustment problem it is necessary to rescale the time:

$$t^* = \frac{\tau}{M} \text{ and } U(\tau, \xi, \eta, p) = U^*_0(t^*, \xi, \eta, p) + MU^*_1(t^*, \xi, \eta, p) + \dots \quad (11.82)$$

In such a case from the Kibel primitive equations (11.35a, b, c, d) we derive the following adjustment equations (again when: $Bo = 1$, $Ro = 0$ and $\beta = 0$) for $\boldsymbol{v}_0^*$, ω_0^* and H_1^*:

$$\frac{\partial \boldsymbol{v}_0^*}{\partial t^*} + \boldsymbol{D}\frac{H_1^*}{\gamma} = 0, \; \boldsymbol{D}\cdot \boldsymbol{v}_0^* + \frac{\partial \omega_0^*}{\partial p} = 0, \quad (11.83a)$$

$$\frac{\partial}{\partial t^*}\left(\frac{\partial H_1^*}{\partial p}\right) + \frac{K_0(p)}{p^2}\omega_0^* = 0. \quad (11.83b)$$

Then, through a straightforward computation, we may derive a single equation for H_1^*, namely:

$$\frac{\partial^2}{\partial t^{*2}}\left\{\frac{\partial}{\partial p}\left[\frac{p^2}{K_0(p)}\frac{\partial H_1^*}{\partial p}\right]\right\} + \boldsymbol{D}^2\left(\frac{H_1^*}{\gamma}\right) = 0. \quad (11.84)$$

If we assume that $K_0(p) \equiv K_{00} = \textbf{const}$ this equation (11.84) may be treated in an analogous way to the equation for the adjustment to geostrophy and as a consequence we obtain:

$$H_1^* \cong exp\left(\frac{\zeta}{2}\right)T^{-3/2}g_1^*,\ \zeta = log\left(\frac{1}{p}\right),\ T = \left(\frac{K_{00}}{\gamma}\right)^{1/2} t^*, \quad (11.85)$$

from which (since $|g_1^*| = O(1)$) the decay of H_1^* to zero when $T \to \infty$ becomes obvious and adjustment to quasi-nondivergent flow is of the decaying type.

In conclusion we can impose, on the quasi-nondivergent equation (11.76) for $\boldsymbol{v}_0$, the initial condition prescribed at the start for the Kibel primitive equation (11.35a), for $\boldsymbol{v}$.

11.7. THE INFLUENCE OF LOCAL RELIEF: LEE-WAVE AND PRIMITIVE (INVISCID-HYDROSTATIC) PROBLEMS AS INNER AND OUTER EXPANSIONS

First we consider the inviscid and adiabatic equations ($1/Re = 0$) with $\delta \downarrow 0$ (f^o-plane approximation) but we take into account the term with ε.

In this case, in place of the system of equations (11.19a, b, c, d) we obtain the following dimensionless set of *of (inviscid) f^o-plane equations*:

$$\rho\left\{\frac{D\boldsymbol{v}}{Dt} + \frac{1}{Ro}(\boldsymbol{k} \wedge \boldsymbol{v}) + \frac{\varepsilon}{tan\,\phi^o} w\boldsymbol{i}\right\} + \frac{1}{\gamma M^2}\boldsymbol{D}p = 0; \quad (11.86a)$$

$$\rho\left\{\varepsilon^2\frac{Dw}{Dt} - \frac{1}{Ro}\frac{\varepsilon}{tan\,\phi^o}\boldsymbol{v}\cdot\boldsymbol{i}\right\} + \frac{1}{\gamma M^2}\left(\frac{\partial p}{\partial z} + Bo\,\rho\right) = 0; \quad (11.86b)$$

$$\frac{D\rho}{Dt} + \rho\left\{\frac{\partial w}{\partial z} + \boldsymbol{D}\cdot\boldsymbol{v}\right\} = 0; \quad (11.86c)$$

$$\rho\frac{DT}{Dt} - \frac{\gamma - 1}{\gamma}\frac{Dp}{Dt} = 0, \quad (11.86d)$$

with $p = \rho T$.

Concerning the initial conditions (at $t = 0$) for the (non-hydrostatic) equations (11.86a, b, c, d) it is obvious (see, for instance, (11.15a)) that we should assume:

$$t = 0:\ \boldsymbol{v} = \boldsymbol{V}^0,\ \varepsilon w = W^0,\ \rho = R^0,\ p = P^0, \quad (11.87)$$

where the initial data V^0, W^0, R^0, P^0, $T^0 = P^0/R^0$, are given functions of the horizontal position *(x, y),* and altitude *z*.

Now, we assume the existence of a local relief which is simulated by the following dimensionless equation:

$$z = \sigma_0 G\left[\alpha_0 \frac{x - x^\circ}{\varepsilon}, \beta_0 \frac{y - y^\circ}{\varepsilon}\right], \tag{11.88a}$$

where

$$\sigma_0 = \frac{h^\circ}{H^\circ}, \ \alpha_0 = \frac{H^\circ}{l^\circ}, \ \beta_0 = \frac{H^\circ}{m^\circ}. \tag{11.88b}$$

In fact, we assume that the point (x°, y°) on the plane $z = 0$ serves as the origin for the local relief, and (l°, m°) encloses a finite area D° on the plane $z = 0$, while h° is the maximum height of the relief, when $(x, y) \in D^\circ$.

In the inviscid case we have the following dimensionless slip condition on the relief:

$$w = \sigma_0 \mathbf{v} \cdot \mathbf{D}G\left(\frac{x^*}{\varepsilon}, \frac{y^*}{\varepsilon}\right), \text{ on } z = \sigma_0 G\left(\frac{x^*}{\varepsilon}, \frac{y^*}{\varepsilon}\right), \tag{11.89a}$$

where

$$x^* = \alpha_0(x - x^\circ) \text{ and } y^* = \beta_0(y - y^\circ). \tag{11.89b}$$

Below the *main small* parameter is the hydrostatic parameter, $\varepsilon = H^\circ/L^\circ$ (see(11.16a)).

11.7.1. The inviscid hydrostatic limiting process and the primitive (inviscid-hydrostatic) problem

First, we consider the following main "hydrostatic" limit, namely:

$$lim^P = [\varepsilon \downarrow 0 \text{ with } t, x, y, z, \text{ fixed}], \tag{11.90}$$

and we assume that the parameters *Ro, Bo, M,* σ_0, α_0, β_0 are *O(1)* and that the horizontal variables x^* and y^* are also *O(1).*

In this case, as a consequence of (11.90) we derive from the equations (11.86a, b, c, d) the following *primitive (inviscid-hydrostatic)* equations:

$$\rho^P \left\{ \frac{D^P \boldsymbol{v}^P}{Dt} + \frac{1}{Ro} \left(\boldsymbol{k} \wedge \boldsymbol{v}^P \right) \right\} + \frac{1}{\gamma M^2} \boldsymbol{D} p^P = 0; \tag{11.91a}$$

$$\frac{\partial p^P}{\partial z} + Bo \rho^P = 0; \tag{11.91b}$$

$$\frac{D^P \rho^P}{Dt} + \rho^P \left\{ \frac{\partial w^P}{\partial z} + \boldsymbol{D} \cdot \boldsymbol{v}^P \right\} = 0; \tag{11.91c}$$

$$\rho^P \frac{D^P T^P}{Dt} - \frac{\gamma - 1}{\gamma} \frac{D^P p^P}{Dt} = 0, \tag{11.91d}$$

with $p^P = \rho^P T^P$, for

$$(\boldsymbol{v}^P, w^P, p^P, \rho^P) = lim^P (\boldsymbol{v}, w, p, \rho),$$

where

$$\frac{D^P}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}^P \cdot \boldsymbol{D} + w^P \frac{\partial}{\partial z}, \quad \boldsymbol{D} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right).$$

For the limiting equations (11.91a, b, c, d) we can write only the following slip boundary condition at the flat ground, namely:

$$w^P = 0 \text{ on } z = 0, \tag{11.92}$$

since by hypothesis ($D°$ is a bounded limited local area) we have obviously: $G(\infty, \infty) \equiv 0$.

According to (11.87) and with (11.91b), it is clear that we can only require initial conditions for the unsteady equations (11.9la) and (11.91c), namely:

$$t = 0: \boldsymbol{v}^P = \boldsymbol{v}^{P0} \text{ and } \rho^P = \rho^{P0}, \tag{11.93}$$

where $\boldsymbol{v}^{P0}$ is different from $\boldsymbol{V}^0$ and ρ^{P0} is different from $\boldsymbol{R}^0$.

Two of the initial conditions have been lost during the main limiting process (11.90) and two questions arise:

(i) How have these initial conditions been lost?

(ii) How are v^{P0} and ρ^{P0} are related to V^0, W^0, R^0, P^0?

Regarding the first question, the answer is simple. According to the above primitive, inviscid-hydrostatic equations (11.91a, b, c, d), the pressure p^P is related to ρ^P by the equation of hydrostatic balance (11.91b), while w^P (as noticed for the first time by Richardson (1922) in Chapter V of his book) is computed by the process of solution of the problem (11.91a, b, c, d) with (11.92) and (11.93); all this holds true at the initial time as well. We intend to address the second question in Section 11.7.3.

We observe again that the initial data W^0 need not be $O(\varepsilon)$ with respect to V^0. So that in order to consider the most general case, we assume that at the initial time εw is of order $O(1)$ according to (11.87).

11.7.2. The local limiting process and the lee-wave problem

First, we observe that in the framework of the primitive inviscid-hydrostatic equations (11.91a, b, c, d), derived from the main hydrostatic limiting process (11.90), it is no longer possible to take into account the local relief (11.89a, b) under consideration.

Therefore, it is necessary to consider a second *local* limiting process, namely:

$$lim^l = [\varepsilon \downarrow 0 \text{ with } t, x^*, y^*, z, \text{ fixed}], \tag{11.94a}$$

and in this case with

$$(v^l, w^l, p^l, \rho^l) = lim^l\,(v, \varepsilon w, p, \rho), \tag{11.94b}$$

and

$$D = \frac{1}{\varepsilon} D^*,\ D^* = \left(\alpha\frac{\partial}{\partial\xi}, \beta\frac{\partial}{\partial\eta}\right),\ \xi = \frac{x^*}{\varepsilon},\ \eta = \frac{y^*}{\varepsilon}, \tag{11.95a}$$

$$\frac{D^*}{Dt} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\left[v^l \cdot D^* + w^l \frac{\partial}{\partial z}\right], \tag{11.95b}$$

we derive for $(v^l, w^l, p^l, \rho^l)\,(t, \xi, \eta, z)$, the following system of *local, lee-wave steady* equations:

$$\left(v^l \cdot D^*\right)v^l + w^l \frac{\partial v^l}{\partial z} + \frac{1}{\gamma M^2 \rho^l} D^* p^l = 0; \tag{11.96a}$$

$$\left(\boldsymbol{v}^l \cdot \boldsymbol{D}^*\right)w^l + w^l \frac{\partial w^l}{\partial z} + \frac{1}{\gamma M^2}\left[\frac{1}{\rho^l}\frac{\partial p^l}{\partial z} + Bo\right] = 0; \qquad (11.96b)$$

$$\boldsymbol{D}\cdot\left(\rho^l \boldsymbol{v}^l\right) + \frac{\partial}{\partial z}\left(\rho^l w^l\right) = 0; \qquad (11.96c)$$

$$\rho^l\left(\boldsymbol{v}^l \cdot \boldsymbol{D}^* T^l\right) - \frac{\gamma - 1}{\gamma}\left(\boldsymbol{v}^l \cdot \boldsymbol{D}^* p^l\right) + w^l\left\{\rho^l \frac{\partial T^l}{\partial z} - \frac{\gamma - 1}{\gamma}\frac{\partial p^l}{\partial z}\right\} = 0, \qquad (11.96d)$$

with $p^l = \rho^l T^l$.

Now, to the local system of steady lee-wave equations (11.96a, b, c, d) we can assign the following slip condition on the relief according to (11.89a, b) and (11.95a), namely:

$$w^l = \sigma_0 \boldsymbol{v} \cdot \boldsymbol{D}^* G(\xi, \eta), \text{ on } z = \sigma_0 G(\xi, \eta). \qquad (11.97a)$$

On the other hand, it is necessary to write the matching conditions relative to the horizontal variables:

$$lim_{\xi\uparrow\infty,\eta\uparrow\infty}\left(\boldsymbol{v}^l, w^l, p^l, \rho^l\right) = \left(\boldsymbol{v}^P, 0, p^P, \rho^P\right)\Big|_{x°,y°}. \qquad (11.97b)$$

In the above (quasi-stady) lee-wave problem the time t plays the role of a parameter which enters into the local lee-wave problem (11.96a, b, c, d), (11.97a, b), via the outer field associated with short-range, primitive inviscid-hydrostatic, forecasting according to problem (11.91a, b, c, d), with (11.92) and (11.93) at the time t and at the point $(x°, y°)$ of the plane $z = 0$. We note that, with respect to the vertical velocities, (11.94b) suggests that in (11.97b) really:

$$lim_{\xi\uparrow\infty,\eta\uparrow\infty} w^l = 0,$$

of course

$$\left(\frac{\partial p^P}{\partial z} + Bo\rho^P\right)\Bigg|_{x°,y°} = 0; \ \left(p^l - \rho^l T^l\right)\Big|_{x°,y°} = 0. \qquad (11.97c)$$

11.7.3. The problem of the adjustment to hydrostatic balance (11.91b)

The major conclusion of the Guiraud and Zeytounian (1982) paper is that the initial conditions for the short-range, primitive inviscid-hydrostatic, forecasting according to problem (11.91a, b, c, d), with (11.92) and (11.93), may be derived from the full set of initial conditions (11.87) by solving a one-dimensional unsteady problem of vertical motion which takes into account the effect of acoustic waves.

First, it is necessary to rescale the vertical velocity w and time t such that:

$$w^a = \varepsilon w, \ t^a = \frac{t}{\varepsilon}. \tag{11.98}$$

Then we use the limiting process:

$$lim^a = [\varepsilon \downarrow 0 \text{ with } t^a, x, y, z, \text{ fixed}], \tag{11.99a}$$

and consider the limiting values

$$(v^a, w^a, p^a, \rho^a) = lim^a\,(v, \varepsilon w, p, \rho). \tag{11.99b}$$

In such a case it is straightforward to derive the following set of limiting equations:

$$\frac{\partial v^a}{\partial t^a} + w^a \frac{\partial v^a}{\partial z} = 0, \tag{11.100}$$

$$\gamma M^2 \rho^a \left[\frac{\partial w^a}{\partial t^a} + w^a \frac{\partial w^a}{\partial z} \right] + \frac{\partial p^a}{\partial z} + Bo\rho^a = 0,$$

$$\frac{\partial \rho^a}{\partial t^a} + \frac{\partial}{\partial z}\left(\rho^a w^a\right) = 0, \tag{11.101}$$

$$\left[\frac{\partial}{\partial t^a} + w^a \frac{\partial}{\partial z} \right] \frac{p^a}{\rho^{a\gamma}} = 0.$$

These above equations (11.100), (11.101) are decoupled. The system of three equations (11.101) is identical to the equations for one-dimensional vertical motion in the atmosphere. Once w^a has been obtained trough the solution of (11.101) with the initial conditions (see, (11.87)):

$$t^a = 0: w^a = W^0, \rho^a = R^0, p^a = P^0, \quad (11.102a)$$

and proper boundary conditions on the ground and at infinity, we may use the transport equation (11.100) in order to compute $\mathbf{v}^a$ using the initial condition:

$$t^a = 0: \mathbf{v}^a = \mathbf{V}^0. \quad (11.102b)$$

Now we may understand the relation between (11.93) and (11.87) or (11.102a, b). As a matter of fact, we should consider (11.93) as a matching condition between the hydrostatic approximation at $t = 0$, and the initial-time layer, at $t^a = \infty$.

The matching requires:

$$lim_{t^a \uparrow \infty}(w^a, p^a, \rho^a) = (0, p^{PO}, \rho^{PO}), \text{ with } \frac{\partial p^{PO}}{\partial z} = -Bo\rho^{PO}, \quad (11.103a)$$

$$lim_{t^a \uparrow \infty} \mathbf{v}^a = \mathbf{v}^{PO}. \quad (11.103b)$$

Due to nonlinearity (quasi-linearity) it is difficult to make rigorous statements concerning the behaviour of the solution of equations (11.100), (11.101) as $t^a \uparrow \infty$.

Nevertheless it is quite clear, on physical grounds, that the solution, as well as entropy waves governed by the last equation of (11.101), consists of acoustic waves radiating towards infinity, leaving behind them a state of hydrostatic equilibrium: $p^a_\infty(z)$, $\rho^a_\infty(z)$, $T^a_\infty(z)$ with no vertical velocity and some distribution of horizontal velocity $\mathbf{v}^a_\infty(z)$, that we call the limiting state of the initial-time layer. Accordingly we expect that the problem of adjustment to hydrostatic balance (11.91b) pertains to the class of unsteady adjustment problems when one may have a tendency towards a limiting steady state. Of course we cannot rely on any mathematical theorem for a justification of our assertion. Our argument is a physical one and relies on a test of internal consistency.

Indeed, according to Guiraud and Zeytounian (1982), the final phase of unsteady adjustment by a process of linearization from the hydrostatic balance, is described by the equation of telegraphy with coefficients depending on the vertical coordinate (height-altitude) z. Assuming that the temperature of the standard (hydrostatic) atmosphere is constant, the corresponding initial boundary value problem is solved by the usual techniques (see §5 in Guiraud and Zeytounian (1982)), and it may be shown that the acoustic waves described by the equations (11.101) of unsteady adjustment to hydrostatic balance *decay in time like* $(1/t^a)^{3/2}$. The result may be stated very simply by saying that,

the horizontal components of the velocity and the specific entropy in the two sets of initial conditions are merely shifted vertically by the amount of vertical displacement during the whole process of the vertical, one-dimensional, unsteady motion.

This analysis does not explain how gravity waves with short horizontal wave- length are filtered out when ones goes from the full atmospheric Euler equations (equations of §11.2, where $\mu = 0$, $k = 0$, and $Q = 0$, written in dimensionless form) to the f^o-plane inviscid equations (11.86a, b, c, d).

A final comment concerning the fact that in the first equation of (11.101) the time derivative is multiplied by the Mach number M. In any realistic situation $M << 1$ and we have to deal with a new singular perturbation problem as is even more clearly revealed from the derived equation of telegraphy where the second derivative relative to t^a is multiplied by M^2! As a consequence the true time of adjustment is $Mt^a = M\varepsilon t$, at least under the limiting process which is considered here, $\varepsilon \downarrow 0$ with M fixed and then $M \downarrow 0$. It would be enlightening to examine the simultaneous limiting process:

$$\varepsilon \downarrow 0, M \downarrow 0, \text{ with some similarity relation } M = M^* \varepsilon^\lambda,$$

where λ is some real, positive exponent to determine, and $M^* = O(1)$. Usually a realistic meteorological situation corresponds to $\lambda = 1/2$.

According to Outrebon's (1981) thesis, the process of the adjustment to hydrostatic balance is composed of three main phases. During the first phase of adjustment, typical profiles of the vertical velocity, pressure and temperature are characterized by several reversals of the direction of the vertical velocity and rather strong perturbation in the pressures and temperatures and a shock wave is formed. There is a second phase during which the vertical velocity decays to zero with several additional reversals and decreases in amplitude, while the temperature and pressure profiles approach the equilibrium states values. The third phase is the ultimate phase of adjustment during which convergence to steady state is achieved.

Indeed, we can be more precise, defining the amount of vertical displacement $\Delta(t^a, z^0)$ during the whole process of the vertical one-dimensional unsteady adjustment motion, such that $\Delta(t^a, z^0)$ is the solution of the following equation:

$$\frac{\partial \Delta(t^a, z^0)}{\partial t^a} = w^a(t^a, \Delta(t^a, z^0)), \text{ with } \Delta(0, z^0) = z^0, \qquad (11.104)$$

and z^0 is the initial $(t^0 = 0)$ position of the function $\Delta(t^a, z^0)$. As a consequence of the above discussion, the limiting value, $\Delta_\infty(z^0)$, when $t^a \uparrow \infty$, exists and we may define an inverse function $Z^0(z)$ in the following way:

$$\Delta_\infty(z^0) = z^\infty \Rightarrow z^0 = \Delta_\infty^{-1}(z^\infty) = Z^0(z^\infty), \qquad (11.105)$$

which yields a correspondance between the initial and final altitudes of the particles during the unsteady adjustment process.

Then, the last evolution equation of the systen (11.101) shows that we have:

$$\frac{p^a{}_\infty(z)}{[\rho^a{}_\infty(z)]^\gamma} = \frac{P^0(Z^0(z))}{[R^0(Z^0(z))]^\gamma} = exp[S^0(Z^0(z))], \qquad (11.106)$$

where S^0 is the initial value of specific entropy related to the system of non-hydrostatic equations (11.86a, b, c, d).

By analogy, from the evolution equation (11.100) we have:

$$v^0(z) = v^a{}_\infty(z) = V^0(Z^0(z)). \qquad (11.107)$$

According to Outrebon's (1981) results the vertical shift is a quite significant phenomenon. Unfortunately twenty years later the problem of the practical interest of this result, for the short-range forescasting according to primitive inviscid-hydrostatic equations, has not been not cleared up! In fact, the best argument for considering the unsteady adustment to hydrostatic balance rests on the investigation of stability of the primitive inviscid-hydrostatic equations. What the computations by Outrebon (1981) tell us is that there is a quite active mechanism built into the equations which drives the atmosphere back to a state of hydrostatic balance and that the transient time associated with to this mechanism is less than the time necessary for a sound wave, starting from the ground, to cross back and forth the whole of the troposhere.

But the hydrostatic approximation should fail at high altitude! We note also that the decay in time like $(1/t^a)^{3/2}$ for w^a is non-uniform at infinity and is related to the double limiting process:

$lim_{t^a \to \infty}$ applied first, and then $lim_{z\uparrow\infty}$.

11.8. A SYSTEM OF MODEL EQUATIONS FOR LEE-WAVES IN THE WHOLE TROPOSHERE

We consider below the system of dimensionless Euler equations for the velocity components, u, v, w and the perturbations ω, π, θ, of the density, pressure and temperatue relative to the standard thermodynamic functions $\rho^*(z^*)$, $p^*(z^*)$ and $T^*(z^*)$, namely:

$$St\frac{D\omega}{Dt}+(1+\omega)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\frac{1+\omega}{T^*(z)}\left[1-\Gamma^*(z)\right]w;$$

$$(1+\omega)St\frac{Du}{Dt}+\frac{T^*(z)}{\gamma M^2}\frac{\partial\pi}{\partial x}=0;$$

$$(1+\omega)St\frac{Dv}{Dt}+\frac{T^*(z)}{\gamma M^2}\frac{\partial\pi}{\partial y}=0;$$

$$(1+\omega)St\frac{Dw}{Dt}+\frac{T^*(z)}{\gamma M^2}\frac{\partial\pi}{\partial z}-\frac{1+\omega}{\gamma Ma^2}\theta=0; \tag{11.108}$$

$$(1+\omega)St\frac{D\theta}{Dt}-\frac{\gamma-1}{\gamma}St\frac{D\pi}{Dt}+\frac{1+\pi}{T^*(z)}\left\{\frac{\gamma-1}{\gamma}-\Gamma^*(z)\right\}w=0;$$

$$\pi=\omega+(1+\omega)\,\theta.$$

When we assume that $Bo\equiv 1$ and in this case $z^*\equiv z$.

In this case $\Gamma^*(z) = -dT^*/dz$, in the troposphere is very close to $(\gamma-1)/\gamma$ and below we assume that

$$\Gamma^*(z) = -\frac{dT^*}{dz} = \frac{\gamma - 1}{\gamma} + M^2\Phi^*(z), \qquad (11.109)$$

the function $\Phi^*(z)$ being a known function of z which takes into account a weak stratification with altitude z of the standard atmosphere in the troposphere and $|\Phi^*(z)| = O(1)$.

With the assumption (11.109) and when M *tends to zero* (again low-Mach numbers asymptotics) with t, x, y, z and St, γ **fixed,** we assume the following asymptotic representation applies:

$$(u, v, w) = (u_a, v_a, w_a) + O(M), \qquad (11.110a)$$

$$(\omega, \pi, \theta) = M^2(\omega_a, \pi_a, \theta_a) + o(M^2). \qquad (11.110b)$$

In this case, from the above exact Euler equations (11.108) we derive in a straigthforward way a set of equations which are the so-called "deep convection" equations (Zeytounian (1979)) and are very similar to the so-called "*anelastic*" equations of Ogura and Phillips (1962). Namely:

$$\gamma T_p(z)\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w_a}{\partial z}\right) = w_a; \qquad (11.111a)$$

$$St\frac{D_a u_a}{Dt} + \frac{T_p(z)}{\gamma}\frac{\partial \pi_a}{\partial x} = 0; \qquad (11.111b)$$

$$St\frac{D_a v_a}{Dt} + \frac{T_p(z)}{\gamma}\frac{\partial \pi_a}{\partial y} = 0; \qquad (11.111c)$$

$$St\frac{D_a w_a}{Dt} + \frac{T_p(z)}{\gamma}\frac{\partial \pi_a}{\partial z} - \frac{\theta_a}{\gamma} = 0; \qquad (11.111d)$$

$$St\frac{D_a \theta_a}{Dt} - \frac{\gamma - 1}{\gamma} St\frac{D_a \pi_a}{Dt} + \frac{1}{T_p(z)}\Phi^*(z) w_a = 0; \qquad (11.111e)$$

$$\pi_a = \omega_a + \theta_a. \qquad (11.111f)$$

In equations (11.111 a, b, c, d, e) we have

$$\frac{D_a}{Dt} = \frac{\partial}{\partial t} + u_a \frac{\partial}{\partial x} + v_a \frac{\partial}{\partial y} + w_a \frac{\partial}{\partial z},$$

$$T_p(z) = 1 - \frac{\gamma - 1}{\gamma} z. \tag{11.112}$$

11.8.1. The steady two-dimensional case

In, the 2D case the continuity equation is:

$$\gamma T_p(z)\left(\frac{\partial u_a}{\partial x} + \frac{\partial w_a}{\partial z}\right) = w_a,$$

and can be integrated if we introduce the following generalized stream function $\psi_a(x, z)$, such that:

$$u_a = -exp\left[\frac{1}{\gamma}\int \frac{dz}{T_p(z)}\right]\frac{\partial \psi_a}{\partial z}, \tag{11.113a}$$

$$w_a = +exp\left[\frac{1}{\gamma}\int \frac{dz}{T_p(z)}\right]\frac{\partial \psi_a}{\partial x}. \tag{11.113b}$$

Then the equation:

$$\left(u_a \frac{\partial}{\partial x} + w_a \frac{\partial}{\partial z}\right)\left[\theta_a - \frac{\gamma - 1}{\gamma}\pi_a\right] + \frac{1}{T_p(z)}\Phi^*(z) w_a = 0,$$

or

$$\left(\frac{\partial \psi_a}{\partial x}\frac{\partial}{\partial z} - \frac{\partial \psi_a}{\partial z}\frac{\partial}{\partial x}\right)\left\{\theta_a - \frac{\gamma - 1}{\gamma}\pi_a + \int \frac{\Phi^*(z)}{T_p(z)} dz\right\} = 0,$$

leads to the following first integral:

$$\theta_a - \frac{\gamma - 1}{\gamma}\pi_a + \int \frac{\Phi^*(z)}{T_p(z)}dz = \Theta(\psi_a), \qquad (11.114)$$

where the function Θ is arbitrary and depends only on ψ_a.

Furthermore, from the two equations:

$$u_a\frac{\partial u_a}{\partial x} + w_a\frac{\partial u_a}{\partial z} + \frac{T_p(z)}{\gamma}\frac{\partial \pi_a}{\partial x} = 0,$$

$$u_a\frac{\partial w_a}{\partial x} + w_a\frac{\partial w_a}{\partial z} + \frac{T_p(z)}{\gamma}\frac{\partial \pi_a}{\partial z} - \frac{\theta_a}{\gamma} = 0,$$

or from the following vorticity equation

$$u_a\frac{\partial \Omega_a}{\partial x} + w_a\frac{\partial \Omega_a}{\partial z} + \frac{1}{T_p(z)}\Omega_a w_a = \frac{1}{\gamma}\frac{\partial}{\partial x}\left\{\theta_a - \frac{\gamma - 1}{\gamma}\pi_a\right\},$$

using the above steady 2D continuity equation, we derive a second first integral, namely:

$$exp\left[\frac{1}{\gamma}\int\frac{dz}{T_p(z)}\right]\Omega_a - \frac{1}{\gamma}\frac{d\Theta}{d\psi_a}z = \Sigma(\psi_a), \qquad (11.115)$$

where the function Σ is again arbitrary and depends only on ψ_a. We note that $[(\partial\psi_a/\partial x)\, \partial/\partial z - (\partial\psi_a/\partial z)\, \partial/\partial x]\,(d\Theta/d\psi_a) = 0$.

But

$$exp\left[\frac{1}{\gamma}\int\frac{dz}{T_p(z)}\right] = \left[T_p(z)\right]^{-\frac{1}{\gamma-1}},$$

and

$$\frac{\partial w_a}{\partial x} = \left[T_p(z)\right]^{-\frac{1}{\gamma-1}}\frac{\partial^2 \psi_a}{\partial x^2},$$

$$\frac{\partial u_a}{\partial z} = -[T_p(z)]^{-\frac{1}{\gamma-1}} \frac{\partial^2 \psi_a}{\partial z^2} - \frac{1}{\gamma}[T_p(z)]^{-\frac{1}{\gamma-1}} \frac{\partial \psi_a}{\partial z}.$$

Therefore, from (11.115) the following single equation for ψ_a is derived:

$$\frac{\partial^2 \psi_a}{\partial x^2} + \frac{\partial^2 \psi_a}{\partial z^2} + \frac{1}{\gamma T_p(z)} \frac{\partial \psi_a}{\partial z} = [T_p(z)]^{\frac{2}{\gamma-1}} \left[\Sigma(\psi_a) + \frac{1}{\gamma} \frac{d\Theta}{d\psi_a} z \right] \quad (11.116)$$

with

$$T_p(z) = 1 - \frac{\gamma - 1}{\gamma} z.$$

The arbitrary functions of ψ_a, in equation (11.116), $\Sigma(\psi_a)$ and $\Theta(\psi_a)$, must be determined (in the case of the lee-waves problem) from the boundary conditions far away from the relief, at upstream infinity. This problem is considered in detail in Chapter 13 of Zeytounian (1990).

11.8.2. The unsteady adjustment equation

Again in the system of deep convection equations (11.111a, b, c, d, e, f) the time derivative relative to ω_a is absent! The implied unsteady adjustment problem for the full quasi-linear unsteady Euler equations (11.108) is at the present time still an open problem. Below, we consider only the linear case.

In the linear case, in place of the set of deep convection equations (11.111a, b, c, d, e, f), we can derive a single equation for: $\partial u_a/\partial x + \partial w_a/\partial z \equiv D_2$, namely

$$St^2 \frac{\partial^2}{\partial t^2} \left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{2\gamma - 1}{T_p(z)} \frac{\partial}{\partial z} + \frac{\Phi^*(z)}{T_p(z)} \frac{\partial^2}{\partial x^2} \right\} D_2 = 0, \quad (11.117)$$

which is only a *second-order equation in time t,* whereas the associated linear equation derived from the full linear Euler equations (11.108) even with hypothesis (11.109), but with $M = O(1)$, was clearly of fourth order in time t! In §3.2 of Chapter 3 in Zeytounian (1990) the reader can find a general equation for the Euler linear theory.

Once again, there is a loss of the internal acoustic waves just like in the Boussinesq approximation. The difference being that in equation (11.117), there is a term in $\partial/\partial z$ which takes into account a weak compressibility.

To conclude, we wish to point out that various authors (in particular, Voulfson (1981)) have attempted to obtain forms of the deep convection equations which are "better" adapted for the numerical computations so that the latter possesses integral conservation laws. Unfortunately, from the viewpoint of asymptotic modelling, these various attempts cannot be considered consistent since they lead to the retention of certain terms of various orders, in approximate limiting equations, which do not correlate with the main dominant terms.

11.9. SOME COMPLEMENTARY REFERENCES

First we mention the book by Friedlander (1980) which is devoted to an introduction to the mathematical theory of geophysical fluid dynamics. In this book the reader can find the various topics related to geostrophic flow, the Ekman layer, the Rossby waves, vertical shear layers, internal waves, the problem of "spin-down" in a rotating stratified fluid and the some aspects of "baroclinic instability". The basic construction of each mathematical model treated by Friedlander is given in detail in order to provide sufficient information to communicate the essence of the material to an uninitiated reader. A second very interesting book is the book by Pedlosky (1987) where the reader can find various fundamental aspects of geophysical fluid dynamics, namely: the shallow water waves theory, viscous layers theory, quasi-geostrophic motion on the earth sphere, ageostrophic motion and stability (baroclinic instability) theory. In the more recent book by Monin (1990), which is divided into three parts, the reader can find in the first part the fundamental equations of geophysical fluid dynamics, linear wave theory and elements of the hydrodynamic instability; the second part is devoted to wave motions (surface, internal and Rossby-Blinova waves) and some aspects of the geophysical turbulence; finally, the third part concerns the problem of the general circulation of the atmosphere and ocean, the theory of climate and some hydromagnetic problems for planetary interiors.

CHAPTER 12

SINGULAR COUPLING AND THE TRIPLE-DECK MODEL

Keith Stewartson attributed the triple-deck model to "Neiland (1969), Messiter (1970) and Stewartson & Williams (1969)," but his own contributions to the origins and development of their theory were truly formidable (see, for instance, Stuart (1986)). The idea is simply stated: in perturbations to boundary layers, not necessarily small perturbations, there may be a three-layer structure composed of: (III) a thin sublayer on the wall - the lower deck, (II) the main part of the original boundary-layer - the main deck, and (I) the outer (possibly potential) flow - the upper deck. Viewing the matter today, and with knowledge of various papers by F. T. Smith (see, for instance, Smith (1979, 1982), but also the two papers by Bouthier (1972, 1973)), according to Stuart (1986): "we see a clear relation to the known Heisenberg-Tollmien structure of the solution to the Orr-Sommerfeld equation at high-Reynolds numbers".

Indeed, the paper of Lighthill (1953) to which Stewartson referred ["seminal ideas put forward by Lighthill (1953)"] was, (as far as J. T. Stuart knew!) the first to recognize the relevance of the Orr-Sommerfeld equation and of the Heisenberg-Tollmien structure for the solution of the boundary layer problem other than those of stability; in that paper, a Heisenberg-Tollmien layer on the wall was found to transmit disturbances upstream, even though most of the flow was supersonic - the problem was a linear one, and no separation was possible.

The importance and power of Stewartson's insight was to see the layered structure as the basis for a logical, rational scheme for the calculation of perturbed boundary layers, even in nonlinear cases (see, for instance, Stewartson (1974)).

The triple-deck structure is now seen as a useful and, indeed, valuable element in aerodynamic calculation and design. This is a substantial tribute to Stewartson's power and foresight. The triple-deck concept is K. Stewartson's greatest contribution to theoretical fluid mechanics; without doubt it has been immensely influential, perhaps after some delay, and it is, probably, the last giant milestone along the road through the whole story, which originated with Prandtl, and flourished continuously throughout that century. For an pertinent short 'qualitative-phenomenological" introduction to the asymptotics of the triple-deck theory, see Guiraud (1995, pp. 262-271).

For a comprehensive rewiew of the progress made in using the triple-deck concept, see the short but illuminating survey paper by Meyer (1983) and also the review paper by Nayfeh (1991). In the survey paper by V. Ya. Neiland (1981, in Russian), asymptotic studies of the separation and boundary layer/supersonic flow interaction are reviewed and the basic results for the three-layer theory of the separated and attached regions of the compressive and expanded flows with a strong local interaction are presented. The transonic regime was treated by Brillant & Adamson (1974), while the hypersonic one was investigated a bit later and thoroughly covered by Lagrée(1992).

A recent interesting book on the asymptotic theory of separating flows, is the book edited by V.V. Sychev (1987, Russian original edition), where the reader can find a detailed presentation of the so-called ‘Sychev proposal” concerning the criterion (at *Re >> 1)* for laminar separation, derived from compatibility of a perfect fluid analysis with a triple-deck simulation of the boundary-layer near the separation line. An application of Sychev’s proposal, for a vortex sheets separating from a smooth body, is given in Guiraud & Zeytounian (1979). In the more recent “Rapport Technique” by Roget, Brazier and Mauss (1994), a logical way to proceed, in order to obtain the triple-deck structure deductively is presented. In fact, once the asymptotic order of the longitudinal extent is obtained, one finds that a three-tier structure is needed normal to the wall. In particular, J. Mauss (see, also Mauss (1995) and Saintlos & Mauss (1996)) shows that the triple-deck structure is the first perturbation that can both displace the classical Prandtl boundary layer and cause separation of the flow. Above this exists a serie of perturbations, smaller but “stronger”, that cause a separation of the boundary layer without displacing it! This series is limited by the smallest perturbation compatible with the hypothesis of the theory, thus leading to a theory in “double deck”. Finally, in Zeytounian (1987, Chapter IX, pp. 159-229) the reader can find a short but detailed presentation (in French!) of the “Neiland -Stewartson & Williams” model of singular coupling with the various applications in hydro-aerodynamics, environmental fluid dynamics and atmospheric thermal convection problems, up to the beginning of the 1980’s. Unfortunately, further achievements (quite fascinating!) are so extensive that an up-to-date review would be appropriate.

12.1. GENERAL SETTING

The usual derivation of Prandtl's boundary-layer model assumes that the flow is, to some extent, smooth with respect to coordinates along the wall. If something breaks this smoothness one may expect some failure of Prandtl's boundary-layer model. Separation which was the main motivation for Prandtl in his 1904 paper, and shock boundary-layer interaction, as studied by Chapman, Kuehn and Larson (1958), are examples of such a failure.

12.1.1. The Oleinik paradox and the instability of the boundary-layer

Before discussing triple-deck theory, I will touch one the instability of a Blasius boundary-layer. Again we face a paradox, since it has long been suspected that Blasius solution are inherently stable within the Prandtl boundary-layer framework. This has been mathematically proven by Oleinik (1969). What one may state is that if we start from the Blasius solution and we linearize the two-dimensional unsteady Prandtl equations around it, all solutions to the linearized equations decay exponentially with time. On the other hand, the Blasius solution has long been known to be unstable, both from experiment, and as early as Tollmien (1929) from theory. As a matter of fact, all the eigenmodes of instability found by Tollmien (1929) violate the main assumption of Prandtl's boundary-layer that streamwise gradients are $O(Re^{-1/2})$ smaller than cross-stream ones.

Physically, we know that most instabilities arise from the breaking of some mathemathical structure (symmetry) involved in the basic flow. For example, Couette-Taylor instability arizes from the breaking of the invariance with respect to translation along the axis of symmetry. Tollmien-Schlichting waves arise from the breaking the basic assumption which is at the root of Prandtl's boundary-layer. They are, even, at complete variance with the latter, whick looks like

$$u = U_B\left(x, \frac{z}{\sqrt{Re}}\right),\ w = \frac{1}{\sqrt{Re}} W_B\left(x, \frac{z}{\sqrt{Re}}\right),$$

while the eigenmodes of instability, when superposed on the basic solution (U_B, W_B) lead to

$$u = U_B\left(x, \frac{z}{\sqrt{Re}}\right) + Real\left\{ i\alpha\Phi\left(\frac{z}{\sqrt{Re}}\right) exp\left[i(\alpha x - \Omega t) + \sigma t\right]\right\},$$

$$w = \frac{1}{\sqrt{Re}}\left[W_B\left(x, \frac{z}{\sqrt{Re}}\right) + Real\left\{\Phi'\left(\frac{z}{\sqrt{Re}}\right)exp[i(\alpha x - \Omega t) + \sigma t]\right\}\right],$$

but such a form of solution is not compatible with the fact that the basic flow (U_B, W_B) depends on x as well as on $z/\sqrt{Re}$.

The only consistent way would be to start from the basic solution: $u = U_0(z/\sqrt{Re})$, $W_0 = 0$, rather than from (U_B, W_B) and to look for a solution;

$$u = U_0\left(\frac{z}{\sqrt{Re}}\right) + Real\left\{i\alpha\Phi\left(\frac{z}{\sqrt{Re}}\right)exp[i(\alpha x - \Omega t) + \sigma t]\right\},$$

$$w = \frac{1}{\sqrt{Re}}\left[Real\left\{\Phi'\left(\frac{z}{\sqrt{Re}}\right)exp[i(\alpha x - \Omega t) + \sigma t]\right\}\right].$$

But this latter solution for u and w is not the solution to Prandtl equations.

The inconsistency goes further since the time-dependent terms in this solution are solutions of the linearized full Navier (incompressible and viscous) equations, while the basic solution $(U_0, W_0 = 0)$ is not a solution of the Navier equations. One deliberately changes the equations when going from $(U_0, W_0 = 0)$ to the latter solution for (u, w)!

Research on boundary-layer instability went on blithely from Tollmien (1929), but without questioning the above basic inconsistency! Of course the theory met success when compared to experiments, for example those of Schubauer & Skramstad (1943-republished 1947).

Some authors argued that the boundary-layer is slowly varying with respect to x, much more slowly than the eigenmodes of instability involved in the second part of the latter solution for (u, w), so that the inconsistency between the two above forms of solutions for (u, w) corresponds to such a tiny error that it does not change the conclusions extrated from the analysis.

The process for overcoming the above mentioned inconsistency was very long. Finally, the "Oleinik paradox" itself was, partially, resolved by Bouthier (1972, 1973), almost fifty years after the first attempt to predict instability, by Heisenberg (1924). The full asymptotic resolution, via the triple-deck, waited seven years more for the contibutions of Smith (1979), and Bodonyi & Smith(1981). It is fair to mention an early attempt by Lanchon and Eckhaus (1964). In Guiraud (1995, section 2.1 of §2) the reader can find a short but well documented discussion concerning the 'early roots' of the triple-deck theory.

12.1.2. D'Alembert's paradox and the Kutta -Joukowski-Villat condition at the trailing edge

Surprisingly, it was not until 1843 that Stokes obtained the potential flow around a moving sphere (in steady translation) leading to a pressure determined by Bernoulli equation, symmetric fore-and-aft! This Stokes solution confirmed the truth of an assertion of d'Alembert, made 75 years earlier, that: "A solid moving through a fluid, as defined by the Laplace equation, would encouter no resistance at all!". This is because, according to their model, steady potential flows are reversible:

> "To each flow with velocity potential $\Phi(x)$ corresponds a flow with opposite velocity potential $\Phi^*(x) = - \Phi(x)$, hence opposite velocity, $u^*(x) = \nabla\Phi^* = - u(x)$, and yet the same (Bernoulli) pressure distribution and hence the same lift and drag".

Therefore, symmetrical solids (e.g., ellipsoids) should encounter no lift or drag in steady translation! This d'Alembert's *paradox is* also related to the so-called Kutta and Joukowski (Joukovsky) condition. Indeed, this condition was given in an unpublished dissertation dated 1902 by Kutta (but see, Kutta (1902)), which obtained a solution for the two-dimensional flow of an inviscid fluid past a solid surface in the shape of a circular arc, at zero incidence, with circulation around the surface and a finite velocity at the trailing edge. A prior publication is attributed to Joukovsky in 1906 [according to Russian citations, in 1904 - see, for instance, the Russian translation (1951, pp. 121 and 279-280) of Prandtl's (1942) book], but oral tradition tells us that the French Mathematician H. Villat had, at about the same time, the same idea and the reader may find an allusion to this story in Villat's paper (1972; see the end of page 1 and the beginning of the page 2). Here, we only note that these questions are strongly connected with the well-posedness of the fluid flow problems and the behaviour of fluid flows with vanishing viscosity.

Indeed, according to the paper by Stewartson (1981), the mathematical conjectures associated with d'Alembert's paradox can be stated most clearly in terms of the Navier equation for an incompressible viscous fluid. In fact, the steady fluid flow passing close to a bluff body at the forward stagnation point does not remain near the body up to the rear stagnation point, but appears to *break* away from the surface leaving behind an eddying wake of almost constant pressure. This wake can extend to large distances behind the body and is turbulent, but we may infer that in the appropriate laminar solution the fluid also detaches from the body setting up a discontinuity surface. On one side of this surface the fluid has originated from an infinite distance upstream of the body but on the other one it has a different origin. The simplest structure one could ascribe to it is that it is at rest forming, as if were, a dead - water region.

We may therefore think of there being two candidates for the limiting solution to the Navier equation for a steady motion:

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\frac{1}{\rho_0}\nabla p+\boldsymbol{g}=\nu_0\nabla^2\boldsymbol{u},$$

when the constant kinematic viscosity ν_0 tends to zero: one (attached potential flow) being smooth and leading to d'Alembert's paradox, and the other [the so - called, Kirchhoff free - streamline flow; see, for example, the book of Birkhoff and Zarantonello (1957)] being discontinuous and containing a degree of asymmetry enabling us to avoid the paradoxical conclusion. Once discontinuities are admitted many choices can be made relating to the following queries: How do we choose the detachment point? Is the fluid at rest in the wake? How many detachment points are there? Is the wake finite? Those questions cannot all be answered on an inviscid basis only, and the next important step, for their resolution, was taken by Prandtl (1904), who introduced the concept of the boundary-layer. But, the resolution of the d'Alembert paradox and the "proof" of the validity of the K-J-V condition requires us to insert the more sophisticated "triple-deck structure". The classical Prandtl boundary layer is thereby partitioned into two parts, which are usually referred to as the lower deck and the main deck (which plays a passive role). The third, upper deck is the domain of the external flow field which is most significantly affected by rapid changes in the boundary layer. Indeed, the triple-deck structure is an interactive model, and such a structure can be applied to a rather large variety of situations for which the hierarchical structure (as in the boundary-layer theory) does not work.

The reason is that Prandtl's (asymptotic) boundary-layer structure leads to a singularity at the trailing edge, on the downstream side, due to the discontinuous change in the boundary condition, when the flow leaves the wing to enter the wake. On the other hand, the proposal of K-J-V, that the lift coefficient adjusts itself to the value which eradicates the worst singularity, is only true in the limit of infinite Reynolds number, thus one should expect a very tiny region, where the fluid flows around the trailing edge contradicting the condition! But the extent of this region is vanishingly small, being of the order of $\boldsymbol{Re}^{-3/8}$, involving a correction to the lift of the order of $\boldsymbol{Re}^{-1/8}$. Above some angle of attack, the critical value being of the order of $\boldsymbol{Re}^{-1/16}$, a separation occurs which pervades the whole of the lee-side of the aerofoil. It seems that a limited amount of separation near the trailing edge may occur with reattachment quite close to the trailing edge.

The model of flow is even more complex for aerofoils with finite thickness, where hysteresis phenomena may occur, concerning the dependence of the separation bubble, on the small thickness parameter. One may be sceptical, about the possibility of elucidating the structure of trailing-edge angle flows by a full scale simulation of the Navier (incompressible, viscous) model at high Reynolds numbers. The trailing-edge angle has to be small, of the order of $\boldsymbol{Re}^{-1/4}$, in order that there is no separation. The question, which is sometimes raised in the literature, concerning the birth of the vortex, as far as the inviscid fluid model is concerned, is meaningless:

> "if the trailing edge has a finite angle, then the flow is separated and the vortex sheet does not start at the trailing edge".

On the other hand, if the trailing edge is such that separation is avoided, the perfect fluid model is one at infinite Reynolds number with a vanishing trailing-edge angle and the sheet necessarily starts tangent to the common tangent to both sides of the edge.

It seems to us that the three dimensional analysis of Mangler and Smith (1970) overlooked this issue, to which the three dimensionality has no relevance because the variation of the flow, in the vicinity of the edge, is dominated only by *two dimensional effects*. Anyway the problem still largely open to discussion, even when one does not take into account either unsteadiness or turbulence.

It is necessary to note, also, that concerning separation, under subsonic flow outside the boundary layer, the potential for finding a rational explanation through the triple deck was discovered by Sychev (1972) but awaited its achievement through a numerical code until F.T. Smith (1977).

For bodies with a sharp trailing edge, experimental observation of subsonic flows shows that, in general, the action of viscosity causes the flow to leave the trailing edge smoothly, and that a thin wake is formed downstream from the trailing edge by the retarded layers of fluid from the body surface. As the viscosity tends to zero, this situation is idealized by the assumptions that the wake is infinitely thin as it leaves the body, and that the velocity at the sharp trailing edge is finite. This last condition is due to Kutta and Joukovski, but also to Villat (the K-J-V condition). Indeed, this K-J-V condition applies not only at sharp trailing edges in subsonic flows, but *also at sharp subsonic trailing edges in supersonic flows - the condition is found to be satisfied automatically at supersonic trailing edges [Ward (1955)]. The velocity varies rapidly through the thin wake in real flows, so the idealized wake can be taken as a vortex sheet from the time it leaves the trailing edge onwards. At a distance from the body this vortex sheet is supposed to rolled-up under the action of its own induced velocity, until it ultimately assumes the form of distinct vortex cores of finite diameter at large distances downstream. The greater the strength of the vortex sheet the more rapidly it rolls up, and experimental evidence generally supports this assumption, but the action of viscosity ultimately dissipates the vortex cores, so they do not extend downstream to "infinity" in practice.*

In linearized theory, the K-J-V condition is that:

the component of the perturbation velocity, normal to the mean body surface, must be finite in a neighbourhood of any trailing edge, since the usual condition of finite velocity is too *restrictive in linearized theory.*

This causes the appearance in the mathematical solution of surfaces stretching downstream from the trailing edges on which a tangential component of velocity is discontinuous. For a very pertinent discussion of the K-J-V condition, in relation to the d'Alembert paradox, see the mentioned above paper by Stewartson (1981). In fact, we note again that, according to the triple-deck model and contrary to the accepted K-J-V scheme, the flow does not leave the profile right at the trailing edge, but, rather, there is a slight warping of it, with an inviscid stagnation point on the leeward side of the profile, at a distance $O(Re^{-3/8})$ ahead of the trailing edge (where, in fact, $Re >> 1$). At high but finite values of the Reynolds number, for a two-dimensional steady flow, when we have a correction to the value of the lift coefficient = $const\, Re^{-1/16}$, the constant incorporates data from the inviscid Eulerian theory of the K-J-V flow around the profile, from the boundary layers, at the trailing edge, from both sides of the profile. It also incorporates data coming from the numerical solution of a "universal" triple-deck problem. As a consequence, the correction to the K-J-V classical condition, due to viscous effects, is far from being negligible. For a discussion of the various results in this direction and the main issues, see F. T. Smith (1983). For the case of a finite flat plate, due to this correction the leading-edge separation can be avoided for sufficiently small angles of incidence and at the trailing edge, on the other hand, this correction suggests how the self-regulation of circulation by vortex shedding may, if the trailing edge remains sharp, lead to a predictable value for the circulation which can eliminate separation in that region as well. Indeed, an aerofoil shape, with rounded leading-edge and sharp trailing edge, avoids separation not only in symmetrical flow around the aerofoil; it may also, for sufficiently small angles of incidence, avoid separation both at the leading and trailing edges after the circulation has become adjusted to a value for which the velocity at the trailing edge is finite (K-J-V condition!).

Curiously, it seems that it is not necessary to impose a K-J-V condition on calculations with time-dependent Euler fluid models in order to obtain the correct lift on aerofoils with sharp trailing edges. This remarkable result implies the existence of some mechanism in the pseudo-time evolution of the computed Euler solutions, which reproduces and simulates the essential physical phenomena leading to the generation of circulation and lift. At the initial instant the flow behaves in an irrotational manner with a stagnation point *S* on the suction surface inducing a turning of the flow around the sharp trailing edge (at the point *P*). Around the trailing edge, very strong velocity gradients exist since the inviscid incompressible velocity tends to infinity at *P* and the compressible flow will expand up to zero vacuum pressure. By some mechanism, an eddy is formed at *P*, preventing the infinite velocities or the vacuum conditions, and a surface of discontinuity appears, also called a vortex sheet, along which the two flows from the pressure and suction sides merge with a discontinuity in the tangential velocity.

In fact, this surface of discontinuity is a possible weak solution of the Euler equations, in the same way as are shocks. After some short time, the stagnation point has indeed reached the trailing edge and the eddy is transported by convection downstream of the aerofoil. Finally, a circulation appears around the aerofoil, equal and opposite to the circulation around the downstream convected eddy such that the total circulation around any contour enclosing the airfoil and the rolling-up eddy is zero as dictated by Kelvin's theorem. This sequence of events can not be simulated with potential flows, since this isentropic, irrotational flow model does not allow for vortex sheets with a discontinuity in tangential velocities. With the Euler flow model, on the other hand, vortex sheets can be captured by the computations and this transient sequence of events can be simulated numerically and inviscidly as soon as some mechanism exists that would trigger the generation of the trailing-edge eddy. It seems clear that in the Euler computations that do not require the imposition of the K-J-V condition, some mechanism has to exist that generates vorticity around the trailing edge in order to initiate the production of circulation

12.1.2a. The unsteady case

The behaviour of the unsteady triple deck has also attracted attention and interest here has been in the generalization of the K-J-V condition to include unsteady flows and the possible impact on the noise radiated from a finite plate when subject to a turbulent flow field. Concerning the viscous flow about the trailing edge of a thin profile in incidence angle rapidly oscillating, the reader can find an analysis in Brown and Daniels (1975). In this case, with the (large) Reynolds number, *Re* (such that the reference length, L_0, is the chord of the profile and the reference velocity, U_0, is the upstream velocity U_∞), we have also a (large) Strouhal number, $St = L_0\omega/U_\infty$ (based on the frequency of the oscillation ω, which is imposed), and a small parameter α, which characterizes the oscillations of the profile and plays the role of an incidence angle.

Then, when: $Re \to \infty$ and $St \to \infty$, we have a significant situation if: $St = S° Re^{1/4}$, with $S° = O(1)$, and the critical value of α is given by: $\alpha_c = K_{pi} S^{\circ -2} Re^{-1/4}$, where the coefficient K_{pi} is independent of *Re* and *St* (but the numerical value of K_{pi} is not known!). In fact, for high frequencies the mechanism which realizes the K-J-V condition, plays a significant role only for notably reduced amplitudes.

For sufficiently high Strouhal number, the trailing edge generates a thick wake. Brown and Daniels (1975) showed that the triple deck formulation is unaltered if $St << Re^{1/4}$, while if $St = S° Re^{1/4}$, time may be regarded as a parameter except in the lower deck where the BL equations must be generalized to an unsteady form. However, in view of the relation connecting p and u_e, in classical unsteady boundary-layer theory, namely: $\partial p/\partial x = u_e(\partial u_e/\partial x) + \partial u_e/\partial t$, care must be taken upstream of the triple deck. Over the majority of this region only the last term on the right is important, but there is a fore-deck region of length $O(l/St)$ just before the triple deck where both terms are significant. In the upper part of the triple deck, the relation for $\partial p/\partial x$ itself must be modified. Indeed, if $1 \leq St \leq Re^{1/4}$, Brown and Cheng (1981) have found the necessity for yet another precursor region, of thickness $O(l/St^{3/2})$ where the lower deck adjusts to achieve the quasi-steady form required in the triple deck.

It is interesting to note that a very naïve dimensional analysis shows that the above similarity relation: $St = S° Re^{1/4}$, plays a significant role because in the lower (viscous) deck of the triple-deck asymptotic model, the both terms $\partial u/\partial t$ and $u\partial u/\partial x$ are of the same order! Daniels (1978) has studied the viscous correction to the K-J-V condition in detail when: $St = S° Re^{1/4}$, and the fluid below the plate is stagnant, and found that application of the full K-J-V condition leads to consistent results, while the other conditions leads to inconsistencies when the amplitude of the oscillations is large enough, namely: $\geq 1/Re^{7/16}$, and conjectures that in fact they may always be excluded. The study of the flow past a flat plate at a small angle α^* of incidence is complicated by the appearance of irregularities with inviscid flow at both leading and trailing edges, and each may be associated with a type of stall, occuring with practical aerofoil shapes. Leading edge stall has received little attention so far from the point of view of a rational theory, and below we shall avoid it by supposing that the aerofoil is sufficiently fat at the leading edge to prevent it occuring, and only behaves like a flat plate very near the trailing edge. Suppose further that the K-J-V condition is applied at the trailing edge, so that according to inviscid theory, there is no stagnation point nearby on either surface of the aerofoil. Even so, the pressure gradient is infinitely unfavorable on the top side, so that, other things being equal, the boundary layer must have a catastrophic separation before the trailing edge is reached. On the other hand, as the symmetric problem with $\alpha^* = 0$ so clearly demonstrates, the speeding up of the fluid just downstream of the plate, once its restraining influence is removed, generates a favorable anticipatory pressure gradient upstream of the trailing edge. A triple deck analysis can be applied when those effects are comparable so that the adverse, inviscid pressure gradient is swallowed up by the favorable pressure gradient generated by viscous forces, before it becomes too serious. The condition requires that: $\alpha^* \leq O(Re^{-1/16})$. Details of the structure of the flow field near the trailing edge are given in Brown and Stewartson (1970). It is noted that the trailing edge effects again dominate such effects as displacement thickness and wake curvature, contrary to previously-held views! The theory cannot apparently prevent separation from occuring if: $\alpha^* > \alpha_S \, \lambda^{9/8} \, Re^{-1/16}$, with $\alpha_S \sim 0.4$ and where $\lambda = 0.3321$ is the Blasius skin friction. But it is not yet known whether the separation is of the catastrophic type or regular, so permitting an easy extension of the theory to include regions of reverse flows. Indeed, Sychev's theory of separation (1972), discussed below in §12.4, has a close relationship to the above studies and assumes that it is regular! The low power of Re occuring in the above estimates is interesting for it helps us to see why the stall problem appears to be inviscid in character, while its intimate association with the boundary layer demands a fundamental dependence on the Reynolds number. An extension of this study to aerofoils oscillating with period $2\pi/\omega^*$ has been made by Brown and Daniels (1975). The ideas of the triple deck have a ready extension to unsteady problems, and as ω^* increases, a significant effect first appears in the lower deck when: $\omega = \omega^* L/U = O(\varepsilon^{-2})$, when an unsteady term needs addings to the left-hand side of the equation of motion for the horizontal velocity component in the lower deck. Brown and Daniels

(1975) were able to examine how the K-J-V condition is modified from the quasi-steady form when: $\omega\,\varepsilon^2 << 1$ to having a $\pi/4$ phase lag when $\omega\,\varepsilon^2 >> 1$.

12.1.3. A phenomenological approach to the triple-deck theory

Curiously enough scientists waited until Stewartson and Williams (1969) and independently Neiland (1969) in order to have in their hands a proper asymptotic model while it is safe to mention that the essentials of it were contained in the seminal work of Lighthill (1953). Cutting short historical considerations let us try to explore how one can build the model. We assume, at the outset, that some kind of failure in Prandtl's model occurs near a line $\boldsymbol{L}$ drawn along the wall, and that this failure leads to rapid variation of flow characteristics in the vicinity of $\boldsymbol{L}$.

Starting with a steady flow one expects that the asymptotic structure within such a vicinity is plane steady, and normal to $\boldsymbol{L}$ locally. We introduce a basic small parameter ε such that:

$$\varepsilon^q = \frac{1}{Re}, \tag{12.1}$$

and we shall see that the precise value of q has no consequence on the final result provided $q > 0$. We assume that the extent of the looked-for new structure is $O(\varepsilon^\alpha)$ in both directions normal to $\boldsymbol{L}$. We set x and z for the local coordinates such that $z = 0$ is the wall, x is the distance to $\boldsymbol{L}$ in the plane tangent to the wall. At $O(1)$ distance from $\boldsymbol{L}$, $z = O(\varepsilon^{q/2})$ within the Prandtl boundary layer, and the component u of the velocity, parallel to the wall and orthogonal to $\boldsymbol{L}$ is $O(1)$. When approaching $\boldsymbol{L}$ within $x = O(\varepsilon^\alpha)$ we expect u to be perturbed and we try a small perturbation of order $O(\varepsilon^\mu)$ with $\mu > 0$ to be found. We observe that such a small perturbation was the starting point of Lighthill's (1950) work.

The reader should understand that were it not for the assumed failure, the u component would be convected without change over a short distance $O(\varepsilon^\alpha)$. As a consequence one may expect that the linearization of the equations are adequate; but the result of such a linearization depends on the value of α with respect to $q/2$.

In fact, a potentially richer situation may be expected if

$$\alpha < \frac{q}{2}, \tag{12.2}$$

the reason being that, then, the perturbations may be felt outside the boundary layer itself. Over distances $|x| = O(\varepsilon^{\alpha})$, $z = O(\varepsilon^{\alpha})$, the perturbations are governed by linear equations for steady plane perturbations relative to a uniform flow, being incompressible or compressible and either subsonic or supersonic depending on the value of the Mach number of the external flow at $\boldsymbol{L}$, or rather its component normal to $\boldsymbol{L}$. The special case of transonic external flow needs a separate treatment and we exclude it.

Let us try to figure out what is the order of magnitude of the perturbations. Let w be the unsealed component of the velocity normal to the wall, once the perturbation of u is $O(\varepsilon^{\mu})$, the continuity equation forces w to be $O(\varepsilon^{\mu+q/2-\alpha})$ when $z = O(\varepsilon^{q/2})$ and the same order is valid for w when z is just emerging from $O(\varepsilon^{q/2})$ that is from the inside of the boundary layer.

Then we expect that for $|x| = O(\varepsilon^{\alpha})$, $z = O(\varepsilon^{\alpha})$, all perturbations (both unsealed components of the velocity, pressure, density for compressible flow) are of the same order $O(\varepsilon^{\mu+q/2-\alpha})$. Thanks to (12.2) and to the obvious constraint $\mu > 0$, the perturbations are indeed small as they should be if linearization is to be adequate.

Let us now look for the order of perturbations within the boundary layer proper, that is when $|x| = O(\varepsilon^{\alpha})$, but $z = O(\varepsilon^{q/2})$. We assume that the perturbation of pressure within this region remains of the same order $O(\varepsilon^{\mu+q/2-\alpha})$ as outside and (12.2) leads to the conclusion that this perturbed pressure is negligible in comparaison to the $O(\varepsilon^{\mu})$ perturbation of u, which was assumed at the beginning and which remains to be found. Within $|x| = O(\varepsilon^{\alpha})$ the Prandtl boundary layer is so violently perturbed that the inertial variation is not compensated by perturbation of pressure. Such an inertia is $O(\varepsilon^{\mu-\alpha})$ while viscous forces are $O(\varepsilon^{\mu})$. The result is that the perturbations of the velocity within $|x| = O(\varepsilon^{\alpha})$, $z = O(\varepsilon^{q/2})$ are purely inertial. Let $U(zRe^{1/2})$ be the profile, within the boundary layer, above $\boldsymbol{L}$, of the component of the velocity normal to $\boldsymbol{L}$, for the boundary layer assumed to be unaffected by the failure. One may account very simply for inertial perturbation by assuming that u is transported, unchanged, along streamlines: $zRe^{1/2} = \zeta + \varepsilon^{\mu}A(\varepsilon^{-\alpha}x)$, at least when dealing with the incompressible case. This rough reasoning leads us to set down, within $|x| = O(\varepsilon^{\alpha})$ and $z = O(\varepsilon^{q/2})$, the relations

$$u = U(\zeta) + \varepsilon^{\mu}A(\varepsilon^{-\alpha}x)\, U'(\zeta) + o(\varepsilon^{\mu}), \tag{12.3}$$

$$w = -\, \varepsilon^{\mu+q/2-\alpha}\, U(\zeta)\, A'(\varepsilon^{-\alpha}x) + o(\varepsilon^{\mu+q/2-\alpha}), \tag{12.4}$$

and one may check that this agrees with the linearization of the Prandtl's boundary layer equations. We emphasize that the perturbation in (12.3) vanishes outside the boundary layer when ζ tends to infinity while it does not in (12.4). This justifies, *a posteriori,* our initial assumption that w leads the perturbations when $|x|$ and z are both $O(\varepsilon^{\alpha})$.

Now since $U'(0) \neq 0$, (12.3) and (12.4) fail to meet the proper no-slip conditions right on the wall at $\zeta = 0$. Reasoning again in the spirit of Prandtl (1904), and the argument was crucial in the pioneering work by Lighthill (1953), we should bring into the description a viscous inner layer within the inviscidly perturbed (according to (12.3) and (12.4)) boundary layer. Let us name it the inner layer or, as it is usually termed, the "lower deck". We expect that inertia balances shearing stress and if $z = O(\varepsilon^{q/2+\beta})$ within this lower deck such a balance is expressed by the relation $O(|u|^2\varepsilon^{-\alpha}) = O(|u|\varepsilon^{-2\beta})$. We require an appropriate magnitude for u and we extract it from (12.3) by rewriting it as (we introduce $\zeta/\varepsilon^{\beta} = \zeta^*$)

$$u = U'(0)\,[\varepsilon^{\beta}\,\zeta^* + \varepsilon^{\mu} A(\varepsilon^{-\alpha} x)\,] + O(\varepsilon^{2\beta}). \tag{12.5}$$

Let us emphasize that we borrow our argument from a kind of matching by assuming that (12.3) is valid within the outskirts ($\zeta \to 0 \sim \zeta^* \to \infty$) of the lower-deck. We expect that we retain the most general situation with $\beta = \mu$ so that, within the lower deck:

$$|x| = O(\varepsilon^{\alpha}),\ z = O(\varepsilon^{q/2+\beta}) \text{ and } u = O(\varepsilon^{\beta}). \tag{12.6}$$

We remind the reader of the fact that we expect failure of the Prandtl structure where, to leading-order, the external flow drives the boundary layer while the latter drives only perturbations $O(Re^{1/2})$ of the external flow. Here, on the other hand, we try to get (strong) coupling, at leading order, between the lower deck and the external flow.

The latter drives the lower deck perturbations through the action of the pressure and this is effective under the condition that $O(|u^2|)$ within the lower deck is of the same order as the perturbation of the pressure. But, according to what has been argued the latter goes straight from the external region: $|x| = O(\varepsilon^{\alpha})$, $z = O(\varepsilon^{\alpha})$ named the upper deck, to the lower deck one, crossing the so-called main deck $|x| = O(\varepsilon^{\alpha})$, $z = O(\varepsilon^{q/2})$, unchanged. But we already know the order of all perturbations, including pressure, within the upper deck, namely $O(\varepsilon^{\mu+q/2-\alpha})$, which means that $|u|^2$ should be $O(\varepsilon^{\mu+q/2-\alpha})$ within the lower deck. Thanks to (12.6) we get:

$$2\beta = \mu + \frac{q}{2} - \alpha, \tag{12.7}$$

while the balance between inetia: $O(u\varepsilon^{-\alpha}) = O(\varepsilon^{\mu-\alpha})$ and shear stress $O(\varepsilon^{-2\beta})$ allows one to add

$$\mu - \alpha = -2\beta, \tag{12.8}$$

to (12.7) and we remind the reader of our choice, derived from (12.5)

$$\mu = \beta. \tag{12.9}$$

Finally, from (12.7)-(12.9) we derive the following relations:

$$\varepsilon^{\beta} = \varepsilon^{\mu} = \varepsilon^{q/8} = Re^{-1/8},\ \varepsilon^{\alpha} = \varepsilon^{3q/8} = Re^{-3/8}, \tag{12.10}$$

and we check that (12.2) is satisfied. We observe that q has not been found at the end of the process and that its precise value does not change the various orders of magnitude as expressed in terms of powers of Re^{-1}.

12.1.4. The triple-deck vertical structure and strong coupling

Figure 12.1, below, gives a rough sketch of the whole local "triple-deck" vertical structure in the vicinity of the point $\boldsymbol{P}$ (on the x-axis) on $\boldsymbol{L}$. At the bottom we have a straight line representing the section of the wall. The reader can see the three layers, usually named decks, with their respective orders of magnitude, parallel and orthogonal to the wall. The upper deck (I) plays a crucial role because it encroaches on the domain of the Eulerian flow. It is, as far as the order of magnitude is concerned, as wide as it is large. The other two decks ((II) (III)) are rectangular - the main one (II) is simply embedded in the Prandtl boundary layer, while the lower one (III) is thinner than the latter.

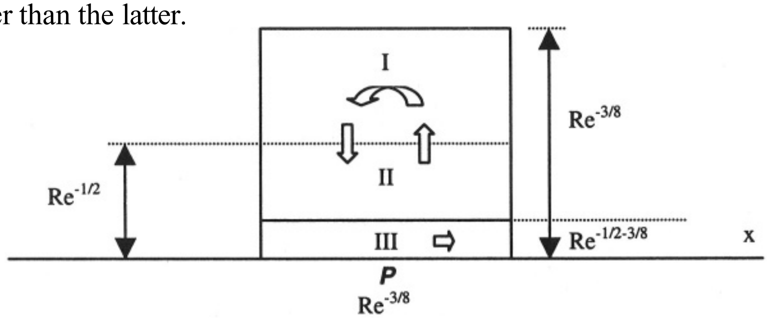


Fig. 12.1 Sketch of the whole local "triple deck" vertical structure in the vici,ity of the point P

The upper (I) deck is a kind of upstream transmitting layer for informations concerning violent perturbations produced by the failure near $\boldsymbol{P} \in \boldsymbol{L}$.

Such a transmitting role would be completely inhibited by the classical, outer-inner, asymptotic Euler-Prandtl non-interactive structure, and this was, to some extent, seized on by a number of scientists and the best we can do for characterizing these efforts is to quote Brown and Stewartson (1969, p.70):

"We want to make a final point about recent attempts to obtain theories of wake flows and interaction problems. These methods lay tremendous emphasis on massive computations on mathematical grounds. While we recognize the need for such a computational effort in the end, *it seems of great importance that a rational approach be adopted to make sure, for example, that terms neglected really are much smaller than those retained.* Until this is done, and even now it is possible in part, it will be difficult to convince the detached and possibly skeptical reader of their value as an aid to understanding".

We indicate the role of the upper deck by the bent arrow. The main deck is almost passive and just conveys information bilaterally from upper to lower (descending arrow) and from lower to upper (ascending arrow).

Finally, most of the information, and in particular all its linear part, goes into the lower deck and as with the Prandtl boundary layer such information cannot travel upstream, a feature which is recalled by the horizontal arrow.

We insist on the coupling schematized by the whole set of arrows drawn on the above figure 12.1. Such a coupling is unthinkable with the Euler-Prandtl structure except through higher order and would be, from first principles, restricted to weak coupling. On the other hand the triple-deck structure can *accomodate itself to a strong coupling,* in the sense that it occurs at leading order not as a small perturbation. The necessity of such a coupling was felt by many researchers.

But, early attempts failed to escape the conceptual restrictions of the Euler-Prandtl boundary layer. Lighthill (1950) tried to do so, but only in his 1953 paper did he succeed in elucidating the role of the viscosity in a thin sublayer. Finally, apparently independently, Stewartson and Williams (1969) and Neiland (1969) succeeded in building an asymptotically consistent structure which is the one we have described schematically.

Let us stress here that, to our knowledge, all the presentations of the triple deck, from an asymptotic point of view, are convincing but lack of a kind of uniqueness as demonstrated by Mauss (1995) using the general concept of significant degeneracy, according to Eckhaus (1979) as a methodology for deriving appropriate asymptotically consistent models, rather than "imagining" them and checking, aftewards, that they meet all the requirements of asymptotic models.

12.2. CLASSICAL TRIPLE-DECK THEORY

In the framework of high Reynolds number asymptotics, considered in Chapter 6, the theory "à la Prandtl" proceeded by the hierarchical method in which the expansions were worked out in each of two or more subregions alternately and term by term. Typically, one might begin with an inviscid flow in which $(1/Re)$ is set equal to zero, and the solution of this inviscid (Euler) problem satisfies the limiting form of the governing equations but not all the boundary conditions! A boundary layer is then introduced which enables the missing condition to be satisfied but leads to a mismatch with the inviscid solution of order $(1/Re)^{1/2}$. The next term in the inviscid solution is then worked out, which removes this mismatch but again fails to satisfy all the boundary conditions, and so on.

This line of approach works well except in two main instances. Whenever corners appear in the flow - and one may mention notably leading edges and trailing edges as examples - the hierarchical approach seems to break down, and apparently in the neighbourhood of the corners one cannot separate out inviscid and boundary layer subregions which can be treated alternately. Also, the onset of separation, typically associated with the vanishing of the skin friction on a fixed body in a steady stream, may lead to the appearance of an irregularity in the solution of the boundary-layer equations (see, for instance, the fundamental paper by Godstein (1948)) and this seems to render impossible the continuation of the solution further downstream. Since it is virtually impossible to construct a flow past a finite body without one or both of these characteristics, the asymptotic theory had severe limitations and few successes.

But, in 1966, the numerical studies of Catherall and Mangler (1966) showed that if the hierarchical, Prandtl, principle is abandoned and the pressure gradient is not imposed in the boundary layer, but instead the two are related in some way (!), then it is possible to proceed smoothly through separation without a singularity.

This idea was put on a rational basis by Stewartson and Williams (1969) and Neiland (1969) for steady supersonic flows in their studies of compressible free interactions. Further similar ideas were used by Stewartson (1969) and Messiter (1970) to overcome the breakdown in the neighbourhood of the trailing edge which occurs as a result of the formal application of hierarchical arguments. Subsequently it has been possible to make rapid strides in the development of an asymptotic theory of viscous flows, at high-Reynolds numbers, in a number of directions.

Pelletier (1972) attacked the key question of the asymptotics of solutions of the classical 2D steady incompressible Prandtl equations and was able to show with considerable generality that, in normal circumstances and for $dp_e/dx \leq 0$, the influence of the initial (in x) condition (specification of the value of $u(x = 0, y)$ for all $y > 0$) is transient and with increasing x, the solution develops an asymptotic character determined by the functions $p_e(x)$ and $u_e(x)$ related by:

$$p_e + \frac{1}{2}u_e^2 = const.$$

In this way, he illuminated the essential character of Prandtl's boundary layer weak-interaction equations as an inhomogeneous system of partial differential equations forced by the given functions dp_e/dx and $u_e(x)$ in the boundary-layer equations and the condition: $u \to u_e$ as $y \to \infty$.

But it is important to note that all the rigorous results of Oleinik (1963), Walter (1970), Nickel (1958), Serrin (1967) and Pelletier (1972), however, amount to only half a loaf because they depend on a 'favorable' pressure gradient, namely $dp_e/dx \leq 0$. When forced by an 'adverse' pressure gradient, $dp_e/dx > 0$, the solutions develop differently, and our knowledge is more tentative. As x increases, the surface value of $\partial u/\partial y$ then decreases, and before it ceases to be positive, the solution of the Prandtl equations tends to develop a singularity which is terminal in the sense that a continuation of the solution beyond it is not possible. Breakaway involves a reversal of flow direction near the aerofoil surface and hence, a negative 'wall shear' $\partial u/\partial y$ and before the solution of the boundary- layer equations can reach that, weak interaction appears to fail decisively!

Indeed, strong interaction poses the formidable difficulty of interdependent non-uniqueness of both the external inviscid flow and the viscous flow in boundary and shear layers. It is fortunate, therefore, that as in a weak interaction, for the first approximation, the external flow remains the classical potential flow past the geometrical airfloil.

In the spirit of the Meyer paper (1983) we can use the following four notions. First, the *upstream condition,* which requires consideration of the classical Prandtl limit with,

$$y_m = Re^{1/2}\, y, \tag{12.11a}$$

but the *localization condition* implies also a stretching function relative to x, namely,

$$\xi = (Re)^a x. \tag{12.11b}$$

The transformation (12.11a, b) is related to the "main-deck" limit.

Since (12.11a, b) fail to account for the *penetration condition* (which is a fundamental characteristic of the external flow), consideration must also be given to another transformation that can bring greater distances from the body surface into view, and this is the "upper-deck" limit. At such greater distances, the upstream condition implies that the streamlines (here we consider only a steady incompressible 2D Navier flow) come from the external flow because the *mass-bound condition* (we assume that the displacement thickness remains of order $(1/Re)^{1/2}$ throughout!) excludes a displacement of streamlines near the body surface by much more than $\delta = O(Re^{-1/2})$. By Kelvin's theorem, therefore, the flow in the upper deck remains an irrotational one, which has no distinguished direction of influence, and accordingly, the upper limit cannot normally involve a different stretch for x and y and must have a transformation:

$$\xi = (Re)^a x, \text{ and } \eta = (Re)^a y, \tag{12.12a, b}$$

with $0 < a < 1/2$, in order that the upper-deck penetrate significantly beyond the main deck. Mainly through the localization condition, the enhanced streamwise changes in the mild interaction are thus seen to overshadow the viscous shear in both decks so that both are governed by inviscid limit equations. As a consequence, a third deck limit is necessary, which is the "lower-deck" limit.

In fact, shear stresses substantially exceeding those of a weak interaction require the following transformation:

$$\xi = (Re)^a x, \text{ and } \zeta = (Re)^b y, \tag{12.13a, b}$$

with $b > 1/2$. Since $b > 1/2$, (12.11a, b) shows that (12.13a, b) resolves the limit

$$y_m \to 0 \text{ of the main deck!} \tag{12.14}$$

Obviously:

$$\zeta = \frac{y_m}{\left(\dfrac{1}{Re}\right)^c}, \text{ with } c = b - \frac{1}{2},$$

and as a consequence $y_m \to 0$ is 'equivalent', asymptotically, when *Re* tends to infinity, to $\zeta \to \infty$. In the lower deck we derive the boundary-layer equations (see, for instance, (12.16)) only if:

$$a + b - 1 = 0 \text{ and } b - \frac{1}{2} + \frac{a}{3} = 0,$$

and this implies

$$a = \frac{3}{8},\ b = \frac{5}{8},\ c = \frac{1}{8}. \qquad (12.15)$$

12.2.1. Flow over a flat plate with a small hump situated downstream of the leading edge

As a simple example, we consider viscous incompressible Navier flow over a flat plate with a small hump situated downstream of the leading edge. The characteristic Reynolds number *Re* based on the free-stream conditions $(\rho^*_\infty, U^*_\infty, \mu^*_\infty)$ and the distance L^* from the leading edge of the plate to the position of the hump is assumed to be large. The response of the boundary-layer flow to such a protuberance has been reviewed by Sedney (1973).

According to Neiland (1969), Stewartson and Williams (1969) and also Messiter (1970), the flow structure predicted by the triple-deck theory can correctly describe the flow field in the region of such a small hump if the streamwise extent of the hump is $O(L^*\varepsilon^3)$ and its height is $O(L^*\varepsilon^5)$, where $\varepsilon = Re^{-1/8}$. This is because for these scalings the important changes in the proporties of the flow take place over a length $O(L^*\varepsilon^3)$, a basic assumption in the triple-deck theory. Furthermore, the hump is embedded in the lower-deck region whose thickness is $O(L^*\varepsilon^5)$, where viscous terms are retained in order to satisfy the wall boundary conditions. The equation of the small hump within the lower deck is assumed in the form: $y = F(x)$, where x and y are the lower deck transformed variables.

As shown by Stewartson (1974) the flow in this lower deck is governed by the classical boundary-layer equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp(x)}{dx} + \frac{\partial^2 u}{\partial y^2};\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (12.16)$$

The boundary conditions on the wall are the usual no-slip and no penetration conditions:

$$u = 0,\ v = 0,\ \text{at } y = F(x). \tag{12.17}$$

As $y \to +\infty$, the solution in the lower-deck should match with the solution in the main-deck and the matching condition yields (where again $\varepsilon = Re^{-1/8}$):

$$u \to y + A(x) + \varepsilon\,[B(x) + a\,p_2(x)] + O(\varepsilon^2), \tag{12.18}$$

The interaction law which relates the pressure to the displacement function, $A(x) + \varepsilon B(x)$ - via the inviscid upper deck - is (in fact, $p = p_2(x) + \varepsilon p_3(x) + O(\varepsilon^2)$):

$$p = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left[\left(\frac{dA}{d\xi} + \varepsilon\frac{dB}{d\xi}\right)\frac{1}{x-\xi}\right]d\xi + \varepsilon b\frac{d^2A}{dx^2} + O(\varepsilon^2). \tag{12.19}$$

We note that $p_2(x)$, $p_3(x)$, $A(x)$, and $B(x)$, are unknown functions of x and

$$a = -1.3058,\ b = 0.60116.$$

Far upstream and downstream of the hump, the disturbance due the hump should decay and the Blasius flow should be recovered there. Therefore,

$$u \to y + O(\varepsilon^2),\ \text{as } x \to -\infty, \tag{12.20a}$$

and

$$p_2(x),\ p_3(x),\ A(x),\ \text{and } B(x) \to 0,\ \text{as } x \to \pm\infty. \tag{12.20b}$$

Equations (12.16), with the relation (12.19) and the conditions (12.17), (12.18) and (12.20a,b) are those which govern the lower-deck problem correct to second order. As in Ragab and Nayfeh (1980), to simplify boundary conditions (12.17), we use the Prandtl transformation theorem (see, for instance, the book by Rosenhead (1963)). We introduce new variables defined by:

$$z = y - F(x) \text{ and } w = v - u\frac{dF(x)}{dx}, \tag{12.21}$$

and under this transformation, the governing equations (12.16) and conditions (12.17), (12.18), and (12.20a) become:

$$u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{dp(x)}{dx}+\frac{\partial^2 u}{\partial z^2};\ \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0. \tag{12.22a}$$

$$u=0,\ w=0 \text{ at } z=0, \tag{12.22b}$$

$$u\to z+F(x)+A(x)+\varepsilon[B(x)+a\,p_2(x)]+O(\varepsilon^2), \text{ as } z\to+\infty, \tag{12.22c}$$

$$u\to z+O(\varepsilon^2), \text{ as } x\to-\infty, \tag{12.22d}$$

since $F(\pm\infty)=0$. Relation (12.19) and condition (12.20b) are unaltered. Solutions of these above equations can be expanded in the form:

$$u=u_1+\varepsilon u_2+O(\varepsilon^2),\ w=w_1+\varepsilon w_2+O(\varepsilon^2),\ p=p_2+\varepsilon p_3+O(\varepsilon^2). \tag{12.23}$$

Substituting these expansions into equations (12.22a, b, c, d) and equating coefficients of like powers of ε, one obtains equations for the first-order problem (u_1, w_1, p_2) and the second-order problem (u_2, w_2, p_3).

For the first-order problem we obtain:

$$u_1\frac{\partial u_1}{\partial x}+w_1\frac{\partial u_1}{\partial z}=-\frac{dp_2(x)}{dx}+\frac{\partial^2 u_1}{\partial z^2};\ \frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}=0.$$

$$u_1\to z+F(x)+A(x), \text{ as } z\to\infty,\ u_1\to z, \text{ as } x\to-\infty, \tag{12.24a}$$

$$u_1=0,\ w_1=0, \text{ at } z=0,$$

$$p_2(x), \text{ and } A(x), \to 0, \text{ as } x\to\pm\infty, \tag{12.24b}$$

$$p_2=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dA}{d\xi}\frac{1}{(x-\xi)}d\xi. \tag{12.24c}$$

The relation (12.24c), for p_2, gives the perturbation in pressure due to the displacement surface, $-A(x)$. Indeed, we identify this last relation as the direct problem in thin aerofoil theory. In the indirect problem, p_2 is given and the displacement body, $-A(x)$, is sought. By using a vortex sheet along the x-axis of strength $\gamma=-2p_2$ per unit length, we obtain the following relation:

$$\frac{dA(x)}{dx} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{p_2}{(x-\xi)}d\xi . \qquad (12.25)$$

The relation (12.24c) between the two functions $A(x)$ and $p_2(x)$, with inverse relation (12.25) is referred to as a Hilbert transform. Differentiating (12.25) with respect to x and integrating the result by parts gives:

$$\frac{d^2A(x)}{dx^2} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dp_2}{dx}\frac{1}{x-\xi}d\xi . \qquad (12.26a)$$

This relation is solved subject to the two boundary conditions

$$A(x) \to 0, \text{ as } x \to \pm\infty. \qquad (12.26b)$$

Finally, the pertinent problem valid to the first-order, for u_1, w_1, and p_2, deduced from the triple-deck theory, is:

$$u_1\frac{\partial u_1}{\partial x} + w_1\frac{\partial u_1}{\partial z} = -\frac{dp_2(x)}{dx} + \frac{\partial^2 u_1}{\partial z^2}; \quad \frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} = 0, \qquad (12.27a)$$

$$u_1 \to z + F(x) + A(x), \text{ as } z \to +\infty, \; u_1 \to z, \text{ as } x \to -\infty,$$
$$u_1 = 0, w_1 = 0, \text{ at } z = 0, A(x) \to 0, \text{ as } x \to \pm\infty. \qquad (12.27b)$$

$$\frac{d^2A(x)}{dx^2} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dp_2}{d\xi}\frac{1}{(x-\xi)}d\xi . \qquad (12.27c)$$

In a numerical procedure, first, we differentiate the equation of motion for u_1 with respect to z to eliminate the unknown pressure and obtain an equation for the shear stress: $\tau(x, z) = \partial u_1/\partial z$. Initially, we guess a value for the shear distribution on the wall $z = 0$ and then, by a marching technique, we find a solution for the shear stress at all mesh points. The pressure gradient is determined from:

$$\frac{dp_2(x)}{dx} = \left(\frac{\partial \tau}{\partial z}\right)_{z=0} . \qquad (12.28)$$

The second step is to solve the relation of (12.27c) (between A and p_2) with the conditions (12.26b) imposed at a finite position of x, for $A(x)$, and to do so it is necessary to obtain a numerical evaluation of the principal value of Cauchy's integral at the ith nodal point.:

$$CI_i = \int_{-\infty}^{+\infty} \frac{dp_2}{dx} \frac{1}{(x_i - \xi)} d\xi . \qquad (12.29)$$

The third step in the numerical procedure is to use the calculated values of A to update the wall shear stress. This is realized by satisfying the condition:

$$u_1 \to z + F(x) + A(x), \text{ as } z \to +\infty,$$

and since

$$u_1 = \int_0^z \tau\, dz \text{, then } \int_0^z \tau\, dz \to z + F(x) + A(x), \text{ as } z \to +\infty. \qquad (12.30)$$

This asymptotic condition is also imposed at a finite value of z. To start the computation, we needed an initial guess for the wall shear stress. We could start with very small heights of the hump for which the linearized solution given by F.T. Smith (1973) can be used. The reader can find the details of the numerical procedure in Ragab and Nayfeh (1980, pp.3-5 and Appendix A).

12.2.2. The triple-deck problem for the trailing edge of a finite plate at zero angle of attack

If we assume that the origin of coordinates is at the trailing edge of the finite plate, then the lower deck is governed by the classical boundary-layer equations (12.16), with the relation (12.19), and the conditions (12.18), (12.20a), but with $\varepsilon = 0$. In place, of (12.17) we have:

$$u_1 = 0,\ v_1 = 0, \text{ at } y = 0, \text{ and } x < 0, \qquad (12.31a)$$

$$v_1 = 0,\ \frac{\partial u_1}{\partial y} = 0, \text{ at } y = 0, \text{ and } x > 0. \qquad (12.31b)$$

It may be shown that as $x \to +\infty$, the solution of this lower-deck problem approaches that of the inner part of Godstein's near wake solution (see the figure 12.2 below). We now see that the triple deck makes a larger contribution to the drag on the plate than that due to what is commonly referred to as the displacement effect, i.e., the modification to the inviscid flow due to the effect of the boundary layer.

It is interesting to note that we have the astonishing result that the asymptotic expansion, strictly valid only when $Re^{-1/8} >> 1$, is in fact correct to within 10% for all $Re \geq 1$!

Our understanding of viscous incompressible (2D steady) flow past a finite plate, at $Re >> 1$, has been revolutionized by the application of the triple-deck theory for the flow near the trailing edge (on figure 12.2, the small parameter $\varepsilon = 1/Re$). It comprises a region around the trailing edge whose extent is of order $Re^{-3/8}$ times the length of the plate, and which matches upstream with the classical Blasius solution and its associated outer flow and downstream with the two-layered wake analyzed by Goldstein (1930).

There is also a small circular core in which, just as in the corresponding vicinity of the leading edge, the full Navier equations apply.

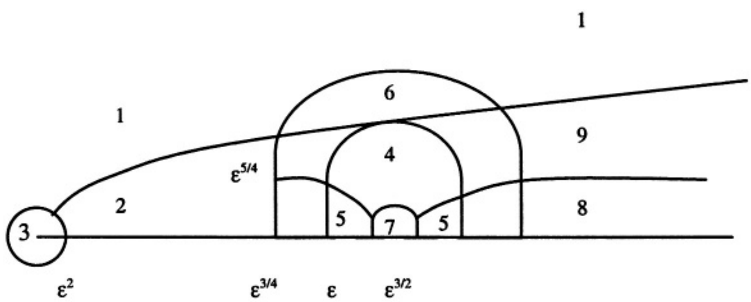


Fig. 12.2.
1- Potential flow; 2- Blasius BL; 3 and 7- Navier region;
4- Main deck, inviscid; 5- Lower deck, vicous; 6- Upper deck, potential flow;
8- Inner Goldstein wake, and 9- Outer Goldstein wake.

For the integrated skin friction on one side of the finite plate, as a consequence of the structure discussed above, we have the following formula:

$$C_F = \frac{1.328}{Re^{1/2}} + \frac{2.654}{Re^{7/8}} + \ldots + \frac{2.226}{Re} + \ldots - \frac{1.192}{Re^{3/2}} \log Re - \frac{4.864}{Re^{3/2}} + \ldots \tag{12.32}$$

In the formula (12.32), where the dots means that there exist probably intermediate terms which are not yet known: the first term is the classical Blasius drag. The above discussed structure, near the trailing edge, contributes a correction to the Blasius drag that is of order $Re^{-7/8}$, and hence slightly more important than the dispacement effects of order $1/Re$ calculated first by Kuo (1953) and Imai (1957) using a global balance of momentum.

The coefficient 2.654 was obtained mumerically by Veldman and van de Vooren (1974) - in fact, these authors find: 2.651 ± 0.003. The coefficient *2.226* was also obtained by a mumerical code (for the full Navier equations) by van de Vooren and Dijkstra (1970). Concerning the last two terms in (12.32)), see the book by Goldstein (1960). In fact, the fourth one comes from the leading order of the second order approximation to the boundary layer and was given by Imai (1957) who was unable to find the fifth term because it must include the contribution of the Navier solution right at the leading-edge. Indeed, it is necessary to note that the last term in (12.32) is given by:

$$-\frac{0.204 + 2C}{Re^{3/2}}, \tag{12.33}$$

according to Imai (1957), and the coefficient: $C \approx 2.33$ was obtained numerically by solving the full 2D steady Navier equations near a region around the leading edge whose extent is of order: 1/Re times the length of the plate [see the paper by van de Vooren and Dijkstra (1970)].

Concerning the $Re^{-7/8}$ dependence of the second term in (12.32), we observe that there is no change in the order of the skin friction when going from the upstream boundary layer to the lower deck, but this one term brings in a perturbation to an $O(Re^{-1/2})$ term integrated over an $O(Re^{-3/8})$ distance which gives an integrated contribution of $O(Re^{-7/8})$.

12.3. DEFINITION OF THE STEADY CANONICAL PROBLEM

Let us try to formalize the previous argument, at least in its simplest version in order to avoid some unnecessary technicalities. We use local Cartesian coordinates, x, y, z with z the distance to the wall, x the distance to $\mathbf{L}$, tangential to the wall, and y completing the orthogonal set.

We also assume locality of the physical phenomenon, again to avoid again technicalities related to the curvature of both $\mathbf{L}$ and the wall $\mathbf{S}$. Again u, v, w are corresponding Cartesian components of the velocity $\mathbf{u}(x, y, z)$. We work with dimensionless quantities as in Chapter 2, and we express asymptotic locality by setting:

$$x = \varepsilon^{\alpha} x^*, y = y^*, z = \varepsilon^{\beta} z^*, u = \varepsilon^{\sigma} u^*, v = \varepsilon^{\sigma} v^*, w = \varepsilon^{\delta} w^*,$$

$$p = P(y^*) + \varepsilon^{\pi} p^*, \rho = \rho^* \text{ and } E = E^*, \tag{12.34}$$

and we shall use three versions of (12.34) to deal with each of the three decks using, I, II, III, as an upper index to distinguish between them.

12.3.1. Upper deck

Starting with the upper (I) deck we have:

$$\alpha = \beta < \frac{q}{2} \text{ and } \sigma = \delta = 0, \tag{12.35a}$$

and

$$u^* = U^I(y^*) + \varepsilon^{\pi} u^{*I} + \ldots, v^* = V^I(y^*) + o(\varepsilon^{\pi}), w^* = \varepsilon^{\pi} w^{*I} + \ldots, \tag{12.35b}$$

$$\rho^* = R^I(y*) + \varepsilon^{\pi} \rho^{*I} + \ldots, p^* = p^{*I} + \ldots, E^* = E^I(y^*) + \varepsilon^{\pi} e^{*I} + \ldots. \tag{12.35c}$$

Substituting into the dimensionless NS-F equations it is straightforward to observe that viscous terms are of higher order, so that it is very easy to derive the following leading-order system for, u^{*I}, w^{*I}, ρ^{*I} and e^{*I}, namely:

$$R^I U^I \frac{\partial u^{*I}}{\partial x^*} + \frac{1}{\gamma M^2} \frac{\partial p^{*I}}{\partial x^*} = 0, \tag{12.36a}$$

$$R^I U^I \frac{\partial w^{*I}}{\partial x^*} + \frac{1}{\gamma M^2} \frac{\partial p^{*I}}{\partial z^{*I}} = 0, \tag{12.36b}$$

$$U^I \frac{\partial \rho^{*I}}{\partial x^*} + R^I \left(\frac{\partial u^{*I}}{\partial x^*} + \frac{\partial w^{*I}}{\partial z^{*I}} \right) = 0, \tag{12.36c}$$

$$R^I U^I \frac{\partial e^{*I}}{\partial x^*} + P \left(\frac{\partial u^{*I}}{\partial x^*} + \frac{\partial w^{*I}}{\partial z^{*I}} \right) = 0. \tag{12.36d}$$

We have ignored the y component of the momentum equation because, starting from $v^* = V^I(y^*) + \varepsilon^{\pi} v^{*I} + \ldots$, one finds $\partial v^{*I}/\partial x^* = 0$ and upstream matching leads $v^{*I} = 0$. One can checks also that the y component of the vorticity, at order ε^{π}, is $\partial u^{*I}/\partial z^{*} - \partial w^{*I}/\partial x^{*} = 0$. Let us put for the dimensionless speed of sound

$$c^{*} = C^{I}(y^{*})\, \varepsilon^{\pi} c^{*I} + \ldots, \tag{12.37}$$

in this case we can introduce a local Mach number M^{*} through

$$M^{*} = U^{I}\frac{M}{C^{I}} = M^{I}(y^{*}), \tag{12.38}$$

from which we extract the usual β, namely:

$$\beta^{2} = |1 - M^{I2}|. \tag{12.39}$$

The reader should recognized in (12.36) the equations of linearized aerodynamics and we may write down without any further argument the following solutions, namely:

$$\text{for } M^{*} < 1,\ \beta u^{*I} - i\, w^{*I} = F(x^{*} + i\,\beta z^{*I}),$$

$$p^{*I} = -\frac{\gamma M^{2}}{\beta} R^{I} U^{I} Real(F); \tag{12.40a}$$

$$\text{for } M^{*} > 1,\ \beta u^{*I} = G(x^{*} - \beta z^{*I}),\ w^{*I} = -\,G(x^{*} - \beta z^{*I}),$$

$$p^{*I} = -\frac{\gamma M^{2}}{\beta} R^{I} U^{I} G(x^{*}). \tag{12.40b}$$

In (12.40a) F is any complex valued function which is holomorphic with respect to its complex argument $x^{*} + i\,\beta z^{*I}$, for $z^{*I} \geq 0$, while in (12.40b) G is any real function of its real argument, but only downstream propagating perturbations have been taken care of. The counterpart of (12.40a, b) when M^{*} is *very close to unity* (transonic case) needs a separate study which can be found in Brillant & Adamson (1974) paper. Let us note that with either (12.40a) or (12.40b) the following holds true:

$$p^{*I} + \gamma M^{2} R^{I} U^{I} u^{*I} = 0. \tag{12.41}$$

12.3.2. Main deck

Considering now the main deck (II), we have to use the same value of α, while

$$\beta^{II} = \frac{q}{2} \text{ and } \sigma^{II} = \delta^{II} = 0, \tag{12.42a}$$

and we get the following perturbation scheme:

$$u^* = U^{II}(y^*, z^{*II}) + \varepsilon^{\mu} u^{II} + \ldots,\ v^* = V^{II}(y^*, z^{*II}) + \varepsilon^{\mu} v^{*II} + \ldots,$$

$$w^* = \varepsilon^{\mu+\beta^{II}-\alpha} w^{*II} + \ldots,\ p^* = p^{*II} + \ldots, \tag{12.42b}$$

$$\rho^* = R^{II}(y^*, z^{*II}) + \varepsilon^{\mu} \rho^{*II} + \ldots,\ E^* = E^{II}(y^*, z^{*II}) + \varepsilon^{\mu} e^{*II} + \ldots. \tag{12.42c}$$

Inserting (12.42b, c) into the NS-F equations one finds once more that viscous terms do not enter into play, at the leading-order, and one easily finds the proper equations:

$$U^{II}\frac{\partial u^{*II}}{\partial x^*} + w^{*II}\frac{\partial U^{II}}{\partial z^{*II}} = 0,$$

$$U^{II}\frac{\partial v^{*II}}{\partial x^*} + w^{*II}\frac{\partial V^{II}}{\partial z^{*II}} = 0,$$

$$\frac{\partial p^{*II}}{\partial z^{*II}} = 0, \tag{12.43}$$

$$U^{II}\frac{\partial \rho^{*II}}{\partial x^*} + R^{II}\left(\frac{\partial u^{*II}}{\partial x^*} + \frac{\partial w^{*II}}{\partial z^{*II}}\right) + w^{*II}\frac{\partial R^{II}}{\partial z^{*II}} = 0,$$

$$R^{II}U^{II}\frac{\partial e^{*II}}{\partial x^*} + P\left(\frac{\partial u^{*II}}{\partial x^*} + \frac{\partial w^{*II}}{\partial z^{*II}}\right) + R^{II}w^{*II}\frac{\partial E^{II}}{\partial z^{*II}} = 0.$$

We have also two relations issuing from the thermostatic equation of state for a perfect gas, first:

$$P(y^*) = (\gamma - 1)R^{II}E^{II}, \tag{12.44a}$$

and then

$$R^{II}\, e^{*II} + E^{II}\, \rho^{*II} = 0. \tag{12.44b}$$

But, the second relation (12.44b) requires the condition $\pi > \mu$, that will be checked later. The last equation of (12.43) may be changed to:

$$U^{II}\frac{\partial \rho^{*II}}{\partial x^*} + w^{*II}\frac{\partial R^{II}}{\partial z^{*II}} = 0\,, \tag{12.45}$$

if we take into account (12.44a) and the first equation of (12.43).

By the end the equations which govern the leading-order perturbations within the main deck are:

$$U^{II}\frac{\partial u^{*II}}{\partial x^*} + w^{*II}\frac{\partial U^{II}}{\partial z^{*II}} = 0\,,$$

$$U^{II}\frac{\partial \rho^{*II}}{\partial x^*} + w^{*II}\frac{\partial R^{II}}{\partial z^{*II}} = 0\,,$$

$$\frac{\partial u^{*II}}{\partial x^*} + \frac{\partial w^{*II}}{\partial z^{*II}} = 0\,, \tag{12.46}$$

$$U^{II}\frac{\partial v^{*II}}{\partial x^*} + w^{*II}\frac{\partial V^{II}}{\partial z^{*II}} = 0\,,$$

$$\frac{\partial p^{*II}}{\partial z^{*II}} = 0\,,$$

with the obvious solutions:

$$u^{*II} = A^{II}\left(x^*\right)\frac{\partial U^{II}}{\partial z^{*II}}\,,\; v^{*II} = A^{II}\left(x^*\right)\frac{\partial V^{II}}{\partial z^{*II}}\,,\; w^{*II} = -U^{II}\frac{dA^{II}\left(x^*\right)}{dx^*}\,,$$

$$\rho^{*II} = A^{II}\left(x^*\right)\frac{\partial R^{II}}{\partial z^{*II}}, \; e^{*II} = -\frac{E^{II}}{R^{II}} A^{II}\left(x^*\right)\frac{dR^{II}}{dz^{*II}}, \qquad (12.47)$$

Matching between (I) and (II) gives:

$$p^{*II}(x^*, y^*) = p^{*I}(x^*, y^*, 0), \; w^{*II}(x^*, y^*, \infty) = w^{*I}(x^*, y^*, 0), \qquad (12.48a)$$

and

$$lim_{z^{*II}\uparrow\infty}\left[U^{II}, V^{II}, R^{II}, E^{II}\right]\left(y^*, z^{*II}\right) = \left[U^{I}, V^{I}, R^{I}, E^{I}\right]\left(y^*\right), \qquad (12.48b)$$

since:

$$\frac{z^{*I}}{\varepsilon^{\phi}} = z^{*II}, \text{ with } \phi = \beta^{II} - \beta^{I} > 0.$$

The function $A^{II}(x^*)$ in the solution (12.47) is arbitrary, up to now, and the whole solution (12.42b, c) with (12.47) is obviously interpreted as convected transport without change of U^{II}, V^{II}, R^{II}, E^{II} along perturbed streamlines:

$$z^{II} = h + \varepsilon^{\mu} A^{II}(x^*),$$

so that $\varepsilon^{\mu} A^{II}(x^*)$ looks like a displacement thickness relative the level h. Notice that the dependence on y^*, which appears as a parameter, has been omitted for simplicity.

12.3.3. Lower deck

We come now to the lower deck, for which we keep α unchanged while we try to meet two requirements: pressure, both terms of the continuity equation and shear stress as well as inertia in the momentum equation, are all equivalent terms and this leads to:

$$\sigma^{III} = q + \alpha - 2\beta^{III}, \text{ and } \delta^{III} = \sigma^{III} + \beta^{III} - \alpha. \qquad (12.49)$$

Previous to inserting quantities into the NS-F equations we rewrite part of (12.42b), (12.47) using rescaled coordinates, according to the lower deck scaling, namely, when $z^{*II} \downarrow 0$:

$$u^* \approx \left.\frac{\partial U^{II}}{\partial z^{*II}}\right|_{z^{*II}=0} \left\{\varepsilon^{\beta^{III}-q/2} z^{*III} + \varepsilon^{\mu} A^{II}\left(x^*\right)\right\}, \tag{12.50a}$$

$$v^* \approx \left.\frac{\partial V^{II}}{\partial z^{*II}}\right|_{z^{*II}=0} \left\{\varepsilon^{\beta^{III}-q/2} z^{*III} + \varepsilon^{\mu} A^{II}\left(x^*\right)\right\}, \tag{12.50b}$$

since $U^{II}(y^*, 0) = V^{II}(y^*, 0) = 0$, and

$$\frac{z^{*II}}{\varepsilon^{\eta}} = z^{*III}\text{, with } \eta = \beta^{II} - \beta^{III} > 0\text{, if } \alpha > \sigma^{III}, \tag{12.51}$$

which is an approximate inequality and will be checked later. In fact, (12.50a, b) allows us to set

$$\sigma^{III} = \mu = \beta^{III} - \frac{q}{2}. \tag{12.52}$$

The reason why we retain (12.52) is that (12.50a, b) will be used for matching between lower and main decks and the general principle of least degeneracy urges us to require that both terms in the right hand side of (12.50a, b) be of the same order as the corresponding one in the lower-deck. In the lower deck we assume the following expansions:

$$u^* = u^{*III} + \ldots,\ v^* = v^{*III} + \ldots,\ w^* = w^{*III} + \ldots,\ p^* = p^{*III} + \ldots, \tag{12.53a}$$

$$\rho^* = R^{II}(y^*, 0) + \ldots,\ E^* = E^{II}(y^*, 0) + \ldots, \tag{12.53b}$$

with the above relations (12.51) and (12.52). We observe also that when $z^{*III} \uparrow \infty$,

$$lim_{z^{*III}\uparrow\infty}\left[U^{III}, V^{III}, R^{III}, E^{III}\right]\left(y^*, z^{*III}\right) = \left[0, 0, R^{II}, E^{II}\right]\left(y^*, 0\right). \tag{12.54}$$

From the NS-F equations, for the functions u^{*III}, v^{*III}, w^{*III}, p^{*III}, ρ^{*III}, e^{*III} which are dependent of variables x^*, y^* (as a parameter) and z^{*III}, we derive the following boundary-layer equations for u^{*III}, v^{*III} and p^{*III}:

$$\frac{\partial u^{*III}}{\partial x^*}+\frac{\partial w^{*III}}{\partial z^{*III}}=0,\ \frac{\partial p^{*III}}{\partial z^{*III}}=0, \tag{12.55a}$$

$$R^{II}(y^*,0)\left[u^{*III}\frac{\partial u^{*III}}{\partial x^*}+w^{*III}\frac{\partial u^{*III}}{\partial z^{*III}}\right]+\frac{1}{\gamma M^2}\frac{\partial p^{*III}}{\partial x^*}=\mu(y^*,0)\frac{\partial^2 u^{*III}}{\partial z^{*III2}}, \tag{12.55b}$$

and the following equation for v^{*III}:

$$R^{II}(y^*,0)\left[u^{*III}\frac{\partial v^{*III}}{\partial x^*}+w^{*III}\frac{\partial v^{*III}}{\partial z^{*III}}\right]=\mu(y^*,0)\frac{\partial^2 v^{*III}}{\partial z^{*III2}}, \tag{12.56}$$

where $\mu(y^*, 0) = \mu(E^{II}(y^*, 0))$.

Above, for the derivation of (12.55a, b) and (12.56) we have assumed that in (12.34) for the pressure p we have:

$$\pi = 2\sigma^{III}, \tag{12.57}$$

a requirement which is essential when one recognizes that the pressure p^{*III} has to be present in (12.55a, b) in order that the up-down mechanism of coupling between the upper deck and lower deck, schematized by the two vertical arrows on the above figure 12.1, operates effectively. We have also taken into account, first, the relation

$$P(y^*) = (\gamma - 1)\, R^{II}(y^*, 0)\, E^{II}(y^*, 0), \tag{12.58}$$

and then, as the wall condition, that the wall is either maintained at constant temperature or thermally isolated.

The equation (12.56) for, v^{*III}, is decoupled from the others and means that v^{*III} is simply advected and diffused like a passive scalar. Certainly the reader may argue, convincingly, that we have just done what is required for getting the result! Indeed, the above system of two equations (12.55a, b) do not govern the boundary-layer flow and this will be felt through the boundary conditions. What deserves paying some attention to it is that equations, which are formally quite similar, rule quite different phenomena).

Of course one is well accustomed to encountering the same mathematical model with the description of quite different physical phenomena (heat transfer versus diffusion of particles, supersonic linearized flow versus electromagnetic propagation, and so on...) but here we encounter the same model when faced with quite different local descriptions of the same global physical phenomenon. This observation is not unique but neither is it common. In fact, one may easily find other examples. For instance, the Stokes equations, considered in the Chapter 9, describe slow viscous motion around a body on the one hand but also motion quite close to the leading edge of a flat plate at high-Reynolds number. This is fair but we make the observation that the Reynolds number as well as the body are different for the two interpretations of the same model. A more convincing example is furnished by the Godstein (1930) similarity solution which governs the wake close to the trailing edge of a finite flat plate (see Section 12.2.2). Curiously it is possible to found another solution to Goldstein's equation but any physical interpretation for that solution seems not possible, while one eventually appeared, even closer to the trailing edge than the region Goldstein was interested in, emerging from a triple-deck analysis of the trailing-edge flow as shown by Brown & Stewartson (1970).

12.3.4. Matching

Now we need to discuss the problem of the matching between the three decks.

12.3.4a. Matching between (I)/(II)

Concerning matching between (I)/(II), using (12.42a) we have a condition on the gauges, namely:

$$\pi = \mu + \frac{q}{2} - \alpha, \tag{12.59}$$

and one on the functions,

$$U^{II} \frac{dA^{II}(x^*)}{dx^*} = Imag\left[F(x^*)\right], \text{ if } M^* < 1, \tag{12.60a}$$

$$U^{II} \frac{dA^{II}(x^*)}{dx^*} = G(x^*), \text{ if } M^* > 1. \tag{12.60b}$$

On the other hand, the matching of the pressure gives:

$$p^{*II} = -\frac{\gamma M^2}{\beta} R^I U^I Real\left(F\left(x^*\right)\right), \text{ if } M^* < 1, \tag{12.61a}$$

and

$$p^{*II} = -\frac{\gamma M^2}{\beta} R^I U^I G\left(x^*\right), \text{ if } M^* > 1, \tag{12.61b}$$

and also the relation (12.48b).

12.3.4b. Matching between (II)/(III)

Let us now consider the matching between the main deck (II) and lower deck (III). First concerning u and v we get the relation (12.52) and the following behaviour when $z^{*III} \uparrow \infty$

$$u^{*III} \approx \left.\frac{\partial U^{II}}{\partial z^{*II}}\right|_{z^{*II}=0} \left[z^{*III} + A^{II}\left(x^*\right)\right]; \tag{12.62a}$$

$$v^{*III} \approx \left.\frac{\partial V^{II}}{\partial z^{*II}}\right|_{z^{*II}=0} \left[z^{*III} + A^{II}\left(x^*\right)\right]. \tag{12.62b}$$

The matching of w is safely realized using an intermediate variable z^*_λ such that

$$z^{*II} = \varepsilon^\lambda z^*_\lambda = \varepsilon^{\beta^{III} - q/2} z^{*III}$$

and thus, the matching condition to order-one is (with z^*_λ fixed):

$$lim_{\varepsilon \downarrow 0}\left\{ -\varepsilon^{\mu+\frac{q}{2}-\alpha+\lambda} \left.\frac{\partial U^{II}}{\partial z^{*II}}\right|_{z^{*II}=0} \frac{dA^{II}\left(x^*\right)}{dx^*} z^*_\lambda - \varepsilon^{\delta^{III}+\lambda-\beta^{III}+q/2} w^{*III} \right\} = 0, \tag{12.63}$$

when (12.42a) has been taken into account. We extract two relations from (12.63): one concerns the gauge

$$\delta^{III} = \mu - \alpha + \beta^{III} \tag{12.64a}$$

the other concerns the behaviour of w^{*III} near the infinity:

$$w^{*III} \approx \left.\frac{\partial U^{II}}{\partial z^{*II}}\right|_{z^{*II}=0} \left[\frac{dA^{II}(x^*)}{dx^*}\right] z^{*III} + \left(w^{*III}\right)^{\infty} + o(1). \tag{12.64b}$$

12.3.5. Definition of the steady canonical problem

Finally, now, we have at our disposal the following relations (12.49), (12.52), (12.57), (12.59), (12.64a), concerning the various gauges, that we sum up:

$$\sigma^{III} = q + \alpha - 2\beta^{III};\ \delta^{III} = \sigma^{III} + \beta^{III} - \alpha;\ \sigma^{III} = \mu = \beta^{III} - \frac{q}{2};$$

$$\pi = \mu + \frac{q}{2} - \alpha;\ \delta^{III} = \mu - \alpha + \beta^{III}. \tag{12.65}$$

From (12.65) we extract the following values for:

$$\alpha = \frac{3q}{8},\ \beta^{III} = \frac{5q}{8},\ \sigma^{III} = \frac{q}{8} = \mu,\ \delta^{III} = \frac{3q}{8};\ \pi = \frac{2q}{8}, \tag{12.66}$$

as we have indicated the value of q has no particular significance and we simply observe that it is convenient to choose $q = 8$, so that the basic small parameter is:

$$\varepsilon = Re^{-1/8}. \tag{12.67}$$

The only meaning of which is that in the simplest circumstances we expect the expansions to include integer powers of ε. Of course, matching the pressure gives

$$p^{*II} = p^{*III} = p^{*I}(x^*, y^*, 0). \tag{12.68}$$

The singular coupling between the upper deck and the lower deck, or, otherwise stated, between the thin viscous sublayer and the relatively thick inviscid one, appears through (12.60a, b), (12.61a, b), (12.62a) and (12.68). As a matter of fact, the function $A^{II}(x^*)$ looks like a displacement thickness, produced by reducing the velocity to zero within the lower-deck, while (12.60a) (respectively (12.60b) when $M^* > 1$) provides a functional relation between the pressure which drives the viscous lower deck and the displacement that slowing down by friction provokes.

This is the asymptotically consistent version of coupling schemes. On the other hand, the numerous and various studies rely on a concept which is at variance with the boundary layer; namely to couple inviscid pressure on the wall and displacement thickness of the boundary layer. Once the boundary-layer concept is taken for granted one accepts that the displacement effect is of higher order. So that coupling both introduces coupling between terms which pertain to different orders of magnitude. Such an attitude is not numerically wrong, but it is asymptotically not consistent and such is the basis of Brown & Stewartson's (1969) criticism, as noted previously.

We have already stressed the remarkable unifying feature underlying the lower-deck equations. But this is not the whole matter because we may go much further towards unification, and we are now concerned with that process.

Let us introduce the so-called Hilbert transform H defined by:

$$Real\,[F(x^*)] = H\,[Imag\,(F(x^*)],\tag{12.69}$$

so that for the subsonic regime, $M^* < 1$, we may write:

$$p^{*III} = -\frac{\gamma M^2}{\beta} R^I U^{I2} H\left[\frac{dA^{II}(x^*)}{dx^*}\right],\tag{12.70a}$$

while for the supersonic one, $M^* > 1$, we rather obtain:

$$p^{*III} = -\frac{\gamma M^2}{\beta} R^I U^{I2} \frac{dA^{II}(x^*)}{dx}.\tag{12.70b}$$

These two formulae (12.70a,b) look quite close to each other, the difference consisting in the replacement of the Hilbert operator by the unit (or identity) operator when shifting from subsonic to supersonic. Concerning the Hilbert operator see, for instance, the relation (12.24c) with (12.25).

All the subsonic regime is condensed in the unique formula (12.70a) while all the supersonic one is condensed in the unique formula (12.70b). Of course there is such a huge difference between (12.70a) and (12.70b) through the presence or absence of H, that one strongly suspects that the transonic regime (M^* close to unity) needs a completely different approach.

Let us go ahead towards unification, we look for a kind of similarity and start by rescaling:

$$x^* = a\,x,\ z^* = b\,z,\ u^{*III} = c\,u,\ v^{*III} = c\,v,\ w^{*III} = b\frac{c}{a}w,$$

$$p^{*III} = d\,p,\ \frac{1}{b}A^{II}(ax) = A(x), \tag{12.71}$$

with:

$$a = \left[\frac{1}{\beta}R^{\infty}U^{\infty 2}\right]^{3/4}\left(\frac{1}{\mu_0}\right)^{1/4}\left(\frac{1}{R_0^{2/5}U'_0}\right)^{5/4}, \tag{12.72a}$$

$$b = \left[\frac{1}{\beta}R^{\infty}U^{\infty 2}\right]^{1/4}\mu_0^{1/4}\left(\frac{1}{R_0^{2/3}U'_0}\right)^{3/4}, \tag{12.72b}$$

$$c = \left[\frac{1}{\beta}R^{\infty}U^{\infty 2}\right]^{1/4}\mu_0^{1/4}\left(\frac{U'_0}{R_0^2}\right)^{1/4}, \tag{12.72c}$$

$$d = \gamma M^2\left[\frac{1}{\beta}R^{\infty}U^{\infty 2}\right]^{1/2}\mu_0^{1/2}(U'_0)^{1/2}, \tag{12.72d}$$

where

$$R^{II}(y^*, \infty) = R^I(y^*) = R^{\infty};\ U^{II}(y^*, \infty) = U^I(y^*) = U^{\infty}, \tag{12.73a}$$

$$R^{II}(y^*, 0) = R_0;\ \left.\frac{\partial U^{II}}{\partial z^{*II}}\right|_{z^{*II}=0} = U'_0\,;\ \mu(y^*, 0) = \mu_0. \tag{12.73b}$$

As a consequence of (12.71) with (12.72a, b, c, d) and (12.73a, b), the equations for the lower deck reduce to a universal form:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} = 0,\ u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial z^2}, \tag{12.74}$$

$$u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} = \frac{\partial^2 v}{\partial z^2}, \tag{12.75}$$

with the boundary conditions:

$$z = 0:\ u = w = v = 0, \tag{12.76a}$$

$$z \to \infty: u - [z + A(x)] \to 0, v - [z + A(x)] \to 0, w \to 0. \qquad (12.76b)$$

We have to, add the coupling condition

$$p = -H\left[\frac{dA}{dx}\right], M^* < 1;\ p = -\frac{dA}{dx}, M^* > 1, \qquad (12.76c)$$

and the upstream conditions

$$x \to -\infty: A(x) \to 0, u - z \to 0, v - z \to 0, \qquad (12.76d)$$

while the matter of conditions at downstream infinity, $x \to +\infty$, is left open and allows for the adequacy of the model with respect to a wide variety of situations.

One may observe that the whole content of the state of upstream boundary layer approaching $\boldsymbol{L}$ is condensed within (12.72a, b, c, d), (12.73a, b), the subsonic or supersonic character of it appearing through the appearance of the Hilbert operator $\boldsymbol{H}$ in (12.76c) applied to *dA/dx*.

The occurrence of negative powers of β in (12.72a, b, c, d) is another indication that the transonic regime requires a separate treatment.

Two main differences will occur; one through the formulae, (12.72a, b, c, d), (12.73a, b), the other through the nonlinear transonic equation for the upper deck and for consistency a coupling condition drastically differing from either (12.70a) or (12.70b). We refer for the details to Messiter, Feo & Melnik (1971) and Brillant & Adamson (1974).

12.4. LAMINAR SEPARATION AND SYCHEV'S PROPOSAL

We come now to what is perhaps the most elegant achievement of the triple-deck strategy, namely the Sychev (1972) proposal concerning two-dimensional steady laminar separation.

Germain (2000) lays emphasis on the fact that most of Prandtl's fundamental 1904 paper is devoted to separation. As a matter of fact, excepting turbulence, separation is the most serious and very difficult challenge that twentieth century fluid mechanicists have had to tackle.

Sychev's proposal is a definitive resolution of one aspect of this fascinating problem. It raises to a climax the Euler-Prandtl concept failure. We limit ourselves to two-dimensional steady incompressible flow. If one excepts highly profiled bodies, the boundary layer cannot accept facing an adverse pressure gradient, which decreases the skin friction downstream and drives the boundary layer towards the catastrophic Goldstein (1948) singularity. What is dramatic with that singularity is that it puts an end to mathematical existence of a solution to Prandtl's model.

The statement is not proven from a mathematical point of view but it is almost certainly true! An apparently obvious way out is to allow the boundary layer to separate ahead of the Goldstein singularity. But any inviscid model including a separating streamline involves an adverse pressure gradient, singular at the point of separation, namely

$$\frac{dp}{dx} = k(x_S - x)^{-1/2}, \tag{12.77}$$

where x is the abscissa along the wall and x_S the abscissa of the separation point. The factor $k \geq 0$ whenever there is separation, while $k < 0$ would mean that the streamline which is supposed to separate penetrates, downstream, inside the body, which is nonsensical! Then the new bounday layer associated to inviscid flow with separation would lead to a new Goldstein singularity ahead of the separation point. The consequence is that we face the paradox that the Prandtl boundary layer rides to a dramatic end of its existence before reaching its separation point. The only way out is to find a location for the separation such that $k = 0$.

12.4.1. From Kirchhoff to Sychev

As a matter of fact that Kirchhoff (1869) idea that a point of separation satisfying $k = 0$ may be found. The Kirchhoff proposal may be considered as the counterpart of the Kutta-Joukowski-Villat condition for streamlined aerofoils. Unfortunately it is very difficult to build a satisfactory inviscid flow involving a wide separated region.

Despite this would it be highly desirable to understand the very process of separation. The analogy between the Kutta-Joukowski-Villat condition and the Kirchhoff proposal is more sensible than one might think. The former says that the viscosity acts close to the trailing edge, due to high gradients, in such a way that the $|x|^{-1/2}$ singularity in pressure gradient is removed. On the other hand the Kirchhoff proposal says that viscosity acts close to the separation point, due to high gradients, in a such a way that the $(x_S - x)^{-1/2}$ singularity in pressure gradient is removed. One may even notice that in either case the singularity in pressure gradient is of the negative square root type. Sychev's (1972) proposal is then very easy to understand.

Imagine that we embed a triple deck of extend $|x_S - x| = O(Re^{-3/8})$, then it will achieve a pressure gradient going like: $Re^{3/8}\,Re^{-1/4} = Re^{1/8}$, where $Re^{3/8}$ comes from $\partial/\partial x$, and $Re^{-1/4}$ from the perturbation in pressure.

But, $k\,|x_S - x|^{-1/2} = O(k\,Re^{3/16})$ and we need only check whether the balance:

$k\,Re^{3/16} \approx Re^{1/8}$ is plausible, which it is provided $k = O(Re^{-1/16})$.

Again the Kutta-Joukowski-Villat condition and the Kirchhoff proposal meet elegantly within the framework of the triple-deck theory due to $\alpha = O(Re^{-1/16})$ for the former and $k = O(Re^{-1/16})$ for the latter. Indeed, the Kutta-Joukowski-Villat condition is valid at high Reynolds number and Kirchoff proposal is also, both provided $Re^{-1/16}$ be sufficiently small.

But the disappointing aspect of these beautiful theories is that they require such a high Reynolds number that the boundary layer is, in most common situations, expected to be turbulent?

Sychev (1972) has formalized the triple deck allowing for separation, and Smith (1977) has completely solved the problem including numerics and asymptotic behaviours upstream and downstream. Guiraud & Zeytounian (1979) have formally extended the analysis to three-dimensional separation along a smooth curve and Riley (1979) has considered separation along a ray of a conical body. In the review paper by Meyer (1983, §9) the reader can find an very pertinent discussion relating to Sychev's proposal and below we give some complementary informations concerning this Sychev's problem.

12.4.2. The Sychev problem for the lower-deck equations

Indeed, it is important to observe that in the connection with the triple-deck asymptotic structure, careful experiments have revealed an interval of 'pre-stall' incidence in which recirculation of the fluid is confined to a thin 'bubble' region adjacent to the aerofoil surface. As a consequence the key-process of 'separation' - *reversal of flow direction and recirculation of fluid close to the body surface* - can occur without full breakaway and without really strong interaction between boundary layer and external Euler flow. An opportunity thus appears of dividing the difficulties of strong interaction by studying first the flow at pre-stall incidence and in particular, by focusing attention on the bubble ends, where a local breakdown of weak interaction seems to occur. But for low-speed flow, it turns out to be a dead-end and in Section 9 of the Meyer paper (1983), an attempt is made to sketch Sychev's ideas on how the impasse might be broken 'to nail down' an important step in the understanding of fluid motion at high-Reynolds number. Indeed, the subsonic bubble-end envisaged at the start of the present account as the main motivation for the mild interaction definition [according to Meyer (1983)] has proven much more recalcitrant than the supersonic case (in this case the supersonic, potential-flow perturbation in the upper deck is governed by the wave equation) for which is derived a much simpler interaction condition,

$$p(x) = -\frac{dA(x)}{dx} \tag{12.78}$$

in the place of:

$$p = -[U_b(\infty)]^2 \frac{1}{\pi\lambda} \int \frac{dA(x')}{dx'} \frac{1}{x - x'} dx' . \tag{12.79}$$

When the qualitative influences upon each other of the mechanisms of potential flow and boundary layer are considered, a change of sign is found to occur at the sonic speed and as a result, the physical argument makes a self-contained, mutual interaction implausible.

The analytical and numerical exploration failed to shake this prognosis and had led to a strong conjecture [Stewartson (1974)] that the lower-deck equations together with the conditions: (12.79) and

$$u \to \lambda y, \text{ as } x \to -\infty \text{ for all } y \geq 0,\ u - \lambda y \to A(x) \text{ as } y \to \infty, \tag{12.80a}$$

where λ is the value of the skin friction coefficient, $\tau(x)$ at the origin of the lower deck coordinates, according to conventional theory, while $A(x)$ has to be found, as indeed does pressure $p(x)$, according to (12.79), and also:

$$u = v = 0, \text{ at } y = 0, \tag{12.80b}$$

does not possess a solution describing flow separation!

In Sychev's proposal (1972) the role of the lower deck is to resolve the apparent singularity of the external-flow limit needed to make the total picture consistent with a locally Kirchhoff model of breakaway. However, this contradicts (12.79) because it makes that integral divergent! Sychev proposes to avoid that difficulty simply by an interpretation of the integral in the finite-part sense.

The introduction of the finite-part involves, of course, a substantial shift of the solution concept into the realm of distributions and thereby mandates a revision from the ground up of the theory outlined in this account.
Specifically, Sychev supposes that:

$$\frac{dp^*}{dx^*} = \frac{1}{2}k^*\varepsilon^{1/2}\left(x_S{}^* - x^*\right)^{-1/2}, \text{ as } x^* \to x_S{}^*, \qquad (12.81)$$

where k^* is a constant. A similar idea has been advanced by Messiter and Enlow (1973). This means that just before separation, the boundary layer has a full profile, and the standard scaling laws of the triple deck can be applied to elucidate its behaviour near $x^* = x_S{}^*$. The only modification needed is that λ is now not necessarily equal to 0.3321 but depends on the skin friction just upstream of the triple deck. We note here that for the lower-deck problem we have:

$x^* = \varepsilon^3 L \lambda^{-5/4} x$, where L is the length of the finite plate.

Then, for the governing equations of the solution in the lower deck the boundary conditions are that:

$$u = v = 0 \text{ at } y = 0,\ u - y \to A(x) \text{ as } y \to \infty, \qquad (12.82a,b)$$

$$u = y + k^\circ (-x)^{1/6} f_1(\eta) + \ldots \text{ as } x \to -\infty, \qquad (12.82c)$$

and

$$p + k^\circ(-x)^{1/2} H(-x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{dA(\xi)}{d\xi} + k^\circ\xi^{1/2}H(\xi)\right]\frac{1}{x-\xi}dx, \qquad (12.82d)$$

in the sense of Cauchy's principal value.

In (12.82c, d), $f_1(\eta)$ is a known function of $h = y/(-x)^{-1/3}$, k° is a numerical constant related to k^*, and $H(x)$ is the Heaviside step (unit) function.

This above problem bears a close relationship to the theory of trailing-edge stall studied by Brown and Stewartson (1970). The two studies can be linked together by regarding Sychev's theory as partially a limiting situation of Brown and Stewartson's in which the angle of incidence α^*, while remaining $O(\varepsilon^{1/2})$, is nevertheless sufficiently large that the separation point has moved upstream, out of the immediate influence of the trailing edge.

It is important to note that, in fact, Sychev suggests that his description of separation, i.e., by a free interaction, is characteristic of incompressible flows and the notion that separation occurs as a result of an adverse pressure gradient distributed over a finite length "does not have any place in actuality".

Clearly the task is to determine whether a numerical solution of the lower-deck equations subject to conditions (12.82a, b, c, d) or its equivalent can be found in which separation occurs and Sychev asymptotic structure holds.

Obviously, the establishment of the correctness of Sychev's view of separation, even in a limited sense, would be of immense importance to the asymptotic theory of viscous flows, as it would open the way to extensive studies paralleling those for supersonic separation. In any case, it intensifies the view that the numerical study of the imcompressible triple deck is a very important area of research in laminar boundary layer theory at present.

But the above Sychev problem is a *homogeneous* problem for the unknown functions $u(x, y)$, $v(x, y)$, $p(x)$, and $A(x)$, and the uppermost question is now clearly:

Whether an eigenvalue γ exists for which the Sychev problem possessses a nontrivial solution?

Sychev was content with showing that such a solution, if there is one, would have asymptotic properties, as $x \to -\infty$, consistent with a description of the flow reversal and recirculation inherent in separation.

This existence question for the Sychev problem was addressed by F T Smith (1977) and in the absence of a mathematical theory for anything resembling to Sychev problem, he attacked it numerically.

It is not hard to appreciate that the task was very difficult, as noted by Meyer (1983), and could not be accomplished with definitive completeness. But Smith (1977) did succeed in establishing very substantial evidence in favour of the conjecture that an eigenvalue of γ exists near 0.44, where $\gamma > 0$ is a constant such that:

$$|x|^{-1/2}\, p \to -\gamma \text{ as } x \to -\infty. \qquad (12.83)$$

12.5. THE TRIPLE-DECK STRUCTURE AS A VERY CHARACTERISTIC DEGENERACY OF THE STEADY NAVIER EQUATIONS

12.5.1. The two-dimensional case

In Mauss (1995) the method of matched asymptotic expansions (MMAE) is used (following Eckhaus (1979)) to explain how the triple deck is a distinguished asymptotic structure arising from a local perturbation to an Euler-Prandtl structure. In the context of a laminar steady Navier flow over a flat plate, a theory is developed to explain the separation due to a disturbance on the wall.

In particular, Mauss shows that the triple-deck asymptotic structure is the first perturbation that can both displace the classical Prandtl-Blasius boundary layer and cause separation of the laminar flow.

In addition this 'first perturbation' related to the triple-deck asymptotic structure, there exists a family of perturbations, smaller than the triple deck but strong in some sense, that can cause a separation of the boundary layer without displacing it. At one end one finds the smallest pertubation compatible with the hypothesis of the theory, thus leading to a theory in *double deck.*

Indeed, little has been done in the development of a proper description of the nature of the triple-deck structure. Following the review paper of Nayfeh (1991), first, a systematic analysis of the Navier equations has been performed to prove the triple-deck theory by Mauss, Achiq and Saintlos (1992) and it was found that a second characteristic asymptotic structure is present in the theory. This structure is smaller since it does not affect the external flow (in this structure the upper deck disappears and we have a 'double-deck structure') and it does not require a triple-deck description even though its perturbation is able to cause boundary-layer separation.

Mauss (1995), for the matching between the main deck and lower deck, define the expansion operator M, for the main-deck, to the order ε^{a} such that

$$Mu = U_B(Y) + \varepsilon^{a}\left(\frac{dU_B}{dY}\right)A(X), \tag{12.84}$$

where $X = x/\varepsilon^{\alpha}$ and $Y = y/\varepsilon^{m/2}$, with $Re = \varepsilon^{m}$. The behaviour of Mu for $Y \downarrow 0$ in the lower deck is given by

$$I^{*} Mu = \lambda\varepsilon^{\alpha/3} y^{*} + \lambda\varepsilon^{a} A(X) + \ldots \tag{12.85}$$

since $U_B(Y) \rightarrow \lambda Y$ as $Y \downarrow 0$ and $y^{*} = y/\varepsilon^{\alpha/3}$ in the lower deck. We observe that: $U_B(Y) \rightarrow 1 + Exp$, as $Y \uparrow \infty$ and $\lambda = 0.33206$. Expansion (12.85) is to be compared with the behaviour when $y^{*} \uparrow \infty$ in the lower deck to the order $\varepsilon^{\alpha/3}$:

$$I^{*}u = \varepsilon^{\alpha/3} u^{*}(X, y^{*}). \tag{12.86}$$

The method of obtaining matching relations follows Van Dyke (1964), using what is called an asymptotic matching principle. In fact, matching is closely related to the problem of constructing a composite expansion and the key is to use Van Dyke's principle *only with expansion operators defined at the same orders* (see, for instance Mauss (1994)). Here, obviously, we must write, at the same order, $\varepsilon^{\alpha/3}$ or ε^{a}: $MI^{*}u = I^{*}Mu$, and now, it is possible to construct a uniform approximation of u such that:

$$C\,u = Mu + I^{*}u - MI^{*}u$$

at the order considered. Comparing first (12.85) and (12.86), we see that no matching is possible for $a < \alpha/3$. Indeed, two cases appear:

(i) $I^{*}Mu = \lambda\varepsilon^{\alpha/3}\,[y^{*} + A(X)]$ for $a = \alpha/3$, leading to the condition for the lower-deck problem:

$$lim_{y^{*}\uparrow\infty}\left(u^{*}\left(X,y^{*}\right)-\lambda y^{*}\right)=\lambda A\left(X\right); \tag{12.87a}$$

(ii) $I^{*}Mu = \lambda\varepsilon^{\alpha/3}\,y^{*}$ for $a > \alpha/3$ with the condition:

$$lim_{y^{*}\uparrow\infty}\left(u^{*}\left(X,y^{*}\right)-\lambda y^{*}\right)=0\,. \tag{12.87b}$$

Now, the problem in the lower deck is well-formulated (see Audounet, Mauss and Saintlos (1991)). In case (i), when $a = \alpha/3$, we need a coupling relation between $A(x)$ and $p^{*} = P(X)$ the pressure in the lower deck (triple-deck structure). In case (ii), when $a > \alpha/3$, $P(X)$ is known from numerical analysis (double-deck structure).

Indeed, the case of the double-deck structure corresponds to the limiting case $\alpha = m/2$, where the equations in the main deck can be written as:

$$U_B(Y)\frac{\partial u_m}{\partial X}+\frac{dU_B(Y)}{dY}v_m=-\frac{\partial p_m}{\partial X}, \tag{12.88a}$$

$$U_B(Y)\frac{\partial v_m}{\partial X}=-\frac{\partial p_m}{\partial Y}, \tag{12.88b}$$

$$\frac{\partial u_m}{\partial X}+\frac{\partial v_m}{\partial Y}=0\,, \tag{12.88c}$$

and in this case:

$$u = U_B(Y) + \varepsilon^a u_m(X, Y) + ..., v = \varepsilon^a v_m(X, Y) + ...,$$

$$p = \varepsilon^a p_m(X, Y) + ..., \tag{12.89}$$

with $a > \alpha/3$. We observe also that $P(X) = p_m(X, 0)$. The corresponding perturbation is the smallest possible which involves separation of the flow and in this case the upper deck disapears. The solution of the equations (12.88a, b, c) is not known analytically. In the case of the double-deck structure the classical equations for the viscous lower deck are resolved with the condition (12.87b).

It is interesting to note that if the pressure gradient in the Navier equations is

$$\frac{\partial p}{\partial x} = O(\varepsilon^{\mu})$$

then with figure 12.3, one visualizes the order of magnitude of the ratio of the pressure gradient to the length of the hump defined by $y = \varepsilon^{\beta} F(x/\varepsilon^{\alpha})$, with $\beta = \alpha/3 + (m/2)$. When α increases, the adverse pressure gradient increases. It becomes comparable to the driving pressure gradient for $\alpha = 3m/10$.

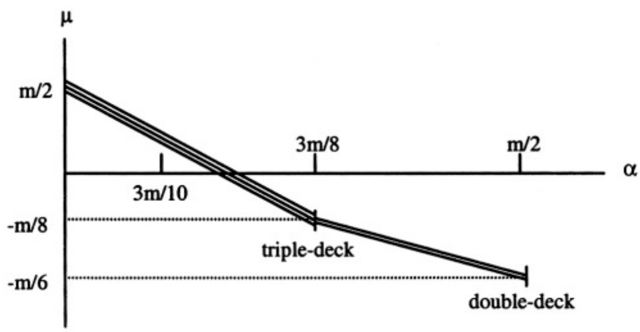


Fig. 12.3 Order of magnitude of the ratio of the pressure gradient to the length of the hump

Then, as μ increases, for $\mu = -m/8$, we have the triple deck and for $\mu = -m/6$, we have the double deck, *which is the last case of the asymptotic theory,* for perturbation ranging outside the standard Prandtl boundary layer.

12.5.2. The three-dimensional case

The three-dimensional perturbation of a Blasius boundary layer has recently been analyzed by Cousteix, Brazier and Mauss and for certain dimensions of the hump (which induce the 3D perturbation of the Blasius boundary layer), a <u>four-deck structure</u> is obtained (private communication of Jean Cousteix).

In 3D flow, the dimensions of the hump are characterized by the powers α, β and γ, such that the equation of the hump is:

$$y = \varepsilon^{\beta} F\left(\frac{x}{\varepsilon^{\alpha}}, \frac{z}{\varepsilon^{\gamma}}\right), \text{ with } \varepsilon = Re^{-m}. \tag{12.90}$$

Indeed, for $\alpha > \gamma$ the flow is, in fact, 2D, and for $\alpha < \gamma$ the "corner-flow" problem is addressed. So fully 3D problems are encountered only for $\alpha = \gamma$.

Depending of the values of α and β values, four zones have been identified in the plane (α, β) in which the flow is multi-deck structured.

The common middle point of these four zones corresponds to the triple-deck structure. Zones 1 and 2 have linear equations in the lower deck, whereas zones 3 and 4, corresponding to higher protuberances, have non-linear equations.

This limit giving the critical height had already been found for 2D humps by Smith, Brighton, Jackson and Hunt (1981).

In the asymptotic analysis, three conditions are imposed:

$$\beta > \frac{m}{2}, \beta > \alpha, \beta > \gamma,$$

and this means (physically) that the protuberance height is less than the oncoming boundary-layer thickness $(Re^{-1/2} = \varepsilon^{m/2})$, and less than its dimensions in the x and z directions. In the asymptotic model for zone 1, the interaction is said to be "weak", because the dispacement function A(X, Z), with $X = x/\varepsilon^{\alpha}$ and $Z = z/\varepsilon^{\gamma}$, and the pressure are unknown *at different orders* while the equations in the lower deck and in the upper deck are solved in the direct mode; the pressure is an output of the upper-deck solution and an input for the lower deck one.

With asymptotic model for zone 2, as in zone 1, the solutions in the different decks are obtained one after the other, because the displacement function is weak but, compared with zone 1, the hierarchy is inverted; the equations in the lower deck and in the upper deck are solved in the inverse mode: the pressure is an ouput of the lower-deck solution and is an input for the upper-deck solution.

The standard 3D model (Smith, Sykes and Brighton (1977), Bogolepov and Lipatov (1985)) is found for the values:

$$\alpha = \gamma = \frac{3m}{8} \text{ and } \beta = \frac{5m}{8},$$

exactly at the crossing point of limiting lines of zone 1 and zone 2. In this case the standard expression of the triple-deck equations as given by Stewartson, for example, is recovered when the function F simulating the hump is zero.

The solution in the three decks can be obtained using an extension of the method proposed by Veldman (1981) in 2D flow.
We precise that the values:

$$\alpha = \gamma = \frac{m}{2} \text{ and } \beta = \frac{2m}{3}, \tag{12.91}$$

corresponds to a *double-deck structure.*

In zones 3 and 4 the *matching of the crooflow* requires a *four-deck structure.* This four-deck structure of the flow: upper deck, main deck, intermediate deck and lower deck is described by a system of four equations, but only in the lower deck are the viscous terms present.

The pressure is constant in the lower, intermediate and main decks and equal to the pressure at the bottom of the upper deck. In the main deck, we have an analytical solution and the solution in the upper deck is such that the pressure at the bottom of this deck satisfies a coupling integral relation, where the integrals are taken as Cauchy principal values.

Taking into account the matching conditions, the upper, main and intermediate decks are strongly coupled. The solution in these decks requires a simultaneous resolution.

As *F(X, Z)* is an input, and ouput of this resolution is the horizontal components of the velocity in the intermediate deck at the bottom of this deck.

These horizontal components of the velocity are used as boundary conditions for the lower deck and in this lower deck the problem consists in solving a standard set of boundary-layer equations with the usual boundary conditions.

As a consequence, the resolution of the four decks is hierarchical and the set of the upper, main and intermediate decks is solved independently from the lower deck. This lower deck is solved afterwards. The displacement function *A(X, Z)* which is present in the explicit solution of the main-deck problem, does not depend on the solution in the lower deck, since *A(X, Z)* is the displacement - of non viscous origin - of the Blasius boundary layer induced by the hump.

The flow three-dimensionality is set up in the viscous lower deck thanks to the existence of the inviscid intermediate deck which is very specific to the 3D case. Indeed, the intermediate deck is a link between the quasi-2D form of the equations in the main deck and the fully 3D form in the lower deck.

12.5.3. Conclusion

The above discussion explains the occurence of least degeneracy for non-linear terms, and strong viscous-inviscid interaction.

The triple-deck asymptotic model thus includes all the properties of the different zones.

On the other hand, a special double-deck structure also appears, at the linear/non-linear crossover point. It corresponds to the situation where the upper-deck thickness becomes equal to the main-deck thickness. The hump is then so small that it does not disturb the inviscid flow.

The double-deck structure was obtained independently by Mauss (1994), see also Mauss, Achiq and Saintlos (1992), and Nayfeh (1991). Smith, Brighton, Jackson and Hunt (1981) also obtained a similar structure but only in the framework of a triple-deck structure which is certainly an unnecessary complication.

It is also interesting to note that in the derivation of a model problem for the local prediction of the temperature in the atmosphere (see for instance Zeytounian (1985, Section 7.2) a fundamental scalar

$$m = \frac{2a}{a-1}, \; 2Ro = (Re)^{-1/a} \tag{12.92}$$

appears in the formulation of the singular problem, where Ro and Re are the local Rossby and Reynolds numbers based on local length-scale l_0 which characterizes the thermal inhomogeneity at the ground surface.

The value $m = 5$ corresponds to triple-deck structure and in such a case the Boussinesq approximation can be utilized. For $m = 4$ or $m = 6$ the Boussinesq approximation is again valid but for $m = 3$ this is not the case and for these three cases we do not have the corresponding asymptotic model.

Finally, we observe that a weak interaction makes it possible to solve alternately a sequence of inviscid and boundary-layer problems in mode direct or inverse, depending on whether the pressure or displacement is known.

On the other hand, the strong interaction requires both problems to be solving simultaneously, and an efficient numerical algotithm should respect this principle closely.

It is interesting to note that, in 3d case, zone 2 requires an inverse method although the flow can not separate.

12.6. APPLICATIONS

Our aim below is to give only some informations concerning the various (mainly recent) applications of the triple-deck asymptotic structure in singular fluid dynamical problems.

The basic issues illuminated by the triple deck are so numerous that I want to select only a (significant?) part of the applications.

First the reader can find in the review paper by Stewartson (1974) the following fundamental problems: transonic free interaction where the main modification occurs in the upper deck, free interactions in supersonic flow with the main properties of flow arising when a shock interacts with a laminar boundary layer, the problem of the trailing edge of a symmetrically disposed flat plate which has been discussed in Section 12.2.2, the problem of the trailing-edge flows for bodies with finite thickness.

As a "complement" to that paper I mention the very pertinent contribution by Stewartson (1981) devoted to 'd'Alembert's paradox', which may now be regarded as largely resolved.

Concerning the separated flow in the vicinity of the trailing edge we note that Stewartson's triple-deck technique has been used by Guiraud (1974) in the case of a three-dimensional thin wing, when the trailing-edge flow is supposed to be dominated by the wedge effect but incidence may be taken care of as a perturbation. In such a case the flow contains two eddies of constant vorticity and it is necessary to consider *two* triple-deck structures fitted together (see also Guiraud & Schmitt (1975) and Schmitt (1976)). Thinness of the wing being referred to by the small parameter ε and the characteristic Reynolds number remaining $>>1$, the similarity parameter

$$\varepsilon Re^{1/4} = R^* \tag{12.93}$$

plays a significant role. For $R^* << 1$ the separation is absent, if $R^* = 1$ then, the separation occurs at distance from the trailing edge of order

$\varepsilon^{3/2} = (1/Re)^{3/8}$. On the other hand when $R^* >> 1$ this separation occurs at distance of the trailing edge of order $\varepsilon^{3/2} R^{*1/2}$.

Concerning laminar separation, an enlightening and thorough review of the problem as given by Brown & Stewartson (1969), Williams,III (1977) and Smith (1986).

A recent, very interesting, paper by Wu, Tramel, Zhu, and Yin (2000), concerns “a vorticity dynamics theory of three-dimensional flow separation” (in particular, a scale analysis under mild assumptions leads to a 3D triple-deck structure near a generic boundary-layer separation line).

We note also the recent paper by F.T. Smith (2000), on physical mechanisms in 2D and 3D separations.

An important step in the application of the triple-deck strategy to hydro/aerodynamics viscous problems is the review paper by Smith (1982), which is divided into three parts and the first one concerns *external flows* (it deals in turn with the classical attached flow strategy and its deficiencies). In a second part the internal flows are considered (in particular, the 3D pipe flows) and the third part deals with flow over an obstacle in a wall layer and also the unsteady breakaway separation.

In Zeytounian (1987, pp. 188-228) the reader can find various applications of the triple-deck asymptotic model and also a discussion related with to ‘unsteady triple-deck model’.

Concerning the unsteady problem, it is necessary, first, to note that an unsteady lower-deck model is derived from the unsteady Navier equations when (with $\varepsilon = Re^{-1/2}$):

$$x = \varepsilon^{3/4} X,\ y = \varepsilon^{5/4} y^*,\ t = \varepsilon^{1/2} t^*, \tag{12.94a}$$

$$u = \varepsilon^{1/4} u^* + \ldots,\ v = \varepsilon^{3/4} u^* + \ldots,\ p = 1 + \varepsilon^{1/2} p^* + \ldots, \tag{12.94b}$$

and in this case we can write an initial condition (at $t^* = 0$), for the horizontal velocity component $u^*(t^*, X, y^*)$, which is a consequence of the initial condition (at $t = 0$) for u in the Navier starting model.

But it is necessary to note also that for the unsteady lower-deck problem we do not have the possibility to write an initial condition for v^* (since $\partial p^*/\partial y^* = 0$ in the lower-deck unsteady equations).

As a consequence of this ‘*singularity in time*’ we have an *inner-initial time layer* at the vicinity of $t = 0$ and an unsteady *adjustment* problem! This unsteady adjustment problem is obviously more interesting in the compressible case, when in fact the *acoustic effects* are ‘almost’ *eliminated* within the lower-deck problem. But, according to our analysis of §7.4, Chapter 7, the unsteady compressible boundary-layer equations are not valid in the vicinity of the initial time $t = 0$ - in place of these BL equations we have the so-called Rayleigh-Howarth, one-dimensional unsteady NS-F equations (where the significant variables are the tune and the coordinate normal to the wall).

As a consequence it is necessary to analyse the Rayleigh-Howarth viscous layer separation in the vicinity of a 'local accident' on the wall in the inner-time region near $t = 0$. Then the derived asymptotic model must match with the triple-deck model valid for t^* fixed.

We oberve that for the Rayleigh-Howarth model the significant local time is $\tau = t/\varepsilon^2 = t^*/\varepsilon^{3/2}$ and when τ tends to infinity (unsteady adjustment process) by matching we can obtain the initial condition (at $t^* = 0$) for u^* as a solution of the lower-deck unsteady problem. *Only when this matching is possible, can we be sure that the unsteady lower-deck problem is a significant model of the NS-F model for high Reynolds number.*

In the book edited by V.V. Sychev (1987) the reader can find various problems considered in the framework of the triple-deck asymptotic model, by the Russian school (Sychev, Neiland, Ruban, Korolev, Timochin, Bogolepov, Ryzhov), and among many Russian papers I mention: Ruban & Sychev (1979), Sychev (1979), Korolev (1980a,b), Sychev & Sychev (1980) and Neiland (1981).

In relation to a triple-deck view of nonlinear stability theory, the so-called vortex-wave interaction theory (see Walton (1995)) as represented by the two fundamental papers of Hall and Smith (1984, 1991), is very significant.
As noticed by Lagrée (1999), thermal effects have not been strictly introduced in the treatments just cited (except in Zeytounian (1985) and (1987, pp.223-228)). Indeed, with a fixed wall temperature or an adiabatic wall the dynamical and thermal problems were decoupled (gravity being neglected).

In hypersonic flows, however, a low temperature is responsible for an effect (see, for instance, Brown & Cheng Lee (1990) and Neiland (1986)) which explains the differences between experiments and theory (Brown, Stewartson and Williams (1975)) - the temperature stratification comes from a low wall temperature or from the hypersonic entropy layer induced by the blunt nose of the plate (see Lagrée (1994)).

The problem of the thermal response within an incompressible Blasius boundary layer has been posed by Zeytounian (1987). Méndez, Treviño and Linan (1992) examined this problem but with the retroaction through a variable density (perfect gas) and viscosity (model fluid); Sykes (1978, 1980) looked at buoyancy effects in a stratified flow, the stratification being in the perfect fluid. In these works, only the forced convection without gravity has been examined. Without external flow, the problem is a free (natural) convection problem driven by the buoyancy, which has been investigated recently by El Hafi, in his thesis of the University of Toulouse (1994), who considers the natural convection flow along a vertical plate with a small hump leading to a special triple-deck problem. Concerning mixed convection (when the effects of both buoyancy and forced convection are present), see the recent paper by Lagrée (1994, 1995), and also: Daniels (1992), Schneider & Wasel (1985) and Zeytounian (1985).

In more recent paper, Lagrée (1999) considers the influence of a step change of a lower wall temperature in an established Poiseuille flow at high Reynolds and Froude numbers: the mixed convection is localized in a thin layer near the wall. Lagrée concentrates his investigation on a special range of longitudinal scales coherent with the triple-deck asymptotic model.

On the other hand the inviscid-viscous interaction on triple-deck scales in a hypersonic flow with strong wall cooling, is considered in Brown, Cheng & Lee (1990).

Recently various papers have been devoted to unsteady boundary-layer separation, which occurs at high Reynolds number in large number of flow configurations [Cowley, Van Dommelen & Lam (1990); Peridier, Smith & Walker (1991); Smith *et al.* (1991); Doligalski, Smith &Walker (1994); Smith and Walker (1995)].

In Degani, Walker & Smith (1998) the reader can find a pertinent discussion concerning various aspects of this problem. An example of a fruitful combination of asymptotics with hard computational difficulties is the results obtained in Degani *et al.* (1998), concerning violent eruptions of viscous flow into an inviscid flow.

In particular, the Lagrangian description of unsteady separation is discussed in Van Dommelen (1991) and by Van Dommelen & Cowley (1990).

The hypersonic boundary-layer separation on a cold wall is considered by Kerimbekov, Ruban & Walker (1994).

On the other hand the laminar separation at a corner point in transonic flow has been investigated in the recent paper by Ruban and Turkyilmaz (2000) in the framework of an asymptotic analysis of the Navier equations at large values of the Reynolds number.

The transonic trailing-edge flow has being considered recently by Bodonyi and Kluwick (1998). Concerning 'mathematical methods in boundary-layer theory' we mention, again, the recent book by Oleinik and Samohin (1997, Russian ed.). In the recent paper by Ryzhov and Bogdanova-Ryzhova (1998), the forced generation of solitary-like wave related to unstable boundary layers, is considered and the reader can find in this review paper various interesting discussion concerning the application of the triple-deck concept (soliton emission in forced BDA and KdV systems, and the onset of random disturbances

Finally, concerning 'turbulent shear flows over hills' in the paper by Hunt, Leibovich & Richards (1988), the reader can find various information, results and references related to the applications of ideas taken from triple-deck asymptotic analysis. In fact, the general structure of the solution is defined by dividing the flow into two regions, each of which is divided into two sublayers: an inviscid outer region composed of an upper layer in which there is potential flow when the atmosphere is neutrally stable, and a middle layer in which the wind shear dominates: and an inner region of thickness $l << L$ (the half-length of the hills) in which the effects of perturbation shear stresses are confined. The latter region being also divided into two.

I shall close this Chapter by quoting a few lines extrated from a recent paper by Paul Germain (2000).

"But, again, I express the view that, even a genius would not have been able to build the whole of the triple-deck model without the help of the matched asymptotic expansion techniques. Triple-deck theory is now a very important building stone in the new 'Fluid Dynamics inspired by asymptotics' and it may be fully included within the heritage of Prandtl. A good rewiew of different contributions issued from this theory may be found in Stewartson (1974) and Smith (1982) That one may include it in the heritage of Prandtl is provided by the contributions of Sychev (1972) and Smith (1977), who gave a quite satisfactory answer to Prandtl's query about separation".

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